MECH 576
Geometry in Mechanics

September 16, 2009

Parametric and Bézier Curves

1 Parametric Cubics and Quintics

Before studying the following material, review and absorb the introductory ideas dealt with in **MECH 289, Design Graphics, Module M2, section 6 Parametric Curves and Surfaces**, pp.56-65. The invariant direct and inverse transformations between geometric and algebraic forms will be derived using the algebraic coefficient vectors $\mathbf{a}_j$ and the standard geometric form vectors $\mathbf{p}_0,\ \mathbf{p}_1,\ \mathbf{p}_0^u$ and $\mathbf{p}_1^u,\ \mathbf{p}_0^{uu}$ and $\mathbf{p}_1^{uu}$. The transformation operators are 4×4 and 6×6 coefficient matrices that pre-multiply the geometric form vectors, *E.g.*,

$$\begin{bmatrix} \mathbf{a}_5 & \mathbf{a}_4 & \mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_0 \end{bmatrix} \begin{bmatrix} u^5 \\ u^4 \\ u^3 \\ u^2 \\ u^1 \\ u^0 \end{bmatrix} = \left[\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_0^u \\ \mathbf{p}_1^u \\ \mathbf{p}_0^{uu} \\ \mathbf{p}_1^{uu} \end{bmatrix} \right]^T \begin{bmatrix} u^5 \\ u^4 \\ u^3 \\ u^2 \\ u^1 \\ u^0 \end{bmatrix} \tag{1}$$

Eq. 1 provides the 36 algebraic coefficients in the parametric quintic equations, for a spatial curve, with three equations -one for each of the three Cartesian coordinates of any point on the curve segment defined by the parameter u in the range $0 \geq u \leq 1$- in a set of three. The first is the explicit algebraic form

$$\mathbf{a}_5 u^5 + \mathbf{a}_4 u^4 + \mathbf{a}_3 u^3 + \mathbf{a}_2 u^2 + \mathbf{a}_1 u^1 + \mathbf{a}_0 u^0 = \mathbf{p} \tag{2}$$

Note that $u^0 = 1$. The second and third are the first and second derivatives of Eq. 2.

$$5\mathbf{a}_5 u^4 + 4\mathbf{a}_4 u^3 + 3\mathbf{a}_3 u^2 + 2\mathbf{a}_2 u^1 + \mathbf{a}_1 u^0 = \mathbf{p}^u,\ \ 20\mathbf{a}_5 u^3 + 16\mathbf{a}_4 u^2 + 6\mathbf{a}_3 u^1 + 2\mathbf{a}_2 u^0 = \mathbf{p}^{uu} \tag{3}$$

Standard form geometric to algebraic transformation of parametric cubics is similar except $\mathbf{a}_5,\ \mathbf{a}_4,\ u^5,\ u^4, \mathbf{p}^{uu}$ are absent and the matrix $[a_{ij}]$ is reduced to 4×4.

1.1 Other Geometric Forms

Recall Eq. (23) on p.62 of the review material recommended, above, in section **1** and reproduced below.

$$[\mathbf{A}] = [\mathbf{P}]_{4P}[\mathbf{U}]^{-1} = [\mathbf{P}]_{SGF}[\mathbf{U}]_{SGF}^{-1},\ \text{ therefore }\ [\mathbf{P}]_{SGF} = [\mathbf{P}]_{4P}[\mathbf{U}]_{4P}^{-1}[\mathbf{U}]_{SGF}$$

This permits the conversion of *any* invariant matrix to standard form. *E.g.*,

$$\begin{bmatrix} a_{5x} & a_{4x} & a_{3x} & a_{2x} & a_{1x} & a_{0x} \\ a_{5y} & a_{4y} & a_{3y} & a_{2y} & a_{1y} & a_{0y} \\ a_{5z} & a_{4z} & a_{3z} & a_{2z} & a_{1z} & a_{0z} \end{bmatrix} \begin{bmatrix} u_a^5 & u_b^5 & u_c^5 & u_d^5 & u_e^5 & u_f^5 \\ u_a^4 & u_b^4 & u_c^4 & u_d^4 & u_e^4 & u_f^4 \\ u_a^3 & u_b^3 & u_c^3 & u_d^3 & u_e^3 & u_f^3 \\ u_a^2 & u_b^2 & u_c^2 & u_d^2 & u_e^2 & u_f^2 \\ u_a^1 & u_b^1 & u_c^1 & u_d^1 & u_e^1 & u_f^1 \\ u_a^0 & u_b^0 & u_c^0 & u_d^0 & u_e^0 & u_f^0 \end{bmatrix} = \begin{bmatrix} p_{ax} & p_{bx} & p_{cx} & p_{dx} & p_{ex} & p_{fx} \\ p_{ax} & p_{bx} & p_{cx} & p_{dx} & p_{ex} & p_{fx} \\ p_{az} & p_{bz} & p_{cz} & p_{dz} & p_{ez} & p_{fz} \end{bmatrix} \tag{4}$$

Eq. 4, essentially Eq. (19) on p.60 of the review material recommended, above, in section **1**, represents six given point position vectors $\mathbf{p}_a, \ \ldots, \ \mathbf{p}_f$ that are defined at six given values of u, *i.e.*, $u_a, \ \ldots, \ u_f$. Let us examine the specification corresponding to standard geometric form where only two points, one at either end of the segment to be connected, are given, *i.e.*, at $u = 0$ and $u = 1$, while the remaining four are the first and second derivative vectors. so Eqs. 2 and 3 contain the numerical coefficients, in the sequence given there and taking into account $u = 0$ and $u = 1$ taken to the various exponent powers. These coefficients are written into the columns of the middle matrix below.

$$\begin{bmatrix} a_{5x} & a_{4x} & a_{3x} & a_{2x} & a_{1x} & a_{0x} \\ a_{5y} & a_{4y} & a_{3y} & a_{2y} & a_{1y} & a_{0y} \\ a_{5z} & a_{4z} & a_{3z} & a_{2z} & a_{1z} & a_{0z} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 5 & 0 & 20 \\ 0 & 1 & 0 & 4 & 0 & 12 \\ 0 & 1 & 0 & 3 & 0 & 6 \\ 0 & 1 & 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} p_{0x} & p_{1x} & p_{0x}^u & p_{1x}^u & p_{0x}^{uu} & p_{1x}^{uu} \\ p_{0y} & p_{1y} & p_{0y}^u & p_{1y}^u & p_{0y}^{uu} & p_{1y}^{uu} \\ p_{0z} & p_{1z} & p_{0z}^u & p_{1z}^u & p_{0z}^{uu} & p_{1z}^{uu} \end{bmatrix}$$

Geometric specification means that matrix $[\mathbf{P}]$ is given. All that is needed, to compute points along the curve segment from end to end, suitable for plotting, is matrix $[\mathbf{A}]$. This can be produced with the inverse of $[\mathbf{U}]^{-1}$, the numerical matrix above. $[\mathbf{U}]$, then is

$$\begin{bmatrix} -12 & 12 & -6 & -6 & -1 & 1 \\ 30 & -30 & 16 & 14 & 3 & -2 \\ -20 & 20 & -12 & -8 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the standard geometric form parametric cubic we get

$$[\mathbf{U}]^{-1} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{U}] = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

2 Join-the-Dots

Even with modern computer graphics and CAD software concepts, for the modeling of curves and surfaces, like the various splines, including NURBS, the designer experiences a certain user-unfriendliness when faced with artifacts imposed by mathematical expediency, *i.e.*, control points and tangent vector magnitudes. Joining points, *on* a desired curve or surface, smoothly is not always easy and convenient. Ways to adapt cubic curves to do this will be discussed after introducing the idea of rotated coordinate, explicit cubics and adapting the Bézier-de Casteljau algorithm models to compute points on a parametric cubic curve segment in terms of the parameter u.

2.1 Subsets, Algorithms and Efficient Computation

Examine Fig. 1.

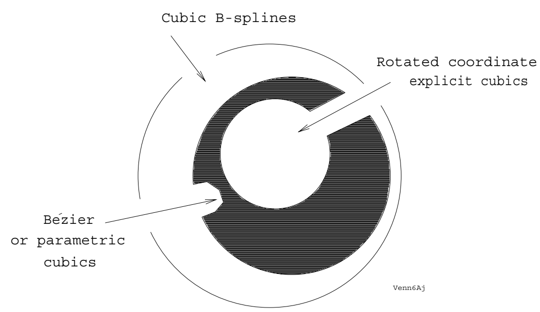

Figure 1: Subset Relations

Parametric cubics, introduced in section **1**, occupy a subset between the general concept of B-splines, widely discussed in computer graphics literature, and what will be discussed in some detail, later; join-the-dots cubics (JtDC), that we call rotated coordinate explicit cubics, are particular cases of parametric cubic curves (PCC). JtDC can always be expressed as PCC but not necessarily *vice-versa*. Before going on to treat some useful variants of JtDC look at Fig. 2. Explained therein is the cubic Bézier curve and the algorithm to compute points on a curve segment, algebraically and constructively, in terms of parameter u. This algorithm is conveniently adaptable to the advanced computer architectural features of pipelining and parallel processing, as shown in Fig. 3.

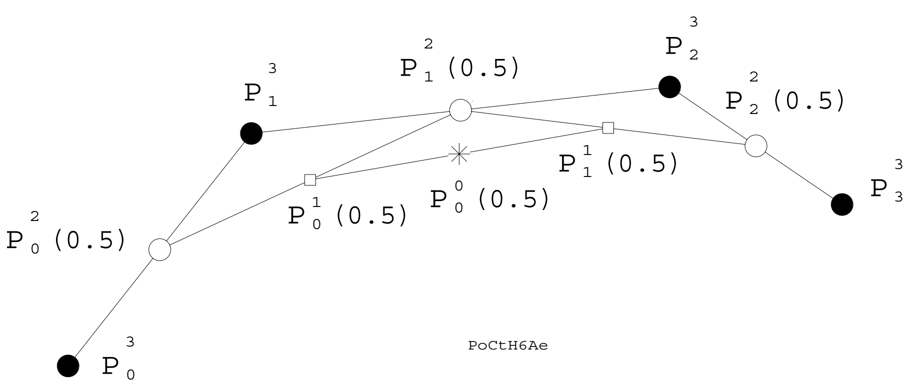

Figure 2: Bézier Algorithm and de Casteljau Construction

Cartesian tensor representation for the recursive construction, based on point position vectors, is given in Eq. 5.

$$P_i^j(u) = (P_{i+1}^{j+1} - P_i^{j+1})u + P_i^{j+1} = uP_{i+1}^{j+1} + (1-u)P_i^{j+1} \tag{5}$$

Finally, by substitution and collecting terms the familiar parametric Bézier cubic expression is obtained as Eq. 6.

$$P_0^0 = P_3^3u^3 + 3P_2^3(1-u)^2 + 3P_1^3(1-u)^2u + P_0^3(1-u)^3 \tag{6}$$

Fig. 4 shows how unsatisfactory geometric specification can lead to unwanted tangent discontinuity and double points on the resulting curve. The parameter u is an additional dimension and, depending on how a smooth curve in the space with this extra dimension is projected into Euclidean space, cusps and loops may appear. Parametric parabolæ, cubics and quintics are merely extensions of the parametric straight line segment concept based on segment limiting point position vectors and a single free parameter.

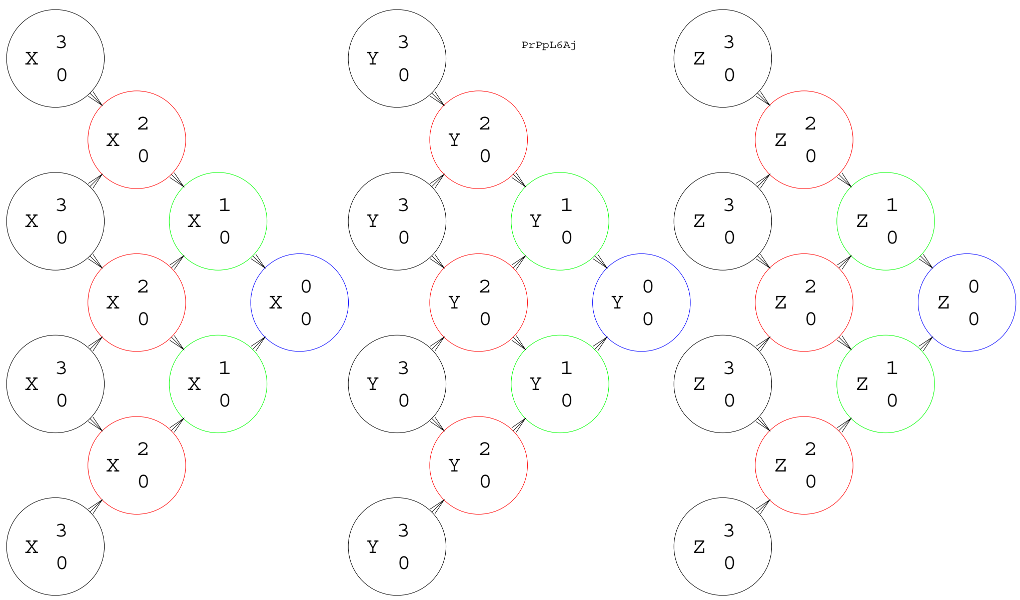

Figure 3: Pipelining and Parallel Processing

$$\mathbf{p} = \mathbf{a} + u(\mathbf{b} - \mathbf{a})$$

This is illustrated in Fig. 5. This generalization allows the two simpler forms to share, as degenerate species, a common data base with cubics and quintics. That makes for efficient processing in the computer at the graphics card level. Furthermore one sees why polynomials of even degree, *e.g.*, 2 or 4, should be avoided. They occupy a odd numbered homogeneous vector space so it is not possible to assign the same number of standard geometric form constraint vectors to both ends of a segment.

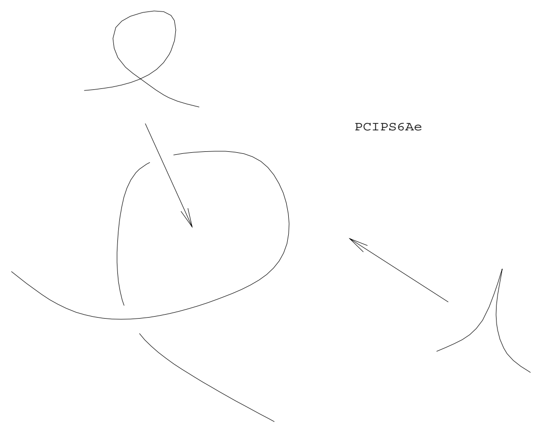

Figure 4: Loops and Cusps

2.2 Funiculum Based Slope Vectors

The following treatment is confined to JtDC a special family of parametric cubics. One sees some ways to systematically and consistently generate tangent vectors, including their magnitude, by using the sequence of straight line segments formed by joining points along a desired curve. Note that closed curves are easily and unambiguously defined by incorporating the last point as the backward point in the first segments and the first point as the forward point in the last. In short, in all schemes tangent vector direction at a given point is based on some combination, either adjacent segment direction bisecting or adjacent segment length weighted, of the properties of the segments that meet at that point. All four schemes are based on a kernel represented by a four point, three segment sequence $P_{i-1}, P_i, P_{i+1}, P_{i+2}$. Fig. 6 bears considerable attention because the concepts illustrated there will be repeated, with small modification, for three other cases.

- It illustrates the original scheme where unit tangent vectors $\mathbf{t}_0$ and $\mathbf{t}_1$ at P_i and P_{i+1}, that are points corresponding to initial and terminal point position vectors $\mathbf{p}_0 \equiv \mathbf{p}_i$ and $\mathbf{p}_1 \equiv \mathbf{p}_{i+1}$, are in the forward bisector direction so that

$$\mathbf{t}_0 = \frac{\frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{\|\mathbf{p}_i - \mathbf{p}_{i-1}\|} + \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|}}{\|\frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{\|\mathbf{p}_i - \mathbf{p}_{i-1}\|} + \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|}\|}, \quad \mathbf{t}_1 = \frac{\frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|} + \frac{\mathbf{p}_{i+2} - \mathbf{p}_{i+1}}{\|\mathbf{p}_{i+2} - \mathbf{p}_{i+1}\|}}{\|\frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|} + \frac{\mathbf{p}_{i+2} - \mathbf{p}_{i+1}}{\|\mathbf{p}_{i+2} - \mathbf{p}_{i+1}\|}\|} \tag{7}$$

- Tangent vector magnitudes k_0 and k_1 to produce

$$\mathbf{p}_0^u = k_0 \mathbf{t}_0, \quad \mathbf{p}_1^u = k_1 \mathbf{t}_1$$

  are obtained by projecting $\mathbf{t}_{0,1}$ onto a "local x-axis" that in this case is in the direction of the forward angle bisector between $\mathbf{t}_0$ and $\mathbf{t}_1$.

- The unit vector $\mathbf{i}$ along this x-axis is

$$\mathbf{i} = \frac{\mathbf{t}_0 + \mathbf{t}_1}{\|\mathbf{t}_0 + \mathbf{t}_1\|} \tag{8}$$

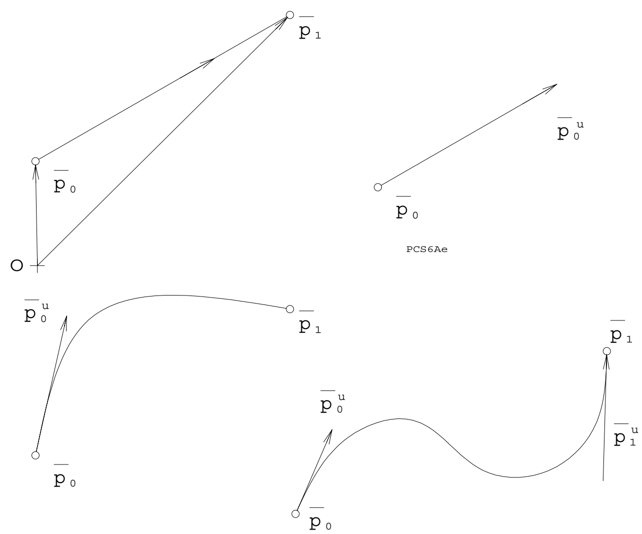

Figure 5: Standard Geometric Form parametric Line, Parabola and Cubic

- Tangent vector magnitudes k_0 and k_1 are given by

$$k_0 = k_1 = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i) \cdot \mathbf{i})}{\mathbf{t}_0 \cdot \mathbf{i}} \tag{9}$$

- This, rotated *single* x-axis, explicit form is a special parametric form with $a_{3x} = a_{2x} = a_{0x} = 0,\ a_{1x} = 1$ for *some* x, y, z.

- For planar cubics of this type $a_{3z} = a_{2z} = a_{1z} = a_{0z} = 0$.

- For *double* x-axis varieties it may be interesting to investigate their explicit form if it exists.

- Parametrization along a Cartesian axis, peculiar to JtDC, might allow more efficient computation of finite arc lengths. This is a very important consideration in certain video games and interactive computer simulations where the user controls the speed of an "avatar" but its image on a known trajectory must be deduced from its speed-time integral in real time.

- Having determined vectors $\mathbf{p}_0^u$ and $\mathbf{p}_1^u$, in this or some other way, the two intermediate Bézier control points P_1^3 and P_2^3, shown in Fig. 2, and again as P_1 and P_2 in Fig. 6, are obtained by chopping the tangent vector $\mathbf{p}_0^u \equiv P_i^u$, radiating *away* from P_i, at $t = 1/3$. Note that t is just the parameter u which has been reserved in these diagrams as an exponent to indicate differentiation. Similarly, the tangent vector P_{i+1}^u at the other end P_{i+1} radiates *toward* $P_{i+1} \equiv P_3$. The Bézier control point P_2 is obtained at $t = 2/3$ or measured one-third of the way *backward* from $P_{i+1} \equiv P_3$ along the vector $\mathbf{p}_1^u$

In Fig. 7 one sees a scheme to determine tangent vectors $\mathbf{p}_0^u$ and $\mathbf{p}_1^u$ almost identical to that illustrated in Fig. 6. The only difference is that the respective tangent vector directions are determined by simply adding segment vectors. $P_{i-1} \rightarrow P_i + P_i \rightarrow P_{i+1}$ and $P_i \rightarrow P_{i+1} + P_{i+1} \rightarrow P_{i+2}$ are taken to be in the direction of $\mathbf{t}_0$ and $\mathbf{t}_1$, respectively.

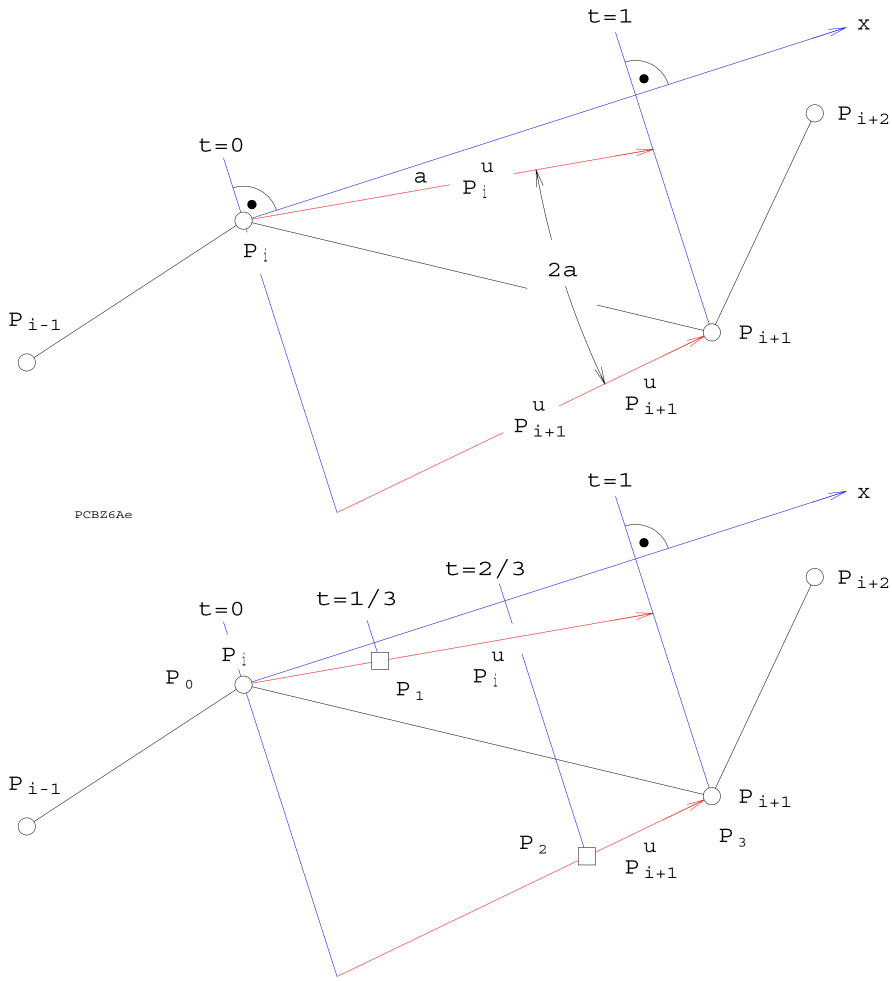

Figure 6: Bézier and Parametric Cubic

- Computing the unit tangent vectors is simpler.

$$\mathbf{t}_0 = \frac{\mathbf{p}_{i+1} - \mathbf{p}_{i-1}}{\|\mathbf{p}_{i+1} - \mathbf{p}_{i-1}\|}, \quad \mathbf{t}_1 = \frac{\mathbf{p}_{i+2} - \mathbf{p}_i}{\|\mathbf{p}_{i+2} - \mathbf{p}_i\|} \tag{10}$$

- The unit vector $\mathbf{i}$ along this x-axis is identical to that given in Eq. 8.

$$\mathbf{i} = \frac{\mathbf{t}_0 + \mathbf{t}_1}{\|\mathbf{t}_0 + \mathbf{t}_1\|} \tag{11}$$

- Tangent vector magnitudes k_0 and k_1 are given by

$$k_0 = k_1 = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i) \cdot \mathbf{i})}{\mathbf{t}_0 \cdot \mathbf{i}} \tag{12}$$

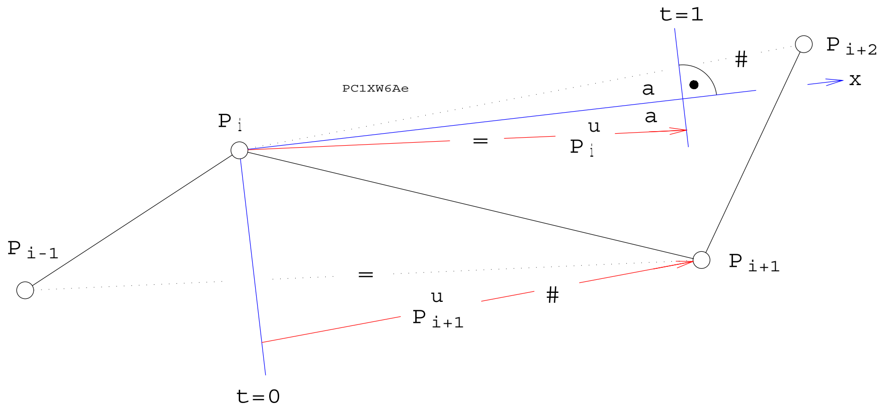

Figure 7: Length Weighted Tangents

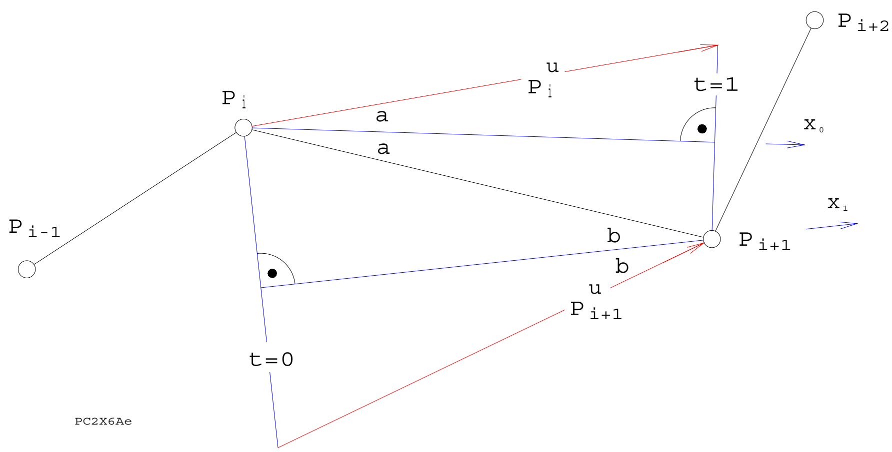

Figure 8: Two Axes to Accommodate "Sharp Returns"

In Fig. 8 we see a scheme that entails *two* local x-axes, one at either end of the line segment supporting the curve segment. This scheme retains the computation of unit tangent vectors expressed by Eq. 7. However the x-axis directions at either end are computed by bisecting the angle between each tangent and the supporting line segment. Local x-axes are given by

$$\mathbf{s} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\|\mathbf{p}_{i+1} - \mathbf{p}_i\|}, \quad \mathbf{i}_i = \frac{\mathbf{t}_0 + \mathbf{s}}{\|\mathbf{t}_0 + \mathbf{s}\|}, \quad \mathbf{i}_{i+1} = \frac{\mathbf{t}_1 + \mathbf{s}}{\|\mathbf{t}_1 + \mathbf{s}\|} \tag{13}$$

while the tangent vector magnitudes are given below.

$$k_0 = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i) \cdot \mathbf{i}_i}{\mathbf{t}_0 \cdot \mathbf{i}_i}, \quad k_1 = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i) \cdot \mathbf{i}_{i+1}}{\mathbf{t}_0 \cdot \mathbf{i}_{i+1}} \tag{14}$$

Double x-axis schemes were developed to overcome "mushrooming" of the curve at a very acute funicular apex. In fact if a pair of funicular segments in the sequence overlapped, the curve, based on single x-axis schemes, would climb the "stem" a bit before wandering off to infinity in one direction, return as a very wide mushroom top, go off in the opposite direction, finally returning to the stem and resuming a credible approximation to a JtDC until another very acute corner. The

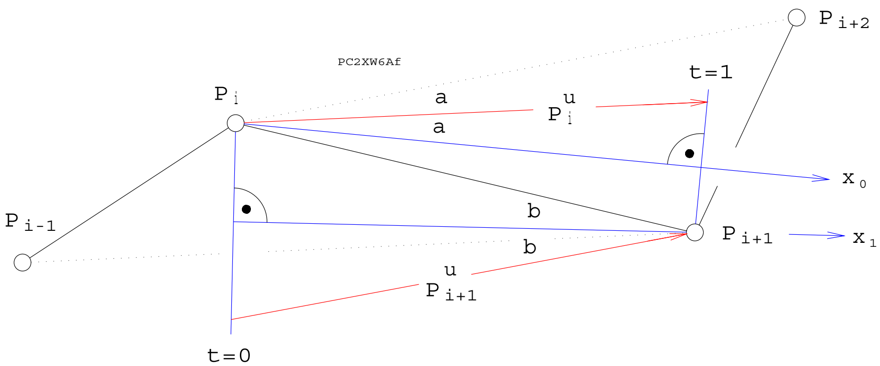

Figure 9: Two Axis Length Weighted Tangents

final scheme, shown in Fig. 9, is a combination that uses the length weighted tangent directions computed with Eqs. 10 and the local x-axis and tangent vector magnitude computations in Eq. 14.

2.2.1 Comparison Example

To compare the four schemes described above a 4×1 rectangle was chosen to have its four vertices connected by four closed curves, each generated according to one of the four scheme rules, respectively. Fig. **??** shows three of the four corners' position and tangent vectors. These corners are at $(-32, 8)$, $(32, 8)$ and $(32, -8)$. The tangent vectors and local x-axes for all schemes are also illustrated there. One may sum up the geometric form data for these examples with four matrix

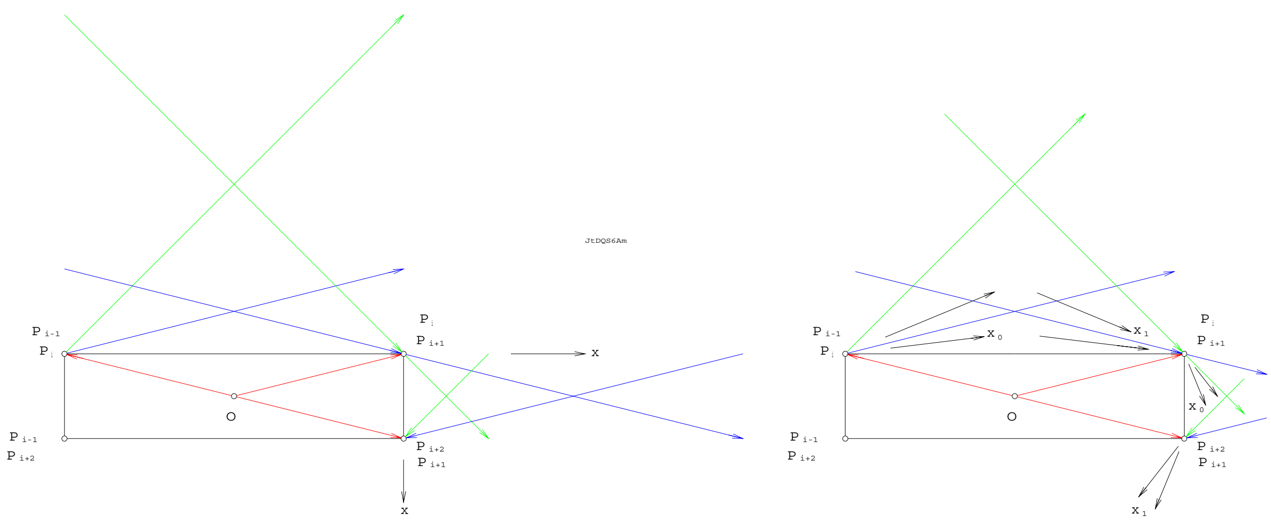

Figure 10: Rectangle Corners, Tangent Vectors and Local x-Axes

pairs containing standard geometric form column vectors. Each pair encodes data for the right half of the long horizontal side of the rectangle and the adjoining top half of the short right side, respectively.

$$1XTB \equiv \begin{bmatrix} -32 & 32 & 64 & 64 \\ 8 & 8 & 64 & -64 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 32 & 32 & 16 & -16 \\ 8 & -8 & -16 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$1XW \equiv \begin{bmatrix} -32 & 32 & 64 & 64 \\ 8 & 8 & 16 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 32 & 32 & 64 & -64 \\ 8 & -8 & -16 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2XTB \equiv \begin{bmatrix} -32 & 32 & 64/\sqrt{2} & 64/\sqrt{2} \\ 8 & 8 & 64/\sqrt{2} & -64/\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 32 & 32 & 16/\sqrt{2} & -16/\sqrt{2} \\ 8 & -8 & -16/\sqrt{2} & -16/\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2XW \equiv \begin{bmatrix} -32 & 32 & 256/\sqrt{17} & 256/\sqrt{17} \\ 8 & 8 & 64/\sqrt{17} & -64/\sqrt{17} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 32 & 32 & 64/\sqrt{17} & -64/\sqrt{17} \\ 8 & -8 & -16/\sqrt{17} & -16/\sqrt{17} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The results appear in Fig. 11. Corresponding schemes and curves are as follows.

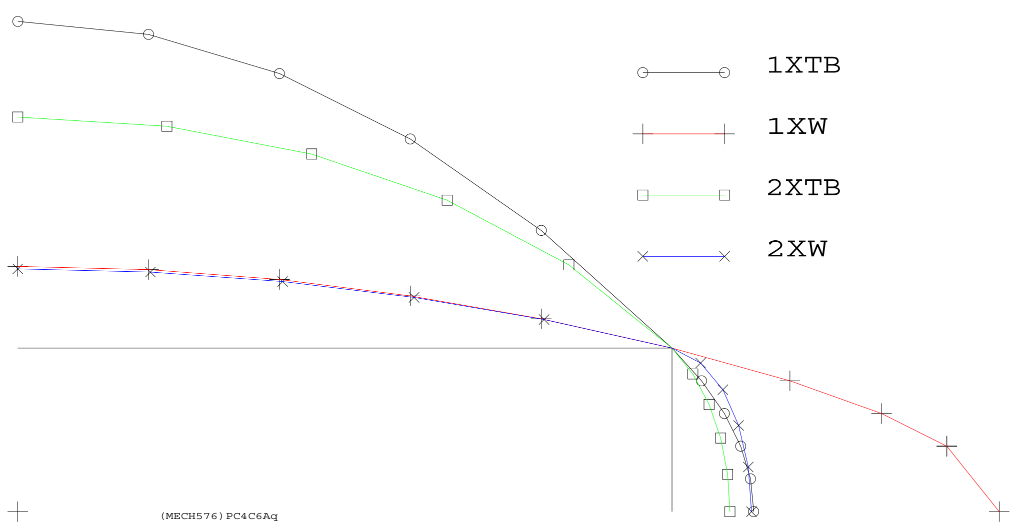

Figure 11: Four Pairs of Curve Segments around a Quarter of the Rectangle

- Curve identified as $1XTB$ corresponds to Fig. 6.
- Curve identified as $1XW$ corresponds to Fig. **??**.
- Curve identified as $2XTB$ corresponds to Fig. **??**.
- Curve identified as $2XW$ corresponds to Fig. 9.

Note that weighted schemes "hug" the rectangle more closely while tangent directions that bisect adjacent funiculum segment directions tend to sweep more smoothly around the rectangle, more closely approximating a circumscribing ellipse. These are in no way ellipses. Cubics do a lousy job in mimicking conics, hence the need for NURBS or Non-Uniform-Rational-"B"-Splines, a planed extension to these notes.