MECH 576
Geometry in Mechanics
Shortest Distance from Line to Coplanar Conic

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1 A Conic and Two Tangent Lines

Examine Fig. 1. Shown there is a given conic a, represented by a general ellipse, and a given coplanar line $g\{G_0 : G_1 : G_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on a that is closest to g.

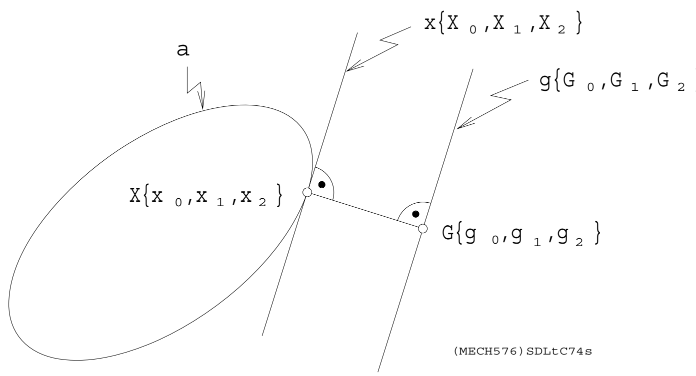

Figure 1: Line and Conic

An equivalent problem, the one chosen to be solved, is to find tangent lines $x\{X_0 : X_1 : X_2\}$ parallel to g and on X. Clearly there are two of these, the one shown and another, not shown, that is *farthest* from g, on the opposite side of a.

2 Constraint Equations

Since one seeks a solution in the Euclidean plane $x_0 = 1$ and the scalar equation, expressing $X \in a$, is Eq. 1.

$$[1\ x_1\ x_2] \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = a_{00} + 2a_{01}x_1 + 2a_{02} + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = 0 \qquad (1)$$

The second constraint equation, Eq. 2, defines the coefficients or homogeneous coordinates of the polar line x tangent to a on X.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{00} + a_{01}x_1 + a_{02}x_2 \\ a_{01} + a_{11}x_1 + a_{12}x_2 \\ a_{02} + a_{12}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} \mu G_0 \\ G_1 \\ G_2 \end{bmatrix} \qquad (2)$$

The last vector states that x and g are parallel and the last two rows of Eq. 2 provide, together with Eq. 1, the three constraints necessary to handle the three variables $x_1,\ x_2,\ \lambda$. Rewriting these last two rows and eliminating λ yields a second equation, Eq. 3, in only x_1 and x_2.

$$\begin{aligned} (a_{01} + a_{11}x_1 + a_{12}x_2) - \lambda G_1 &= 0 \\ (a_{02} + a_{12}x_1 + a_{22}x_2) - \lambda G_2 &= 0 \\ (G_2a_{01} - G_1a_{02}) + (G_2a_{11} - G_1a_{12})x_1 + (G_2a_{12} - G_1a_{22})x_2 &= 0 \end{aligned} \tag{3}$$

3 A Univariate Quadratic

Eliminating x_2 from Eq. 1 with Eq. 3 produces the quadratic, Eq. 4.

$$Ax_1^2 + Bx_1 + C = 0 \tag{4}$$

where

$$A = (G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{11}a_{22} - a_{12}^2)$$
$$B = 2(G_1^2a_{22} - 2G_1G_2a_{12} + G_2^2a_{11})(a_{01}a_{22} - a_{02}a_{12})$$
$$C = (2G_1G_2a_{12} - G_1^2a_{22}^2)a_{02}^2 + G_2^2(a_{01}a_{22} - 2a_{02}a_{12})a_{01} + [G_1^2a_{22}^2 + G_2(G_2a_{12} - 2G_1a_{22})a_{12}]a_{00}$$

4 Normal Line on X

All that remains, after finding the two points X, is to find $G\{g_0 : g_1 : g_2\}$, via Eq. 6, the points of intersection of lines $y\{Y_0 : Y_1 : Y_2\}$, found with Eq. 5, and g. Line y is on X and normal to g. This is outlined below.

$$Y_0 + Y_1x_1 + Y_2x_2 = 0,\ \ Y_0 = -(Y_1x_1 + Y_2x_2),\ \ Y_1 = -G_1,\ \ Y_2 = G_2 \tag{5}$$
$$g_0 = G_1Y_2 - Y_1G_2,\ \ g_1 = G_2Y_0 - G_0Y_2,\ \ g_2 = G_0Y_1 - G_1Y_0 \tag{6}$$

The minimum and maximum distances s to be compared are given by Eq. 7.

$$s = \sqrt{(x_1 - g_1/g_0)^2 + (x_2 - g_2/g_0)^2} \tag{7}$$

5 An Alternate Method Using Conic/Line Intersection

First the intersection points between a and g are found by eliminating, say, x_2 between the conic and line equations.

$$
\begin{aligned}
a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0 \\
G_0 + G_1x_1 + G_2x_2 = 0 \\
(G_2^2a_{11} - 2G_1G_2a_{12} + G_1^2a_{22})x_1^2 \\
+2[G_0(G_1a_{22} - G_2a_{12}) + G_2(G_2a_{01} - G_1a_{02})]x_1 \\
+(G_1^2a_{00} - 2G_0G_1a_{01} + G_0^2a_{11}) = 0
\end{aligned}
$$

Then the midpoint F between the two intersection points are found along with the conic centre point C.

$$
\begin{aligned}
F\{G_1(G_1a_{22} - 2G_2a_{12}) + G_2^2a_{11} : G_0(G_1a_{22} - G_2a_{12}) + G_2(G_2a_{01} - G_1a_{02}) \\
: G_0(G_2a_{11} - G_1a_{12}) + G_1(G_1a_{02} - G_2a_{01})\} \\
C\{a_{11}a_{22} - a_{12}^2 : a_{01}a_{22} - a_{02}a_{12} : a_{01}a_{12} - a_{02}a_{11}\} \qquad (8)
\end{aligned}
$$

The line $f\{F_0 : F_1 : F_2\}$ on $F\{f_0 : f_1 : f_2\}$ and $C\{c_0 : c_1 : c_2\}$ intersects the conic on the two points whereon the tangents to the conic are parallel to g.

$$f\{c_1f_2 - c_2f_1 : c_2f_0 - c_0f_2 : c_0f_1 - c_1f_0\}$$

6 Numerical Example

Consider Fig. 2.

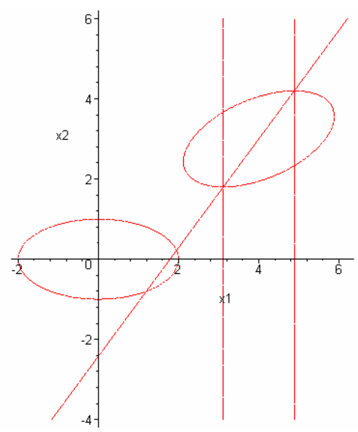

Figure 2: Shortest and Farthest Distances Example

The symmetric coefficient matrix of a standard form ellipse $-4x_0^2 + x_1^2 + 4x_2^2 = 0$ has been rotated counterclockwise by an angle $\tan^{-1}\frac{5}{12}$ and its centre translated to $(4, 3)$ by the transformation

$$169\left\{\begin{bmatrix} 1 & -4 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & -\frac{5}{13} \\ 0 & \frac{5}{13} & \frac{12}{13} \end{bmatrix}\begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{12}{13} & \frac{5}{13} \\ 0 & -\frac{5}{13} & \frac{12}{13} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 4317 & -436 & -1083 \\ -436 & 244 & -180 \\ -1083 & -180 & 601 \end{bmatrix}$$

Notice premultiplications of the standard form coefficient matrix by cofactor (dual) matrices of point vector rotation (inner to effect pre-rotation) and point vector translation (outer to effect post-translation). Postmultiplications are in the same inner/outer sequence and these are the transposes of their premultiplier counterparts. The scalar multiplier $13^2 = 169$ renders the resulting matrix, that produces the ellipse in a general disposition, free of fractional elements.

$$4317x_0^2 - 872x_0x_1 - 2166x_0x_2 + 244x_1^2 - 360x_1x_2 + 601x_2^2 = 0$$

Intersecting the given line $x_2 = 0$ with the general ellipse gives the complex conjugate point pair $\{1 : \frac{109}{61} \pm \frac{13}{122}\sqrt{1277}i : 0\}$. The midpoint between these two is real however; $\{1 : \frac{109}{61} : 0\}$. Joining it to the conic centre point $\{1 : 4 : 3\}$ produces the line

$$-327x_0 + 183x_1 + 353x_2 = 0$$

that intersects the conic on points that are, respectively, the closest to and farthest from the given line. Alternately, putting the three line coordinates $G_0 = -327,\ G_1 = 183,\ G_2 = 353$ in Eq. 8 and solving for two values of x_1 yields the vertical lines $x_1 - 4.88641 = 0$ and $x_1 - 3.11359$. These establish the location of the same two points on the conic.

7 Distance via Discriminant

Recall the quadratic univariate, in terms of x_1, that results from the intersection of the given line and quadric.

$$\begin{aligned} &(G_2^2a_{11} - 2G_1G_2a_{12} + G_1^2a_{22})x_1^2 \\ +2[G_0(G_1a_{22} - G_2a_{12}) + G_2&(G_2a_{01} - G_1a_{02})]x_1 \\ &+(G_2^2a_{00} - 2G_0G_2a_{02} + G_0^2a_{22}) = 0 \end{aligned}$$

Taking the derivative of this univariate with respect to x_1 yields a second equation. Eliminating x_1, removing factor

$$2[2G_2^2(G_2^2a_{11} - 2G_1G_2a_{12} + G_1^2a_{22})]$$

devoid of G_0 produces an expression for G_0.

$$\begin{aligned} (a_{11}a_{22} - a_{12}^2)G_0^2 + 2[(G_2a_{12} - G_1a_{22})a_{01} + 2[(G_1a_{12} - G_2a_{11})a_{02}]G_0 \\ +(G_2^2a_{11} - 2G_1G_2a_{12} + G_1^2a_{22})a_{00} + G_2(2G_1a_{02} - G_2a_{01})a_{01} - G_1^2a_{02}^2 = 0 \end{aligned}$$

Assuming that one obtains two real roots G_0' and G_0'' one may configure three versions of the line equation.

$$G_1x_1 + G_2x_2 + \begin{pmatrix} G_0 \\ G_0' \\ G_0'' \end{pmatrix} = 0$$

Normalizing on the line normal direction numbers, *i.e.*, Dividing these three equations by $\sqrt{G_1^2 + G_2^2}$ leaves the differences, say,

$$|G_0 - G_0'| > |G_0 - G_0''|$$

so that one obtains the distances to the closer and farther tangent lines to the conic, parallel to the original line, the equation that contains G_0.

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