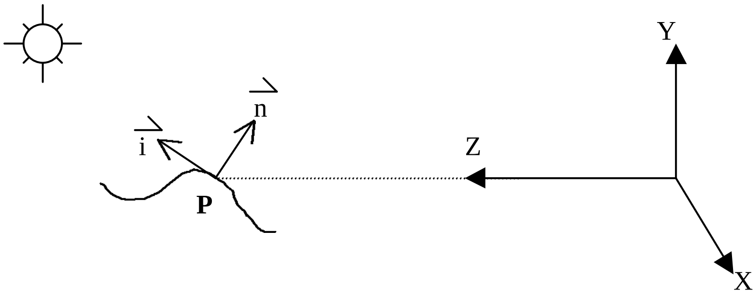
Shape From Shading

SFS uses the pattern of lights and shades in an image to infer the shape of the surfaces in view.
Lambertian Surfaces – uniformly illuminated Lambertain surface appears equally bright from all viewpoints.



i = illuminant direction (points to light source.
n = the surface normal at P
ρ = the effective albedo (real albedo times the intensity of illuminant.

Recall, Scene Radiance is: $L(P) = \rho < i,n >$ (Under the Lambertian assumption ρ is a constant)

Now assume the "Weak Perspective" model with: $x= (f/Z)X$ and $y=(f/Z)Y$
Let $R_{\rho,i}(n) = \rho < i, n >$ be the reflectance map of the surface

Image Irradiance Equation: (denoting with $p=[x, y]^T$ the image of P) $\mathbf{E(p) = L(P)\ (\Pi/4)\ (d/f)^2 \cos^4\alpha}$

E(p) – the brightness measured on the image plane at p.

Assume: 1)neglect the constant term **(Π/4)**, and assume system has been calibrated to offset the $\mathbf{\cos^4\alpha}$
2)All visible points of the surface receive direct illumination.

Therefore: $E(p) = R_{\rho,i}(n)$ –**FUNDAMENTAL EQUATION OF SHAPE FROM SHADING**

Further assume 3)The visible surface is far away from the viewer
4)and the visible surface can be described as Z=Z(X,Y)

This enables us to adopt the weak-perspective camera model and obtain:

$$x = f(X/Z_0) \quad y = f(Y/Z_0)$$

Z_0 – the average distance of the surface from the image plane.
Z = Z(x,y) Z is a function of the (x,y) coordinates on the image plane

A point P is now described by (x, y, Z(x,y))
Now compute the surface slopes by taking the x and y partial derivatives of $[x, y, Z(x,y)]^T$

Let $\vec{V1} = [1, 0, \partial Z/\partial x]^T$
and $\vec{V2} = [0, 1, \partial Z/\partial y]^T$

$$\vec{V1} \times \vec{V2} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}$$

where $p = \partial Z/\partial x$
and $q = \partial Z/\partial y$

$$= + \vec{i}(-p) - \vec{j}(q) + \vec{k}(1) = \boxed{\begin{bmatrix} -p \\ -q \\ 1 \end{bmatrix} \Big/ (1 + p^2 + q^2)}$$

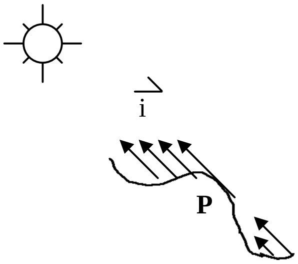
$\vec{n}$ unit vector in the normal direction

Now plug $\vec{n}$ into $\rho \langle \vec{i}, \vec{n} \rangle = E(p)$ to obtain:

$$\boxed{E(p) = \frac{\rho}{\sqrt{1 + p^2 + q^2}} \langle \vec{i}, [-p\ -q\ 1] \rangle}$$

→ Starting point of many shape from shading techniques

Problem: for certain illuminants, some image locations violate the assumption that the entire surface receives direct illumination, as in the diagram below:



Therefore if the image brightness would become negative, simply set it to 0.

$$E(x,y) = \max\{0, \frac{\rho}{\sqrt{1 + p^2 + q^2}} i^T [-p, -q, 1] \}$$

The number of unkowns here may seem to suggest that this equation does not provide enough constraints to reconstruct p and q at all pixels.

Summary of assumptions:

1) The acquisition system is calibrated so that the image Irradiance, E(p) equals the scene radiance, L(P), with $p=[x,y]^T$ image of the 3D point $P=[X,Y,Z]^T$
2) All the visible surface points receive direct illumination
3) The surface is imaged under weak perspective
4) The optical axis is the Z axis of the camera, and the surface can be parameterized as Z=Z(x,y)

Now, our problem is to determine albedo and illuminant direction from a single image of the surface making further assumptions, namely:

5) The surface imaged is Lambertian, and
6) The direction of the surface normals are distributed uniformly in 3D space

Method for Recovering Albedo and Illuminant Direction

1 Compute averages of image brightness, $\langle E \rangle$, and its square, $\langle E^2 \rangle$.

2 Compute the spatial image gradient $[E_x, E_y]^T$. Compute the average of both components $\langle \widehat{E_y} \rangle$ and $\langle \widehat{E_x} \rangle$

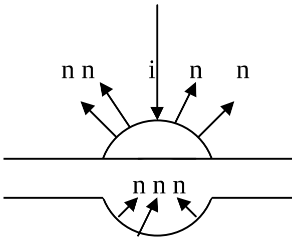
3 Estimate ρ, $\cos\sigma$, and $\tan\tau$ via the equations:

$$\rho = \Upsilon/\Pi \qquad \cos\sigma = 4 \langle E \rangle/\Upsilon \quad \text{and} \quad \tan\tau = \langle \widehat{E_y} \rangle / \langle \widehat{E_x} \rangle$$

where $\Upsilon = (6\Pi^2 \langle E^2 \rangle - 48 \langle E \rangle^2)^{1/2}$

Even under Lambertian assumptions, $E(p) = \frac{\rho}{\sqrt{1 + p^2 + q^2}} \langle i, [-p\ -q\ 1] \rangle$ requires a nonlinear partial differential equation in the presence of uncertain boundary conditions which could lead to an ill posed condition.

$R(p,q) = E = \frac{\rho}{\sqrt{1 + p^2 + q^2}} \quad < \vec{i}, [-p, -q, 1] >$ is an ill-posed question



Ill-posed question: 1) Solution may not exist
2) Solution may not be unique
3) May not vary continuously with input data

one way to circumvent conditions 1 and 2 is to allow for some small deviation between image brightness and the reflectance map, and enforce a smoothness constraint.

ie, minimize the functional: $\varepsilon = \int \partial x \partial y \, \{ \underbrace{(E(x,y) - R(p,q))^2}_{\text{closeness to data term}} + \underbrace{\lambda(p^2_x + p^2_y + q^2_x + q^2_y)}_{\text{smoothness term}} \}$

- -smoothness constraint given by the sum of spatial derivatives of p and q
- -λ is always positive and controls the relative influence of the two terms in the minimization process
- -large λ equates to a very smooth solution, not close to the data
- -small λ equates to a more irregular solution, closer to the data

For a functional ε which depends on two functions, p and q of two real variables x and y and on their first order spatial derivatives, we have the **Euler-Legrange** equations:

$$\frac{\partial \varepsilon}{\partial p} - \frac{\partial}{\partial x}\frac{\partial}{\partial p_x} - \frac{\partial}{\partial y}\frac{\partial \varepsilon}{\partial p_y} = 0$$

and

$$\frac{\partial \varepsilon}{\partial q} - \frac{\partial}{\partial x}\frac{\partial}{\partial q_x} - \frac{\partial}{\partial y}\frac{\partial \varepsilon}{\partial q_y} = 0$$

Note: since R(p,q) depends on p and q, but NOT p_x, p_y, q_x, q_y
E(x,y) does not depend on p or q, NOR p_x, p_y, q_x, q_y

Thus, the **Euler-Legrange** Equations become:

$$-2(E - R)\frac{\partial R}{\partial p} - 2\lambda\, p_{xx} - 2\lambda\, p_{yy} = 0$$

and

$$-2(E - R)\frac{\partial R}{\partial q} - 2\lambda\, q_{xx} - 2\lambda\, q_{yy} = 0$$

or in simplified form:

$\Delta p = -1/\lambda\,(E - R)\,\partial R/\partial p$	$\Delta q = -1/\lambda\,(E - R)\,\partial R/\partial q$

p,q unknown

Δp and Δq denoting the Laplacian of p and q (ie $\Delta p = p_{xx} + p_{yy}$)

Solving this is easier in the discrete case as opposed to the Continuous case, thus denote $p_{i,j}$ and $q_{i,j}$ as the average of 4 neighbors of p and q at location i,j.

	i,j+1	
i-1,j	i,j	i+1,j
	i,j-1	

by letting, $\overline{p}_{i,j} = p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} / 4$ **and** $\overline{q}_{i,j} = q_{i+1,j} + q_{i-1,j} + q_{i,j+1} + q_{i,j-1} / 4$

we can create an iterative scheme, starting at an initial configuration for $p_{i,j}$ and $q_{i,j}$

$$\boxed{\begin{aligned} \overline{p}_{i,j}^{\;k+1} &= p_{i,j}^{\;k} + 1/4\lambda\,(E - R)\,\partial R/\partial p\,\big|^{k} \\ \overline{q}_{i,j}^{\;k+1} &= q_{i,j}^{\;k} + 1/4\lambda\,(E - R)\,\partial R/\partial q\,\big|^{k} \end{aligned}} \qquad \text{for } k = 0 \text{ to } N$$

In many cases, the boundary conditions are unknown, thus an attempt must be made to impose natural boundary conditions or cyclic boundary conditions

→ P,q are wrapped around the image boundary

→ P,q are constant over image boundary

Conclusion: Algorithm for Shape From Shading

Input: image of unknown surface Z
Reflectance Map of surface
ρ - Surface Albedo
i – Illuminant (intensity and direction)

- surface slopes are assumed to wrap around the image boundaries
- until a suitable stopping criterion is met, iterate the following

1.) Update p and q via $\overline{p}_{i,j}^{\;k+1} = p_{i,j}^{\;k} + 1/4\lambda\,(E - R)\,\partial R/\partial p\,\big|^{k}$
and
$\overline{q}_{i,j}^{\;k+1} = q_{i,j}^{\;k} + 1/4\lambda\,(E - R)\,\partial R/\partial q\,\big|^{k}$

2.) Compute FFT of p and q, estimate Z and p' and q'.
Set p = p' and q = q'

Output: is an estimate of Z, p and q.