**Introduction to Statistics**

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October 31, 2007

# 1 Wheeler & Ganji, pp.118-179

You may pass, maybe even with a good grade, the statistics module of MECH 262 by reading these pages in the text and doing all problems assiduously. If you want to skip the lectures on Fridays; no hard feelings just be sure to show up for the mid-term on 07-03-02 at 09H30.

# 2 Broad Issues

Statistics is about *inference* concerning *expectation* to allow us to make wise -if not lucky- *planning decisions*. There is a lot of "extra reading" for those who are interested. They appear via some of the blue hyper-links in the course outline and inherited from my predecessor, Dr. Martin Beuhler. Do not take these too seriously. These are like the appendix, a vestigial organ whose function we don't quite understand. This is *not* the course where you can get deeply into statistics to which only a mere single credit is allotted.

# 3 Am./Can. Idol?

It's pretty safe to say if there was a TV show called "Mechanical Engineering Idol" that Sir Isaac Newton would rank pretty high, based on his contribution to dynamics and optics. But did he do statistics? Examine Fig. 1.

## 3.1 Binomial Distributions

Newton's problem will provide an opportunity to introduce an important topic in statistical probability. A binomial distribution is pretty much like the proverbial "bell", normal or Gaussian but the former is discrete while the latter is continuous. Let us first illustrate via Fig. 2 what is meant by a discrete distribution by looking at the throwing of only a *pair* of dice.
Now let us go on to examine option **A**, that of the probability of throwing one or more sixes with a single toss from the leather cup containing six dice. This is a sufficiently simple problem, having but six levels, that it can be practically represented with the complete *binary decision tree* in Fig. 3. Nevertheless it is complicated enough to convey two important statistical notions.

- The relation among "less than", "greater than" and "exactly" problem specifications and
- How the general binomial probability formula is derived.

To see this, examine Fig. 3 carefully, maybe drawn to a larger scale, and trace out the "direct" and "inverse" problem solution paths. The structure and application of the general formula are detailed in the sample problem below. After studying this carefully, do Newton's dice problem for all three cases, **A**, **B** and **C**.

Meanwhile Pepys, who found his own mysteries in London's clubs and gaming tables, came to Newton for advice on a matter of recreational philosophy: "the Doctrine of determining between the true proportions of the Hazards incident to this or that given Chance or Lot." He was throwing dice for money and needed a mathematician's guidance. He asked:

> A—has 6 dice in a Box, with which he is to fling a 6.
> B—has in another Box 12 Dice, with which he is to fling 2 Sixes.
> C—has in another Box 18 Dice, with which he is to fling 3 Sixes.
> Q. whether B & C have not as easy a Taske as A, at even luck?[9]

Newton explained why A has the best odds and gave Pepys the exact expectations, on a wager of £1,000, in pounds, shillings, and pence.

Figure 1: Gleick, J. (2003) *Isaac Newton*, Vantage, ISBN 1-4000-3295-4, p.146.

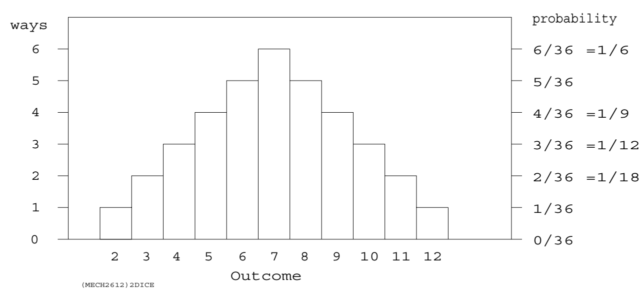


Figure 2: Two Dice Throw Outcome Probability

# 4 Binomial Probability

Here the reader is referred to pp.129-131 in Wheeler & Ganji. Additional examples are cited there. Consider the probability of there being $x$ or *more* rotten eggs in a sample of $n$ eggs. Assume a binomial distribution with a mean probability of $\mu$. The probability $p(r)$ of there being *exactly* $r$ is given by Eq. 1.

$$p(r) = {}_nC_r p_\mu^r q_\mu^{n-r} \tag{1}$$

Note that $p(r) + q(r) = 1$. Recall from the definition of permutations ${}_nP_r$ and combinations ${}_nC_r$ that

$${}_nP_r = \frac{n!}{(n-r)!}, \quad {}_nC_r = \frac{{}_nP_r}{r!}$$

${}_nP_r$ and ${}_nC_r$ appear as buttons on an ordinary scientific pocket calculator. The specific problem assigned asked us to estimate the probability of there being more than 2 bad eggs in a sample of

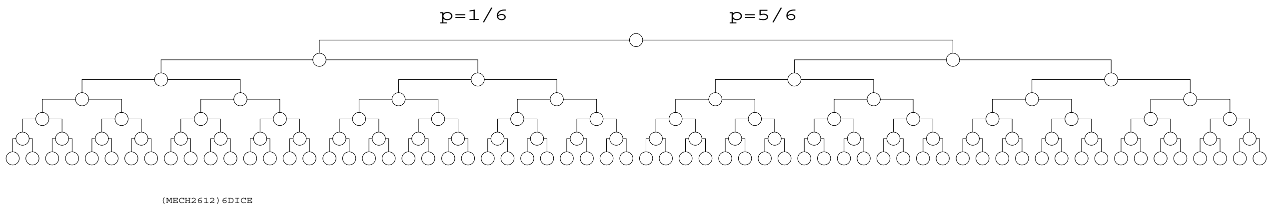


Figure 3: All 64 Possible Outcomes upon Tossing One Die Six Times or Six Dice Simultaneously in Order to Get a "Six"

12, *i.e.*, a dozen where the mean probability of finding bad eggs in a very large sample is one in twenty or $p_\mu = \frac{1}{20}$. One might solve the problem $p(>2)$ as

$$p(>2) = \sum_{i=3}^{i=12} p(i)$$

This is a sum of $12 - 2 = 10$ expressions like Eq. 1. On the other hand only three terms are necessary if we choose instead to use the complement.

$$p(>2) = q(\leq 2) = 1 - p(\leq 2) = 1 - \sum_{i=0}^{i=2} p(i)$$

$$1 - \left( {}_{12}C_0 p_\mu^0 q_\mu^{12} +_{12} C_1 p_\mu^1 q_\mu^{11} +_{12} C_2 p_\mu^2 q_\mu^{10} \right)$$

Putting in the numbers produces

$$1 - \left[ 1 \times \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^{12} + 12 \times \left(\frac{1}{20}\right)^1 \left(\frac{19}{20}\right)^{11} + 66 \times \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{10} \right]$$

or about

$$1 - 0.540_4 - 0.341_3 - 0.098_8 = 1 - 0.980_5 = 0.018_5$$

So we expect about 2% of the dozens to exceed the limit of two bad eggs.

## 4.1 A Somewhat Similar Problem

As a child I lived in a village where my father worked as a design engineer in a factory producing small arms ammunition, for the World War II effort, at the rate of millions of rounds per month. Acceptable product was deemed not to *exceed*, on average, one defective round per 1000. Sample batches of 100 were taken and test fired. If the desired quality standard was being maintained overall, what was the probability of finding two or more "duds" in a batch? Hint: This can be reduced to a problem identical to the formulation of the "rotten egg" one, above.

$$1 - \left( {}_{100}C_0 p_\mu^0 q_\mu^{100} +_{100} C_1 p_\mu^1 q_\mu^{99} \right)$$

# 5 Variability & Randomness

Measurement *variability* -controllable- and *randomness* -uncontrollable- affect conclusions drawn therefrom, hence usefulness. Statistics nevertheless permits us to *plan* experiments and *interpret* the results. Because we are lucky enough to have fairly good theoretical models, unlike the poor social "scientists", we have fewer uncontrollable effect to contend with.

# 6 Models & Correlation

*Correlative relation reliability* will be encountered in other courses like MECH 220, 240, 331, 346. Experimental correlation occurs by comparing measurements to models -phenomenological or heuristic- to be verified. Here's a couple you may have encountered.

$$PV = mNRt, \quad Nu(h) = CRe^{\alpha}Pr^{\beta}$$

Measured data scatter about expected model predicted characteristics leads to the requirement for least squares regression to obtain parameters that best fit data to characteristics.

# 7 Assignment, Bins & Histograms

**At the moment the following is not a current assignment in 2007.** Go to my web-page and find and down-load the entry "MECH 262 Uniform Random Number Assignment". It's due 06-01-13. Study Wheeler & Ganji, pp.118-122, until section **6.2.2**

## 8 Regression on x versus y; a Comparison

Examine Fig. 4. The idea is that the independent variable, conventionally plotted along the $x$-axis, is given unequivocally, *i.e*, it is "known" like the weights used to calibrate the load cell assignment example. So we only need to minimize the sum of the squares of the $y$-deviations, $y_i$. The exercise to minimize sum of squares of $x_i$ is done afterwards to show that when the independent variable becomes $y$ in the normal right-handed Cartesian frame one must be careful to correctly distinguish between the left and right hand!

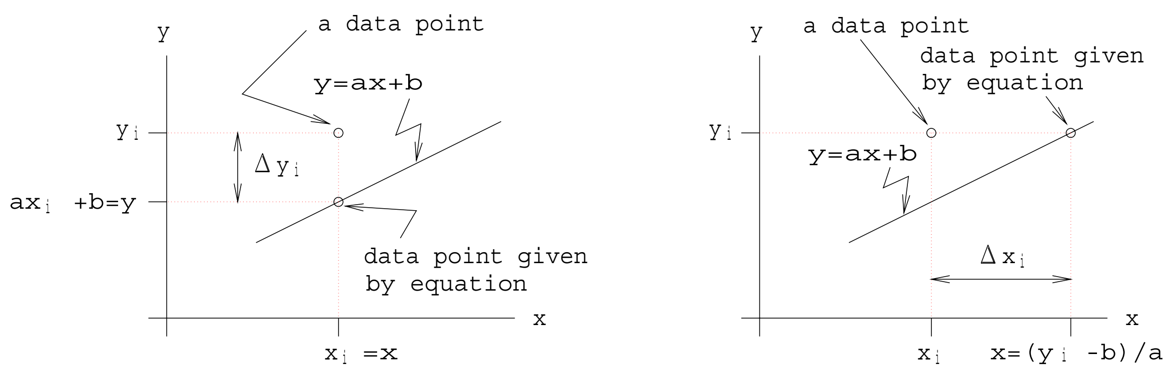


Figure 4: Deviations in x and y

## 9 Regression on Deviations in y

Examine the graph on the left in Fig. 4. The deviation of the single visible data point with respect to line $y = ax + b$ is

$$\Delta y_i = y_i - ax_i - b$$

The problem is to determine $a$ and $b$ such that the expression

$$\sum_{i=1}^{n} (\Delta y_i)^2$$

is minimized with respect to a sample population of $n$ given data points. The simplest redundant system is represented by $n = 3$.

$$\Delta y_1 = y_1 - x_1 a - b$$
$$\Delta y_2 = y_2 - x_2 a - b$$
$$\Delta y_3 = y_3 - x_3 a - b$$

Squaring and collecting the sum of the three right-hand sides and noting the recursion implicit in a sum of only three points yields the following conic equation, $F(a,b)=0$, in $a$ and $b$. Recall that a conic is any equation, in two variables, of order two.

$$F(a,b)=\left(\sum x_i^2\right)a^2+\left(2\sum x_i\right)ab-\left(2\sum x_iy_i\right)a+nb^2-\left(2\sum y_i\right)b+\left(\sum y_i\right)^2=0$$

Take partial derivatives $\frac{\partial F(a,b)}{\partial a}=0$ and $\frac{\partial F(a,b)}{\partial b}=0$ and solve the resulting, now linear, two equations for $a$ and $b$.

$$\begin{array}{llll}\left(\sum x_i^2\right)a & \left(\sum x_i\right)b & -\left(\sum x_iy_i\right) & =0\\ \left(\sum x_i\right)a & +nb & -\left(\sum y_i\right) & =0\end{array}$$

This gives

$$a=\frac{n\sum x_iy_i-\sum x_i\sum y_i}{n\sum x_i^2-(\sum x_i)^2},\quad b=\frac{\sum y_i\sum x_i^2-\sum x_i\sum x_iy_i}{n\sum x_i^2-(\sum x_i)^2}$$

A simpler expression for $b$ is obtained by taking the result for $a$, substituting it in the second linear equation above and rearranging.

$$b=\frac{\sum y_i-a\sum x_i}{n}$$

To better understand the principle of the inertia matrix method, satisfy yourself that the sum of the distances of all data points, represented by unit masses, to $x_G=0$, a vertical line, is zero; a first moment equilibrium with respect to $x$.

## 10 Regression on Deviations in x

Examine the graph on the right in Fig. 4. The deviation of the single visible data point with respect to the line $x=(y-b)/a$ is

$$\Delta x_i=x_i-x=x_i-\frac{y_i}{a}+\frac{b}{a}$$

With a change of variable $A=1/a,\ \ B=-b/a$ the equation above becomes

$$\Delta x_i=x_i-y_iA-B$$

By similarity to the results in the previous section

$$A=\frac{n\sum x_iy_i-\sum x_i\sum y_i}{n\sum y_i^2-(\sum y_i)^2},\quad B=\frac{\sum x_i-A\sum y_i}{n}$$

To better understand the principle of the inertia matrix method, satisfy yourself that the sum of the distances of all data points, represented by unit masses, to $y_G=0$, a horizontal line, is zero; a first moment equilibrium with respect to $y$. Moreover the two lines obtained with the same data point cloud intersect on its centroid.

# The Inertia Matrix in Least-Squares Fitting

### Inertia Matrix & Line on 3 Points

- Fit to Line; Induction & Differentiation
- Induction
- Differentiation
- The Points & Centroid
- Moments & Product of Inertia

### Plane on 6 Points

- Points & Centroid
- Matrix Elements
- Eigenvectors, Eigenvalues and Fitted Plane

## 11 Induction

Examine the upper diagram on Fig. 5.

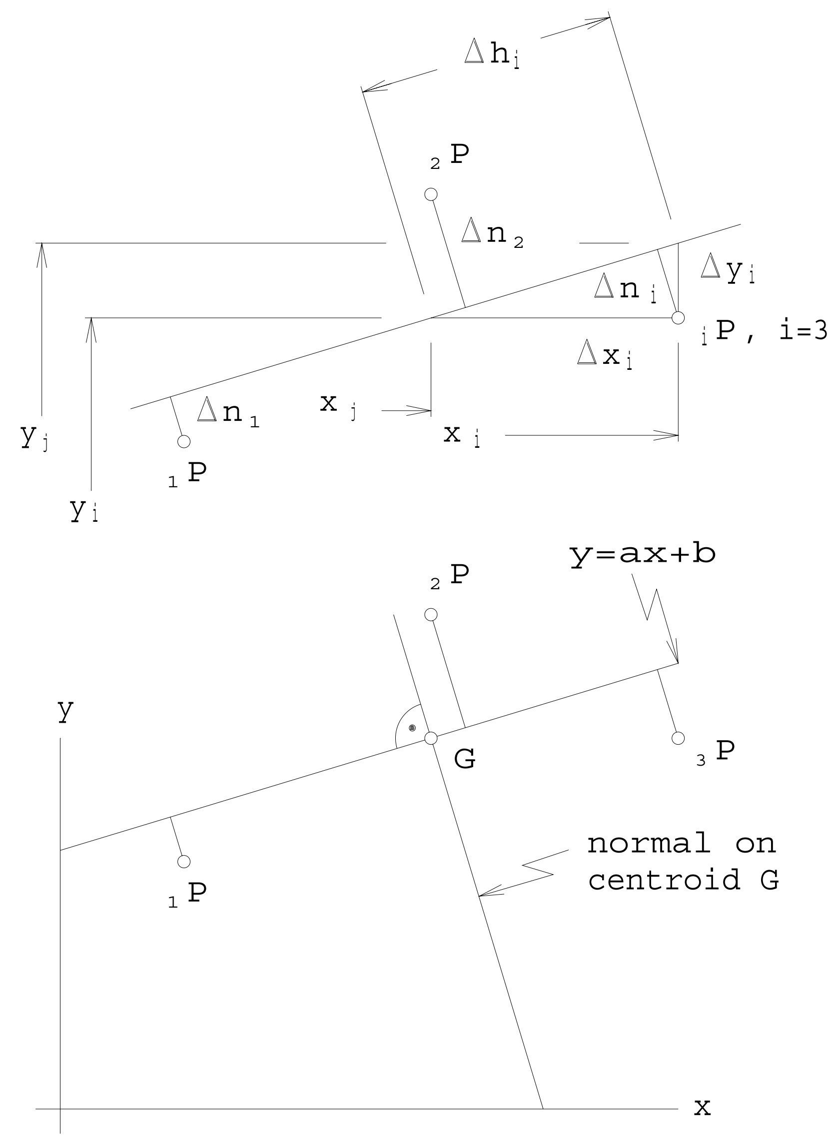


Figure 5: Line with Minimum Least Squares Normal Distance to Three Given Points

The line to be fit to the three points ${}_iP$ is given by $y = ax+b$ which can be inverted as $x = (y-b)/a$. The horizontal and vertical distances from a point to the line are given by

$$\Delta x_i = x_i - \left(\frac{y_i - b}{a}\right) \text{ and } \Delta y = y_i - (ax_i + b)$$

By similar triangles the normal distance $\Delta n$ is

$$\frac{\Delta n_i}{\Delta y_i} = \frac{\Delta x_i}{\Delta h_i}$$

Since the intention is to minimize the sum of the squares of the normal distances the contribution of each point is

$$\Delta n_i^2 = \frac{\Delta x_i^2 \Delta y_i^2}{\Delta h_i^2} = \frac{\Delta x_i^2 \Delta y_i^2}{\Delta x_i^2 + \Delta y_i^2} = \frac{\left\{\left[x_i - \left(\frac{y_i - b}{a}\right)\right](y_i - ax_i - b)\right\}^2}{\left[x_i - \left(\frac{y_i - b}{a}\right)\right]^2 + (y_i - ax_i - b)^2}$$

For three points this expands to

$$\sum_{i=1}^{3} \Delta n_i^2 = \Delta n_1^2 + \Delta n_2^2 + \Delta n_3^2 = \frac{\left\{\left[x_1 - \left(\frac{y_1 - b}{a}\right)\right](y_1 - ax_1 - b)\right\}^2}{\left[x_1 - \left(\frac{y_1 - b}{a}\right)\right]^2 + (y_1 - ax_1 - b)^2}$$

$$+ \frac{\left\{\left[x_2 - \left(\frac{y_2 - b}{a}\right)\right](y_2 - ax_2 - b)\right\}^2}{\left[x_2 - \left(\frac{y_2 - b}{a}\right)\right]^2 + (y_2 - ax_2 - b)^2} + \frac{\left\{\left[x_3 - \left(\frac{y_3 - b}{a}\right)\right](y_3 - ax_3 - b)\right\}^2}{\left[x_3 - \left(\frac{y_3 - b}{a}\right)\right]^2 + (y_3 - ax_3 - b)^2} \tag{2}$$

and this must be minimized with respect to $a$ and $b$ so

$$\frac{\partial \left(\sum_{i=1}^{3} \Delta n_i^2\right)}{\partial a} = 0 \text{ and } \frac{\partial \left(\sum_{i=1}^{3} \Delta n_i^2\right)}{\partial b} = 0$$

The inductive aspect of this process becomes clear upon taking derivatives of the triple sum, Eq. 2.

# 12 Differentiation

The derivative with respect to $a$, after multiplication by $(1 + a^2)^2/2$ and collection of terms, is

$$\frac{\partial \left(\sum_{i=1}^{3} \Delta n_i^2\right)}{\partial a} = [(x_1y_1 + x_2y_2 + x_3y_3) - (x_1 + x_2 + x_3)b^2]a^2 + \{[(x_1^2 + x_2^2 + x_3^2) - (y_1^2 + y_2^2 + y_3^2)]$$

$$+2(y_1 + y_2 + y_3)b - 3b^2\}a - [(x_1y_1 + x_2y_2 + x_2y_3) - (x_1 + x_2 + x_3)b] = 0 \tag{3}$$

Similar treatment of the derivative with respect to $b$ yields

$$3b = (y_1 + y_2 + y_3) - (x_1 + x_2 + x_3)a \tag{4}$$

Substituting Eq. 4 into Eq. 3, multiplying the result by 3 and collecting $a$ yield a quadratic.

$$Aa^2 + Ba + C = 0 \tag{5}$$

$$A = (x_1 - x_2)(y_1 - y_2) + (x_2 - x_3)(y_2 - y_3) + (x_3 - x_1)(y_3 - y_1) = -C$$
$$B = [(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2] - [(y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2]$$

Examining Eq. 5 produces the following general relations for $a$ and $b$, valid for $n$, any number of points, not to be confused with $n_i$, the normal distance of a point to the fitted line.

$$a^2 + \frac{\sum_{ik}(x_i^2 - x_k^2) - \sum_{ik}(y_i^2 - y_k^2)}{\sum_{ik}(x_i - x_k)(y_i - y_k)} a - 1 = 0$$

$$nb = \sum_i y_i - \left(\sum_i x_i\right) a$$

Note that all possible differences and differences of squares must be taken in the sums $\sum_{ik}$. Two solutions for $a$, the slope of the fitted line, are obtained, along with a corresponding value of $b$. One of these is the actual line we seek. The other is on the centroid of the given points and normal to this line. There is no way, *a-priori*, to tell which is which unless one is prepared to take second derivatives.

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# 13 Summing Up

- **Q** Why do single (semi-) regression (fitting) in the plane (between pairs of variable data), *i.e.*, on $y$ (or $x$)?
- **A** Calibration, where the independent variable is deemed to be at least an order of magnitude more accurate and precise than the readings provided by the instrument under calibration.
- **Q** Why do double (normal least squares) regression (fitting)?
- **A** Correlation, where we wish to establish *dependency* between independent and dependent (that's an oxymoron, I guess) variables $x$ and $y$, *e.g.*, smoking and lung cancer may well yield a *positive* correlation; a fit line on the graph that has positive slope; exercise frequency -*vs*- heart attack frequency a negative one. However if we plotted Lotto winnings *vs* last digit of the winning ticket number we should expect a null or zero correlation; a *horizontal* line indicates no correlation.
- Correlation coefficient $r$ is used to *compare* correlation trends. If we found that frequency of gobbling a Big Mac was more steeply upward sloping to the right than smoking a cigarette we'd say a Big Mac has a higher correlation to cancer than a cigarette. Be careful, this evidence may or may not mean something in terms of cancer avoidance.

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}},\ S_{xx} = \sum x^2 - (\sum x)^2/n$$

$$S_{yy} = \sum y^2 - (\sum y)^2/n,\ S_{xy} = (\sum xy - \sum x \sum y)/n$$

- In all cases one must assess the *strength* or significance of the correlation, *e.g.*, $\sum n_i^2/N$ for populations or $n-1$ for samples because a population or sample of 2 has a mean $\bar{x}$ or $\mu$ but no variance $v = \sigma^2$ where $\sigma$ is standard deviation.

## 14 The Points and Centroid

The numerical example shown in Fig. 5 is presented below.

$$_1P(1,2),\ \ _2P(3,4),\ \ _3P(5,3)$$

The first moment is calculated to yield $G$ the centroid on the fit line.

$$[(1+3+5)/3, (2+4+3)/3,] \rightarrow G(3,3)$$

The inertia matrix is referred to this origin so

$$_1P'(-2,-1),\ \ _2P'(0,1),\ \ _3P'(2,0),$$

replaces $_iP$ above.

## 15 Moments and Product of Inertia about Principal Axes

The second moments and product are obtained with $_iP'$ as

$$I_{xx} = (-1)^2 + 1^2 + 0^2 = 2,\ \ I_{yy} = (-2)^2 + 0^2 + 2^2,\ \ I_{xy} = (-2)(-1) + 0((1) + 2(0) = 2$$

## 16 Eigenvectors, Eigenvalues and the Fitted Line

The eigenvectors $\mathbf{e}$ and eigenvalues $\lambda$ are defined below.

$$\left[\begin{array}{cc} I_{xx}-\lambda & -I_{xy} \\ -I_{xy} & I_{yy}-\lambda \end{array}\right]\left[\begin{array}{c} e_x \\ e_y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \rightarrow \left|\begin{array}{cc} 2-\lambda & -2 \\ -2 & 8-\lambda \end{array}\right] = 0 \rightarrow \begin{array}{c} \lambda^2 - 10\lambda + 12 = 0 \\ \lambda = 5 \pm \sqrt{13} \end{array}$$

The greater eigenvalue is associated with the eignevector normal to the fitted line we seek. Why? Because the moment of inertia of the points about a line whose direction numbers are given by the ratio $e_x : e_y$ is the greatest with respect to an axis on the centroid $G$. This ratio is given by either of the linearly dependent equations

$$[2-(5+\sqrt{13})]e_x - 2e_y = 0 \text{ or } -2e_x + [8-(5+\sqrt{13})]e_y = 0$$

The other, lesser, eigenvalue produces a vector with the same direction numbers as the fitted line. These lines of maximum and minimum inertia must not be confused with $I_{zz}$, normal to the plane. This is the moment of inertia encountered in elementary planar mechanics.

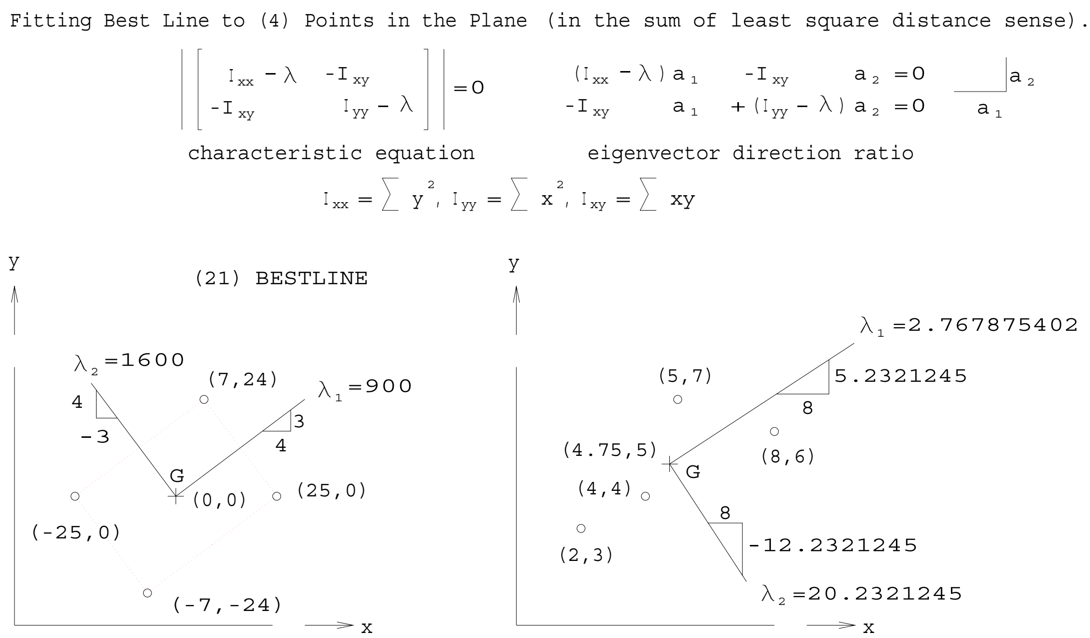


Figure 6: Two 4-Point Line Fits; One Array Is Rectangular, the Other is Random

# 17 Practice

Now work out the two examples shown in Fig. 6 to your satisfaction. If you are confident that you understand how the inertia matrix operates in this respect then continue to the, next, planar fit to six given points. Four could have been chosen and that would be a minimum number that still constituted an over-determined system.

How to fit the six-legged monster's feet, in a fast-moving video game, to a changing surface that can be approximated by planar facets?

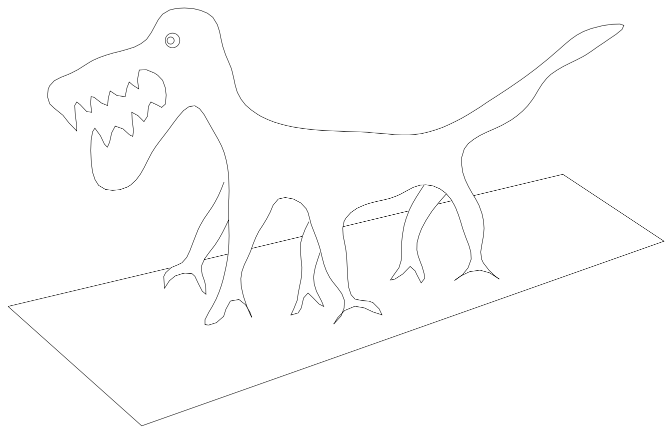

Figure 7: Six Monster Feet and the Plane

# 18 The Points and Centroid

The numerical example shown in Fig. 8 is presented below.

$$ {}_1P(4,3,2),\ \ {}_2P(7,1,4),\ \ {}_3P(2,8,1),\ \ {}_4P(9,6,7),\ \ {}_5P(1,4,6),\ \ {}_6P(12,-3,-5) $$

The homogeneous coordinates of the centroid are $G\{g_0 : g_1 : g_2 : g_3\} \equiv \{6 : 35 : 19 : 15\}$. To obtain the elements of the inertia matrix all ${}_iP$ are multiplied by 6 before subtracting $g_1, g_2, g_3$ from them all to produce

$$ {}_1P'(-11,-1,-3),\ \ {}_2P'(7,-13,9),\ \ {}_3P'(-23,,29,-9), $$

$$ {}_4P'(19,17,27),\ \ {}_5P'(-29,5,21),\ \ {}_6P'(37,-37,-45) $$

# 19 Inertia Matrix Elements

Notice that all six of the following sums of products contain the common divisor 6. Inertia matrix magnitudes are irrelevant because we seek only the direction of eigenvector **e** associated with the greatest eigenvalue $\lambda$.

$$\begin{aligned}
I_{xx} = &\{[(-1)^2+(-3)^2]+[(-13)^2+9^2]+[29^2+(-9)^2]\\
&+[17^2+27^2]+[5^2+21^2]+(-37)^2+(-45)^2]\}/6 = 1010\\
I_{yy} = &\{[(-3^2)+(-11)^2]+[9^2+7^2]+[(-9)^2+(-23)^2]\\
&+[27^2+19^2]+[21^2+(-29)^2]+[(-45)^2+37^2]\}/6 = 1106\\
I_{zz} = &\{[(-11)^2+(-1)^2]+[7^2+(-13)^2]+[(-23)^2+29^2]\\
&+[19^2+17^2]+[(-29)^2+5^2]+[37^2+(-37)^2]\}/6 = 994\\
I_{xy} = &\{[(-11)(-1)]+[(7)(-13)]+[(-23)(9)]\\
&+[(19)(17)]+[(-29)(5)]+[(37)(-37)]\}/6 = 323\\
I_{xz} = &\{[(-3)(-11)]+[(9)(7)]+[(-9)(23)\\
&+[(27)(19)]+[(21)(-29)]+[(-45)(37)]\}/6 = 243\\
I_{xz} = &\{[(-1)(-3)]+[(-13)(9)]+[(29)(-9)]\\
&+[(17)(27)]+[(5)(21)]+[(-37)(-45)]\}/6 = -309
\end{aligned}$$

# 20 Eigenvectors, Eigenvalues and the Fitted Plane

$$\begin{bmatrix} I_{xx}-\lambda & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy}-\lambda & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz}-\lambda \end{bmatrix}\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda^3 - 3110\lambda^2 + 2961505\lambda - 796404408 = 0,$$
$$\lambda = 453.1092563,\ 1244.042205,\ 1412.84854$$

Using the the greatest, $\lambda = 1412.84854$ produces three simultaneous, linearly dependent equations, any two of which can be solved homogeneously. The first two will be used.

$$\begin{aligned}
(1010-1412.84854)e_x \quad &+323e_y &+243e_z &= 0\\
323e_x \quad &+(1106-1412.84854)e_y &-309e_z &= 0\\
243e_x \quad &-309e_y &(994-1412.84854)e_z &= 0
\end{aligned}$$

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} -25242.80478 \\ -45991.1989 \\ 19284.4863 \end{bmatrix}$$

These are direction numbers of normals to the plane. Together with the centroid $G$ we get

$$G_1x + G_2y + G_3z + G_0 = 0,\quad g_0e_xx + g_0e_yy + g_0e_zz - (g_1e_x + g_2e_y + g_3e_z) = 0$$

which becomes, numerically,

$$\begin{aligned}
&(6)(-25242.80478)x + (6)(-45991.1989)y + (6)(19284.4863)z\\
&-[(35)(-25242.80478) + (19)(-45991.1989) + (15)(19284.4863)] \quad = 0
\end{aligned}$$

Fig. 8 shows top and front view of the six spatial points as well as *all three* mutually orthogonal eigenvector *directions* using all three eigenvalues. The vector labelled $\lambda_3$ is associated with the greatest eigenvalue. It does not *appear* as the longest because that would require unitizing the eigenvectors and scaling them to their eigenvalues. Note the edge view of the fitted plane $g$ and the six distances from the points to $g$. These are parallel to the plane normal $\lambda_3$ that appears in true view in the first auxiliary projection. (MECH261-2)StLe16f.tex, (28)NLS42u.tex, (28)IMp6P47s.tex, (MECH261-2)StLe16ft.tex

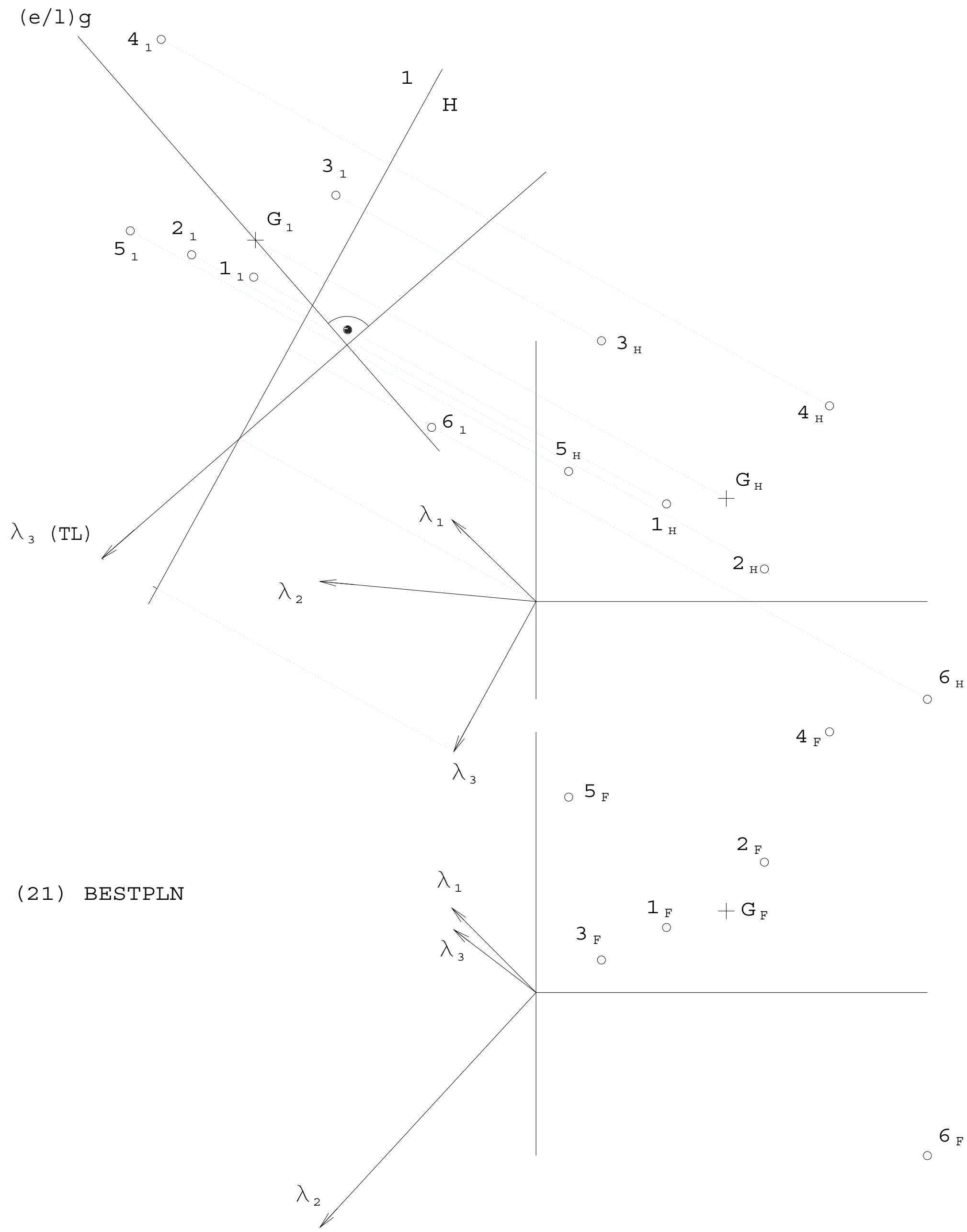


Figure 8: Plane with Minimum Least Squares Normal Distance to Six Given Points