MECH 262 Finding the Line That Minimizes the Sum of Least Squares Normal Distance to a Set of Points in the Plane Using a "Moment of Inertia" Approach

February 20, 2004

1 Two Homogeneous Equations

Consider the two following linear equations.

$$\begin{aligned} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned} \tag{1}$$

In "detached coefficient", matrix-vector form Eqs. 1 may be written as Eq. 2 because we *have* no third equation.

$$\begin{bmatrix} 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Expanding the matrix on top row cofactors produces values of $x_i, \;\; i = 0, 1, 2$ which satisfy Eqs. 1.

$$\begin{aligned} x_0 &= +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \\ x_1 &= -\begin{vmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{vmatrix} = -(a_{10}a_{22} - a_{20}a_{12}) \\ x_2 &= +\begin{vmatrix} a_{10} & a_{11} \\ a_{20} & a_{21} \end{vmatrix} = a_{10}a_{21} - a_{20}a_{11} \end{aligned} \tag{3}$$

2 Projective, Affine & Euclidean

A general projective transformation of a vector, *e.g.*, a point in the plane, maps the vector $\mathbf{u} \to \mathbf{v}$ thus.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{4}$$

An affine transformation leaves $v_0 \neq 0$ if $u_0 \neq 0$ and maintains $v_0 = 0$ if $u_0 = 0$. This is accomplished by the two zeros in the top row of the transformation matrix.

$$\begin{bmatrix} a_{00} & 0 & 0 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{5}$$

A Euclidean transformation is a special affine transformation that always keeps $u_0 = v_0 = 1$ thus.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ v_1 \\ v_2 \end{bmatrix} \tag{6}$$

Because nothing happens up at the top, you are taught that the Euclidean mapping is accomplished by a 2×2 matrix operating on a two element vector.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7}$$

3 The Eigenvalue Problem

In order to get zeros for the solutions of a pair of simultaneous linear equations, the pair must be linearly dependent and *all* of the three 2×2 determinants, shown in Eqs. 3, must vanish. The eigenvalues λ are assigned to make this happen with an otherwise non-linearly dependent pair of equations. We are interested in the λ's because we can then, with the two eigenvalue augmented equations, find two *eigenvectors*, vectors whose *direction number* ratios are not changed under the original transformation. The only 2×2 determinant of any significance in our problem is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \tag{8}$$

To see this the pair of relevant equations is written below.

$$\begin{array}{llll} 0x_0 & +(a_{11} - \lambda)x_1 & & +a_{12}x_2 = 0 \\ 0x_0 & & +a_{21}x_1 & +(a_{22} - \lambda)x_2 = 0 \end{array} \tag{9}$$

These may be solved for λ as

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 + 4a_{12}a_{21}}}{2} \tag{10}$$

Let λ_1 and λ_2 represent the respective $\pm$ roots. Then either *one* of Eqs. 9, together with the first eigenvalue λ_1, yields the vector elements

$$\begin{aligned} (a_{11} - \lambda_1)x_1 + a_{12}x_2 = 0, \quad x_1 = -a_{12}, \quad x_2 = (a_{11} - \lambda_1) \\ a_{21}x_1 + (a_{22} - \lambda_1)x_2 = 0, \quad x_1 = (a_{22} - \lambda_1), \quad x_2 = -a_{21} \end{aligned} \tag{11}$$

In general $a_{12} + (a_{22} - \lambda_1) \neq 0$ and $(a_{11} = \lambda_1) + a_{22} \neq 0$ however the *ratio* $x_1 : x_2$ is identical. To get the two elements of the other eigenvector, say $\mathbf{y}$, the exercise above is repeated with λ_2.

4 What Does Invariance Under Transformation Mean?

To see this consider a conic, *i.e.*, a second order curve, in the plane, whose coefficients may be presented in a symmetric matrix. We will not go into the details but this notion of invariance can be illustrated by the point/line polarity relationship which deals with an origin centred ellipse shown in Fig. (28)PLR. Above, the point P, given by a position vector from the origin, is related to the line p through the two tangent points, on the conic, drawn from P. Notice that this line is *not* normal to the position vector OP so n, the normal to the line, does *not* have direction numbers in the same ratio as, say, $x_1 : x_2$. Hence the transformation of this point was *not* invariant. In the second case, with P infinitely far from O in the direcion of the major axis, OP *is* normal to the line on the two parallel tangents. This point P, at infinity, *is* invariant under transformation by the conic coefficient matrix. It's not enough that a pair of tangents be parallel. They must also be normal to a symmetry axis.

5 What Does This Have to Do With a Normal Least Squares Plane/Line Fitting Problem?

Assume that n points ${}_iP({}_ip_x, {}_ip_y, {}_ip_z)$, $i = 1 \ldots n$, are specified by their three Cartesian coordinates in Euclidean space and it is desired to find a plane $g\{G_0 : G_1 : G_2 : G_3\}$ such that the sum of the squares of the distances of all points ${}_iP$ is minimum. First it is observed that g must be on point G, the centre of gravity of the point cloud, assuming each point is represented by a unit mass. Therefore

$$G(g_x, g_y, g_z) \equiv \left(\frac{\sum_{i=1}^{n} {}_ip_x}{n}, \frac{\sum_{i=1}^{n} {}_ip_y}{n}, \frac{\sum_{i=1}^{n} {}_ip_z}{n} \right)$$

Then the principal inertial magnitudes $\lambda_1, \lambda_2, \lambda_3$ and the axis direction of the greatest, say, λ_3, represented by the absolute point $A\{0 : a_1 : a_2 : a_3\}$, is found. The reason for this choice is evident from the following argument: The sum of the squares of the distances to unit mass points in all directions normal to an axis is maximized if that axis represents maximum moment of inertia of a rigid body whose mass is composed exclusively of such points. This observation proceeds directly from the definition of mass moment of inertia about an axis of rotation. The plane g dual to A is normal to the direction $a_1 : a_2 : a_3$ and on the "origin", *i.e.*, the centre of the point cloud, G. This is the plane we seek. It is easy to show that

$$g\{-(g_1 a_1 + g_2 a_2 + g_3 a_3) : g_0 a_1 : g_0 a_2 : g_0 a_3\}$$

where $g_0 = n$, $g_1 = \sum_{i=1}^{n} {}_ip_x$, $g_2 = \sum_{i=1}^{n} {}_ip_y$ and $g_3 \sum_{i=1}^{n} {}_ip_z$. Finding the coefficients of the plane equation is a three step process.

- Find the inertia matrix in the Cartesian frame of the n given point unit masses.
- The principal moments of inertia, $\lambda_1, \lambda_2, \lambda_3$, the eigenvalues of this matrix, which are the three solutions of the cubic characteristic equation, are computed.
- The eigenvector associated with the greatest eigenvalue λ_3 has elements corresponding to the direction numbers $a_1 : a_2 : a_3 \equiv G_1 : G_2 : G_3$ of the normal to desired plane g.

5.1 Inertia Matrix

$$[\mathbf{I}] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

where

$$I_{xx} = \sum_{i=1}^{n} {}_ip_y^2 + {}_ip_z^2, \quad I_{yy} = \sum_{i=1}^{n} {}_ip_z^2 + {}_ip_x^2, \quad I_{zz} = \sum_{i=1}^{n} {}_ip_x^2 + {}_ip_y^2$$

$$I_{xy} = \sum_{i=1}^{n} {}_ip_x {}_ip_y, \quad I_{xz} = \sum_{i=1}^{n} {}_ip_x {}_ip_z, \quad I_{yz} = \sum_{i=1}^{n} {}_ip_y {}_ip_z$$

5.2 Principal Moments of Inertia

The eigenvalues, which are the principal moments of inertia, are computed as the solutions to the cubic equation which results from evaluation of the following determinant.

$$\left| \begin{bmatrix} I_{xx} - \lambda & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} - \lambda & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} - \lambda \end{bmatrix} \right|$$

$$\begin{aligned} = \lambda^3 - (I_{xx} + I_{yy} + I_{zz})\lambda^2 - (I_{xy}^2 + I_{xz}^2 + I_{yz} - I_{xx}I_{yy} - I_{xx}I_{zz} - I_{yy}I_{zz})\lambda \\ + I_{xx}I_{yz}^2 + I_{yy}I_{xz}^2 + I_{zz}I_{xy}^2 + 2I_{xy}I_{xz}I_{yz} - I_{xx}I_{yy}I_{zz} = 0 \end{aligned}$$

5.3 Eigenvector and Plane Normal

Using the invariance under linear transformation property of eigenvectors one may write.

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This yields the following three homogeneous equations, any two of which may be used to determine the ratios of plane g normal direction numbers $a_1 : a_2 : a_3$. There is no need to normalize these to direction cosines.

$$\begin{array}{rrr} (I_{xx} - \lambda_1)a_1 & -I_{xy}a_2 & -I_{xz}a_3 = 0 \\ -I_{xy}a_1 & +(I_{yy} - \lambda_1)a_2 & -I_{yz}a_3 = 0 \\ -I_{xz}a_1 & -I_{yz}a_2 & +(I_{zz} - \lambda_1)a_3 = 0 \end{array}$$

The first pair of equations above produces numbers in the ratio required.

$$\begin{array}{rrr} a_1 = & I_{xy}I_{yz} & +(I_{yy} - \lambda_1)I_{xz} \\ a_2 = & I_{xy}I_{xz} & +(I_{xx} - \lambda_1)I_{yz} \\ a_3 = & (I_{xx} - \lambda_1)(I_{yy} - \lambda_1) & -I_{xy}^2 \end{array}$$

6 Example

Given the following six points ${}_iP({}_ip_x, {}_ip_y, {}_ip_z)$

$$ {}_1P(4,3,2),\ \ {}_2P(7,1,4),\ \ {}_3P(2,8,1),\ \ {}_4P(9,6,7),\ \ {}_5P(1,4,6),\ \ {}_6P(12,-3,-5) $$

then $G\{g_0 : g_1 : g_2 : g_3\} \equiv \{6 : 35 : 19 : 15\}$. Also, the elements of $[\mathbf{I}]$ can be calculated as follows. Multiplying the six given point coordinates by 6 and subtracting $g_1,\ g_2,\ g_3$ from each triple produces

$$ {}_1P^*(-11,-1,-3),\ \ {}_2P^*(7,-13,9),\ \ {}_3P^*(-23,29,-9) $$
$$ {}_4P^*(19,17,27),\ \ {}_5P^*(-29,5,21),\ \ {}_6P^*(37,-37,-45) $$

Notice that all six of the following sum of six products contains the common divisor 6. The magnitude of inertia matrix elements is irrelevant since we seek only the *direction* of the eigenvector associated with the greatest eigenvalue.

$$ I_{xx} = \{[(-1)^2+(-3)^2]+[(-13)^2+9^2]+[29^2+(-9)^2] $$
$$ +[17^2+27^2]+[5^2+21^2]+[(-37)^2+(-45)^2]\}/6 = 1010 $$
$$ I_{yy} = \{[(-3)^2+(-11)^2]+[9^2+7^2]+[(-9)^2+(-23)^2] $$
$$ +[27^2+19^2]+[21^2+(-29)^2]+[(-45)^2+37^2]\}/6 = 1106 $$
$$ I_{zz} = \{[(-11)^2+(-1)^2]+[7^2+(-13)^2]+[(-23)^2+29^2] $$
$$ +[19^2+17^2]+[(-29)^2+5^2]+[37^2+(-37)^2]\}/6 = 994 $$
$$ I_{xy} = \{[(-11)(-1)]+[(7)(-13)]+[(-23)(29)] $$
$$ +[(19)(17)]+[(-29)(5)]+[(37)(-37)]\}/6 = 323 $$
$$ I_{xz} = \{[(-3)(-11)]+[(9)(7)]+[(-9)(-23)] $$
$$ +[(27)(19)]+[(21)(-29)]+[(-45)(37)]\}/6 = 243 $$
$$ I_{yz} = \{[(-1)(-3)]+[(-13)(9)]+[(29)(-9)] $$
$$ +[(17)(27)]+[(5)(21)]+[(-37)(-45)]\}/6 = -309 $$

The eigenvalue cubic polynomial, *i.e.*, the characteristic equation becomes

$$ \lambda^3 - 3110\lambda^2 + 2961505\lambda - 796404408 = 0 $$

which has three real roots if one ignores small imaginary residues returned in all three solutions.

$$ \lambda_1 = 453.1092563,\ \ \lambda_2 = 1244.042205,\ \ \lambda_3 = 1412.84854 $$

Finally the normal direction to the required plane is obtained with λ_3 by homogeneously solving the first two of the following equations.

$$ \begin{array}{rrrl} (1010-\lambda_3)a_1 & +323a_2 & +243a_3 & = 0 \\ 323a_1 & +(1106-\lambda_3)a_2 & -309a_3 & = 0 \\ 243a_1 & -309a_2 & +(994-\lambda_3)a_3 & = 0 \end{array} $$

$$a_1 = -25242.80478, \quad a_2 = -45991.1989, \quad a_3 = 19284.4863$$

These direction numbers, together with the homogeneous coordinates of point G permit calculation of the required plane equation.

$$G_1x + G_2y + G_3z + G_0 = 0, \quad g_0a_1x + g_0a_2y + g_0a_3z - (g_1a_1 + g_2a_2 + g_3a_3) = 0$$

$$(6)(-25242.80478)x + (6)(-45991.1989)y + (6)(19284.4863)z$$
$$-[(35)(-25242.80478) + (19)(-45991.1989) + (15)(19284.4863)] = 0$$

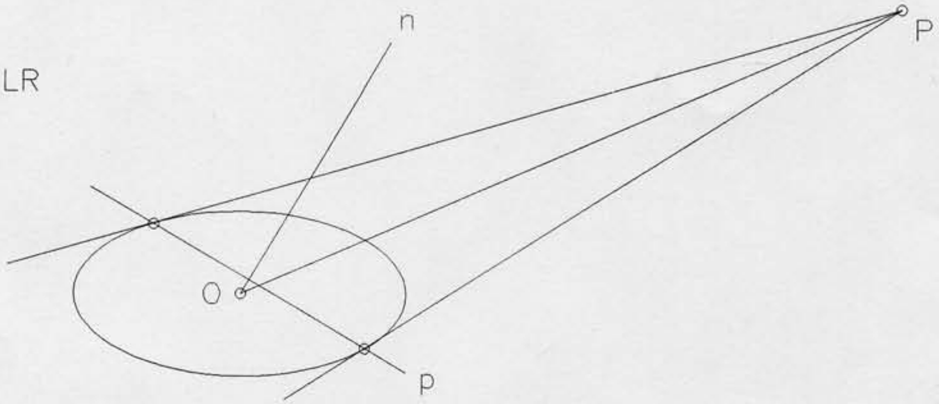
Fig. BESTPLN shows a top and front view of the array of six spatial points as well as the three eigenvector directions. By projecting a true length of λ_3 one sees the edge or line view of the desired plane g on G and normal to λ_3.

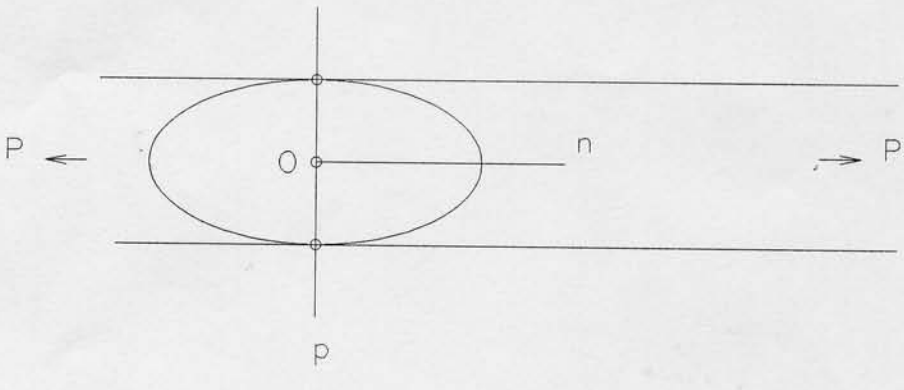
7 Conclusion

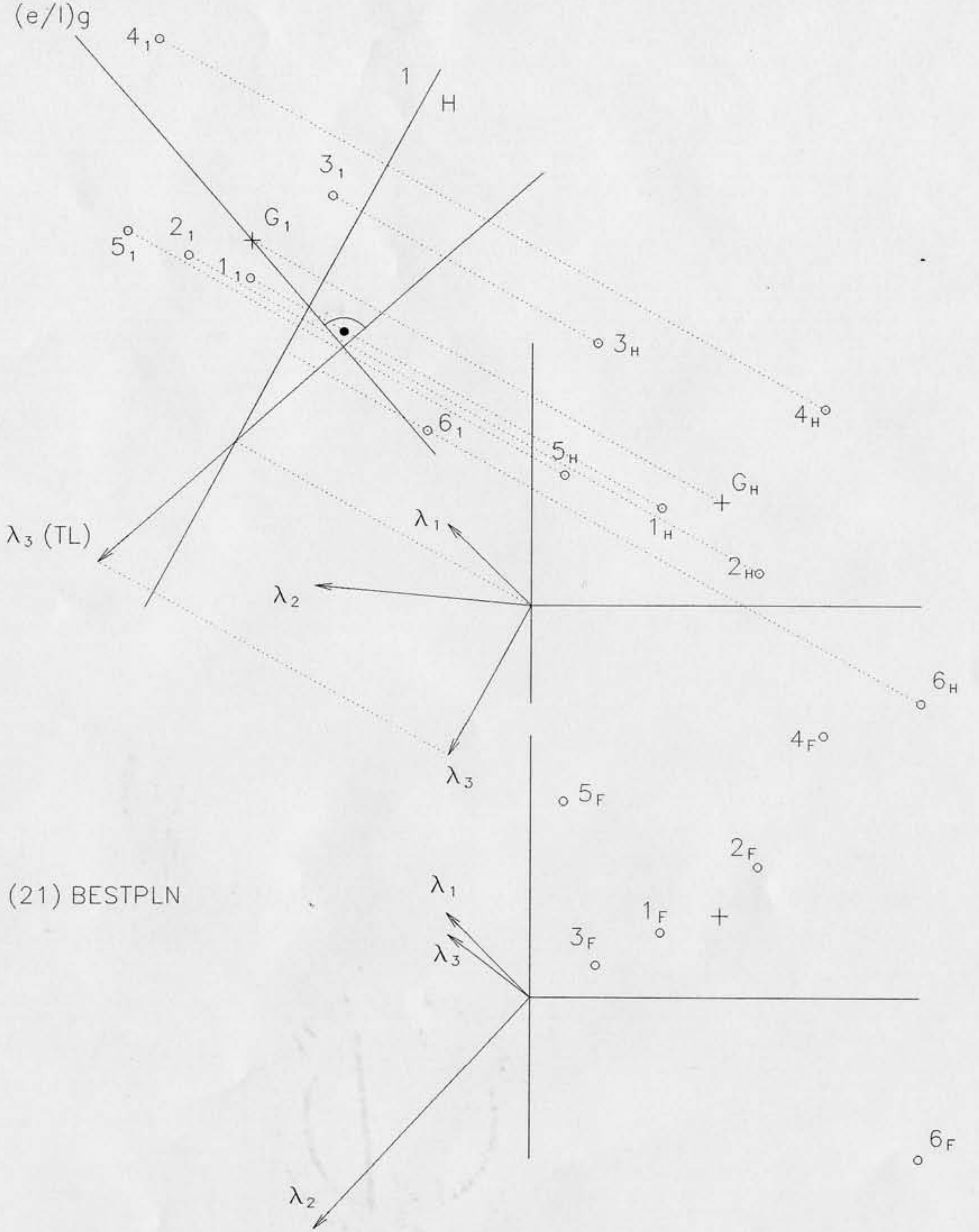
Verification of this method to find best linear fit to a set of given fixed points is effectively demonstrated with simple, planar examples shown in Fig. (21)BESTLINE where lines, not planes are fitted. In the lefthand drawing four given points are presented in a rectangular array. Obviously the line which bisects the pair of shorter sides enjoys minimum sum of squared distances to the points. The greater eigenvalue $\lambda_2 = 1600$ produces an eigenvector with direction numbers -3:4. If a line on G in this direction is struck, then rotation of the rectangular array of four unit masses about this axis will experience the greatest inertia moment. The best fit line, in the direction of the eigenvector computed with λ_1, has homogeneous coordinates $\{0 : -3 : 4\}$ with the origin taken on G. The righthand drawing illustrates a similar fit of four points which have been located more or less randomly.

(28)SLSND42s.tex

(28)PLR







(e/l)g
4_1
1
H
3_1
G_1
5_1
2_1
1_1
6_1
3_H
4_H
5_H
G_H
1_H
2_H
6_H
λ_3 (TL)
λ_1
λ_2
λ_3
4_F
5_F
2_F
1_F
3_F
6_F
λ_1
λ_3
λ_2

(21) BESTPLN

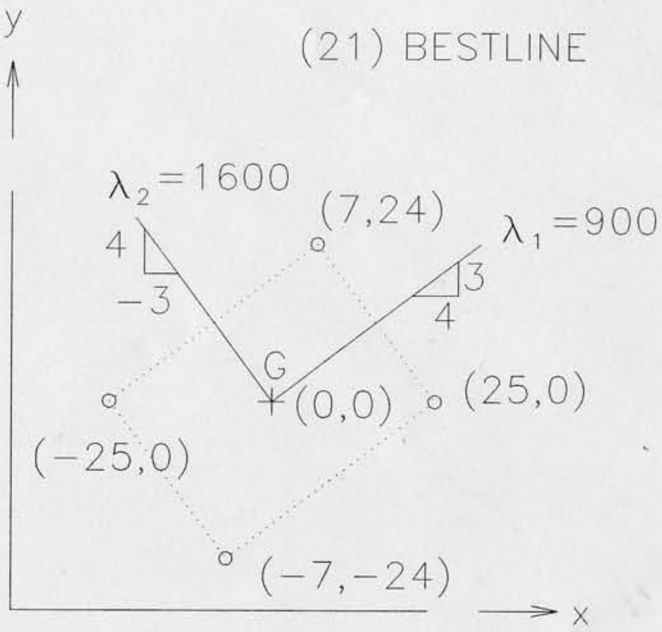
Fitting Best Line to (4) Points in the Plane (in the sum of least square distance sense).

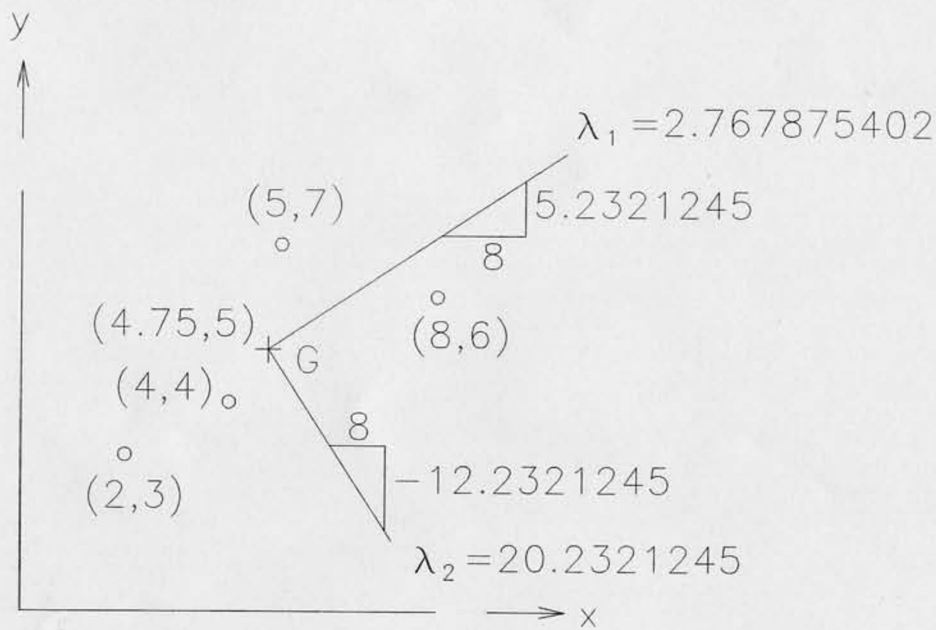
$$\left|\begin{bmatrix} I_{xx}-\lambda & -I_{xy} \\ -I_{xy} & I_{yy}-\lambda \end{bmatrix}\right| = 0 \qquad \begin{aligned} (I_{xx}-\lambda)\,a_1 \quad -I_{xy} \cdot a_2 &= 0 \\ -I_{xy}\,a_1 + (I_{yy}-\lambda)\,a_2 &= 0 \end{aligned} \qquad \frac{a_2}{a_1}$$

characteristic equation eigenvector direction ratio

$$I_{xx} = \sum y^2,\ I_{yy} = \sum x^2,\ I_{xy} = \sum xy$$

(21) BESTLINE





```
> restart:with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
```

```
> evm:=matrix(2,2,[Ixx-lambda,-Ixy,-Ixy,Iyy-lambda]);
```

$$evm := \begin{bmatrix} Ixx - \lambda & -Ixy \\ -Ixy & Iyy - \lambda \end{bmatrix}$$

```
> Ixx:=y1^2+y2^2+y3^2+y4^2:Ixy:=x1*y1+x2*y2+x3*y3+x4*y4:Iyy:=x1^2+x2
  ^2+x3^2+x4^2:
```

```
> x1:=-25:y1:=0:x2:=-7:y2:=-24:x3:=25:y3:=0:x4:=7:y4:=24:Ixx;Ixy;Iyy
  ;
```

$$1152$$

$$336$$

$$1348$$

```
> EVM:=matrix(2,2,[1152-lambda,-336,-336,1348-lambda]);
```

$$EVM := \begin{bmatrix} 1152 - \lambda & -336 \\ -336 & 1348 - \lambda \end{bmatrix}$$

```
> ceq:=det(EVM);
```

$$ceq := 1440000 - 2500\,\lambda + \lambda^2$$

```
> L:=[solve(ceq)];
```

$$L := [\,1600, 900\,]$$

```
> solve((1152-1600)*e1-336*e2,e1);solve(-336*e1+(1348-1600)*e2,e1);
```

$$-\frac{3}{4}\,e2$$

$$-\frac{3}{4}\,e2$$

```
> solve((1152-900)*e1-336*e2,e1);solve(-336*e1+(1348-900)*e2,e1);
```

$$\frac{4}{3}\,e2$$

$$\frac{4}{3}\,e2$$

Eigenvalues and eigenvectors for the "rectangle" on (21)BESTLINE. For your next problem, do the line fit with the 4 random points. (28)BstLn42t.mws 04-02-20