MECH 576
Geometry in Mechanics

September 16, 2009

Planar Kinematic Mapping Fundamentals

1 Introduction

The intent of this article is to explain, in a clear and simple way, what planar kinematic mapping is in terms of elementary homogeneous matrix transformations which "move" points and lines in the plane. Relationship of the image space coordinates to the pole of displacement, the invariant point of a planar motion, is derived. It is believed that these preliminaries are useful preparation to study modern applications of planar kinematic mapping, *e.g*, [1, 2] or to delve deeper into the theory, [3, 4].

2 Notions

For planar displacements consider the transformations which express the homogeneous coordinates of points and lines in the moving, end effector frame EE premultiplied by a transformation matrix and converted to their homogeneous coordinates in the fixed frame FF. Points are transformed as

$$\begin{bmatrix} t \\ u \\ v \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix}$$

Lines are transformed as

$$\begin{bmatrix} T \\ U \\ V \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} W \\ X \\ Y \end{bmatrix}$$

The coefficients will be evaluated by using three ideal elements in the EE frames and their corresponding coordinates in FF. Then these coefficients will be converted and expressed in terms of four homogeneous Blaschke-Grünwald coordinates, a mapping in the kinematic image space. This is an abstract projective 3-space wherein a point represents a displacment of a rigid body in the plane.

$$\{X_0 : X_1 : X_2 : X_3\} \equiv \{2\cos\frac{\phi}{2} : a\sin\frac{\phi}{2} - b\cos\frac{\phi}{2} : a\cos\frac{\phi}{2} + b\sin\frac{\phi}{2} : 2\sin\frac{\phi}{2}\}$$

3 The Point Transformation

Ideal point elements are chosen as the origin,

$$\lambda \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$a_{11} = \lambda, \quad a_{21} = \lambda a, \quad a_{31} = \lambda b$$

the point at infinity which closes the x-axis,

$$\lambda \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$$

$$a_{12} = 0, \quad a_{22} = \lambda\cos\phi, \quad a_{32} = \lambda\sin\phi$$

and the point at infinity which closes the y-axis,

$$\lambda \begin{bmatrix} 0 \\ -\sin\phi \\ \cos\phi \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$a_{13} = 0, \quad a_{23} = -\lambda\sin\phi, \quad a_{33} = \lambda\cos\phi$$

where λ is an arbitrary constant. Populating $[a_{ij}]$ with these results and making the tangent half-angle substitutions

$$\cos\phi = \frac{1 - \tan^2\frac{\phi}{2}}{1 + \tan^2\frac{\phi}{2}}, \quad \sin\phi = \frac{2\tan\frac{\phi}{2}}{1 + \tan^2\frac{\phi}{2}}$$

and multiplying through by $(1 + \tan^2\frac{\phi}{2})$, then by $\cos^2\frac{\phi}{2}$ produce

$$[a_{ij}] = \lambda \begin{bmatrix} (\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}) & 0 & 0 \\ a(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}) & (\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) & -2\cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ b(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}) & 2\cos\frac{\phi}{2}\sin\frac{\phi}{2} & (\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) \end{bmatrix} \tag{1}$$

which becomes, after substitution for $\{X_0 : X_1 : X_2 : X_3\}$ and simplifying and dividing through by 4.

$$[a_{ij}] = \begin{bmatrix} X_0^2 + X_3^2 & 0 & 0 \\ 2(X_0X_2 + X_1X_3) & X_0^2 - X_3^2 & -2X_0X_3 \\ -2(X_0X_1 - X_2X_3) & 2X_0X_3 & X_0^2 - X_3^2 \end{bmatrix}$$

This is easily inverted and after multiplying by $(X_0^2 + X_3^2)^2$

$$[a_{ij}]^{-1} = \begin{bmatrix} X_0^2 + X_3^2 & 0 & 0 \\ -2(X_0X_2 - X_1X_3) & X_0^2 - X_3^2 & 2X_0X_3 \\ 2(X_0X_1 + X_2X_3) & -2X_0X_3 & X_0^2 - X_3^2 \end{bmatrix}$$

The metric is obtained as $[a_{ij}][a_{ij}]^{-1}$.

$$\begin{bmatrix} (X_0^2 + X_3^2)^2 & 0 & 0 \\ 0 & (X_0^2 + X_3^2)^2 & 0 \\ 0 & 0 & (X_0^2 + X_3^2)^2 \end{bmatrix}$$

4 The Line Transformation

Ideal line elements are chosen. The first is the invariant line at infinity.

$$\lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$A_{11} = \lambda, \quad A_{21} = 0, \quad A_{31} = 0$$

Then the coordinates of the x-axis in FF are formed with the origin and the point at infinity.

$$\begin{vmatrix} w & x & y \\ 1 & a & b \\ 0 & \cos\phi & \sin\phi \end{vmatrix} \Rightarrow$$

$$\{a\sin\phi - b\cos\phi : -\sin\phi : \cos\phi\}$$

$$\lambda \begin{bmatrix} a\sin\phi - b\sin\phi \\ -\sin\phi \\ \cos\phi \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$A_{13} = \lambda(a\sin\phi - b\cos\phi), \quad A_{23} = -\lambda\sin\phi, \quad A_{33} = \lambda\cos\phi$$

Finally the coordinates of the y-axis in FF are found with the origin and the point at infinity.

$$\begin{vmatrix} w & x & y \\ 1 & a & b \\ 0 & -\sin\phi & \cos\phi \end{vmatrix} \Rightarrow$$

$$\{a\cos\phi + b\sin\phi : -\cos\phi : -\sin\phi\}$$

$$\lambda \begin{bmatrix} a\cos\phi + b\sin\phi \\ -\cos\phi \\ -\sin\phi \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow$$

$$A_{12} = -\lambda(a\cos\phi + b\sin\phi), \quad A_{22} = \lambda\cos\phi, \quad A_{32} = \lambda\sin\phi$$

Note $X = -1$ making line normal in EE compatible with that in FF. Populating $[A_{ij}]$ with these results and making the tangent half-angle substitutions and multiplying through by $(1 + \tan^2\frac{\phi}{2})$, then by $\cos^2\frac{\phi}{2}$ produce

$$[A_{ij}] = \lambda \left[\begin{array}{cc} \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} & -a(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}) + 2b\cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ 0 & (\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) \\ 0 & 2\cos\frac{\phi}{2}\sin\frac{\phi}{2} \end{array} \right.$$

$$\left. \begin{array}{c} 2a\cos\frac{\phi}{2}\sin\frac{\phi}{2} - b(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) \\ -2\cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ (\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}) \end{array} \right]$$

which, after substitution for $\{X_0 : X_1 : X_2 : X_3\}$, simplification and dividing through by 4, becomes

$$[A_{ij}] = \begin{bmatrix} X_0^2 + X_3^2 & -2(X_0X_2 - X_1X_3) & 2(X_0X_1 + X_2X_3) \\ 0 & X_0^2 - X_3^2 & -2X_0X_3 \\ 0 & 2X_0X_3 & X_0^2 - X_3^2 \end{bmatrix}$$

This is easily inverted and after multiplying by $(X_0^2 + X_3^2)^2$

$$[A_{ij}]^{-1} = \begin{bmatrix} X_0^2 + X_3^2 & 2(X_0X_2 + X_1X_3) & -2(X_0X_1 - X_2X_3) \\ 0 & X_0^2 - X_3^2 & 2X_0X_3 \\ 0 & -2X_0X_3 & X_0^2 - X_3^2 \end{bmatrix}$$

Then the metric

$$[A_{ij}][A_{ij}]^{-1} = [a_{ij}][a_{ij}]^{-1}$$

It is noted with satisfaction that $[A_{ij}] = [[a_{ij}]^{-1}]^T$.

5 Examples

Examining Fig. 1.
One sees a coordinate frame pair with

$$a = 8, \;\; b = 12, \;\; \cos\frac{\phi}{2} = \sqrt{0.9}, \;\; \sin\frac{\phi}{2} = \sqrt{0.1}$$

Substituting these in the definitions of the image space coordinates and dividing by $2\sqrt{0.1}$

$$\{X_0 : X_1 : X_2 : X_3\} = \{3 : -14 : 18 : 1\}$$

Then for the point P

$$[a_{ij}] = \begin{bmatrix} 3^2 + 1^2 & 0 & 0 \\ 2(3 \times 18 - 14 \times 1) & 3^2 - 1^2 & -2 \times 3 \times 1 \\ -2(-3 \times 14 - 18 \times 1) & 2 \times 3 \times 1 & 3^2 - 1^2 \end{bmatrix}$$

Calculating $[a_{ij}]$ and dividing by 2 and using the point P in frame EE

$$\begin{bmatrix} 1 \\ 13.6 \\ 26.2 \end{bmatrix} = \begin{bmatrix} 5 \\ 68 \\ 131 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 40 & 4 & -3 \\ 60 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \\ 8 \end{bmatrix}$$

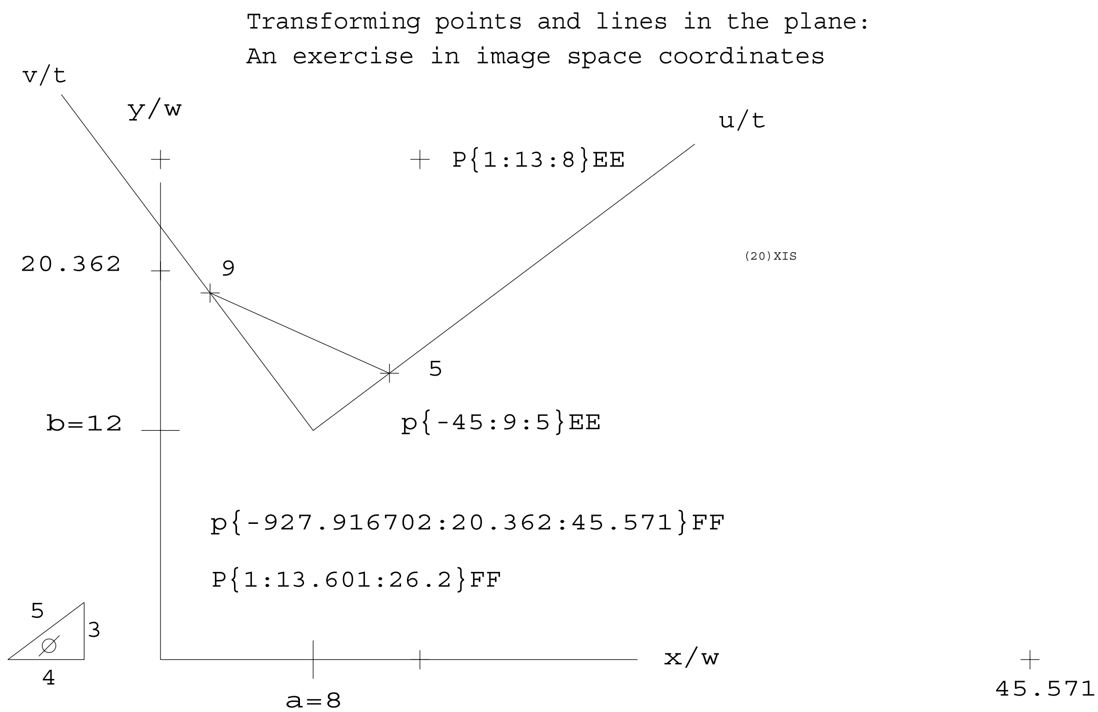

Figure 1: Frames and Elements

Then for the line p

$$[A_{ij}] = \begin{bmatrix} 3^2+1^2 & -2(3\times 18+14\times 1) & 2(-3\times 14+18\times 1) \\ 0 & 3^2-1^2 & -2\times 3\times 1 \\ 0 & 2\times 3\times 1 & 3^2-1^2 \end{bmatrix}$$

Calculating $[A_{ij}]$ and dividing by 2 and using the line p in frame EE

$$\begin{bmatrix} -957 \\ 21 \\ 47 \end{bmatrix} = \begin{bmatrix} 5 & -68 & -24 \\ 0 & 4 & -3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} -45 \\ 9 \\ 5 \end{bmatrix}$$

6 Pole, Half-Angle and Image Space Coordinates

The homogeneous image space coordinates of planar kinematic mapping may be derived using the following parameters which descibe planar displacement of a rigid body. Refer to Fig. 2.

- Cartesian coordinates of a reference point, say, the origin $(0,0)$ in FF which is $(0,0)$ in EE and becomes (a,b) under displacement in FF and
- The angle ϕ between any line in FF and its image in EE after displacement

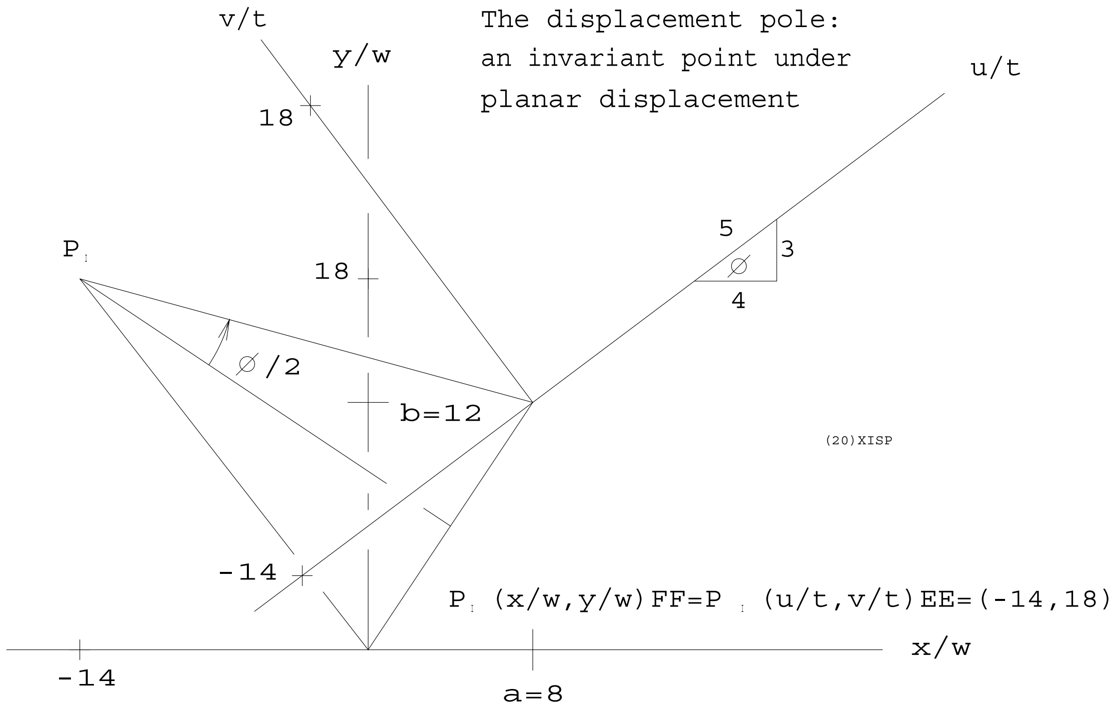

Figure 2: Displacement Pole

Notice that the Cartesian coordinates of a point P_I, the displacement pole, may be expressed in terms of a, b, ϕ and these coordinates are identical in FF and EE.

$$P_I(x/w, y/w) = P_I(u/t, v/t) = \left(\frac{a}{2} - \frac{b}{2}\cot\frac{\phi}{2}, \frac{a}{2}\cot\frac{\phi}{2} + \frac{b}{2}\right)$$

The homogeneous coordinates of P_I in FF are $\{w : x : y\}$ and in EE they are $\{t : u : v\}$. In terms of a, b, ϕ the Cartesian coordinates of P_I may be homogenized as

$$\left\{\frac{a}{2} - \frac{b}{2}\cot\frac{\phi}{2} : \frac{a}{2}\cot\frac{\phi}{2} + \frac{b}{2} : 1\right\} \equiv$$

It does not matter that the ordering has been circular left shifted as $\{u : v : t\}$ and $\{x : y : w\}$ in the representation above. This reordering has been done to minimize subsequent sequence shuffling below while proceeding to the ultimate goal to present the homogeneous planar image space coordinates, ordered as they were initially introduced. But these are just the coordinates of an ordinary *point*. A point in the Cartesian kinematic image space, which must have three coordinates to represent the three degrees of freedom of displacement in the plane, is obtained by dividing by $\cot\frac{\phi}{2}$.

$$\left(\frac{a}{2}\tan\frac{\phi}{2} - \frac{b}{2}, \frac{a}{2} + \frac{b}{2}\tan\frac{\phi}{2}, \tan\frac{\phi}{2}\right)$$

To make this a projective 3-space, four homogeneous coordinates are required

$$\left\{1 : \frac{a}{2}\tan\frac{\phi}{2} - \frac{b}{2} : \frac{a}{2} + \frac{b}{2}\tan\frac{\phi}{2} : \tan\frac{\phi}{2}\right\}$$

Multiplying by $2\cos\phi$ produces the image space coordinates

$$\{X_0 : X_1 : X_2 : X_3\} = \left\{2\cos\frac{\phi}{2} : a\sin\frac{\phi}{2} - b\cos\frac{\phi}{2} : a\cos\frac{\phi}{2} + b\sin\frac{\phi}{2} : 2\sin\frac{\phi}{2}\right\}$$

One may conduct the following simple verification upon the example shown in Fig. 2.

$$a = 8, \quad b = 12, \quad \cos\frac{\phi}{2} = \sqrt{0.9}, \quad \sin\frac{\phi}{2} = \sqrt{0.1}, \quad \cot\frac{\phi}{2} = 3$$

$$\left(\frac{a}{2} - \frac{b}{2}\cot\frac{\phi}{2}, \frac{a}{2}\cot\frac{\phi}{2} + \frac{b}{2}\right) = \left(\frac{8}{2} - \frac{12}{2}\times 3, \frac{8}{2}\times 3 + \frac{12}{2}\right) = (-14, 18)$$

7 Conclusion

Recently planar kinematic mapping has been applied with very encouraging results to

- A unified approach to solving, in compact form, the direct kinematics of all possible varieties of three-legged parallel platforms, [1].
- Solving the five precision pose design problem, [2], using a general algorithm which, when formulated in the projective image space, will reveal the mechanism, whether two-jointed dyad, revolute four-bar, slider-crank or even elliptical trammel, without resort to separate formulation as documented in [3].

The purpose of this short article is to describe, once again, the nature of the projective image space; possibly in a more simple way, more palatable to a wider audience of engineering kinematicians who are yet reluctant to adapt these methods to their own research and teaching. To those who feel that planar kinematics is a well worked over field with little new to offer, it is submitted that a similar rework of spherical and spatial mapping will help us to effectively attack more challenging problems.

References

[1] Zsombor-Murray, P.J., Chen, C. and Hayes, M.J.D. (2002) "Direct Planar Kinematic Mapping for General Planar Parallel Manipulators", *Proc. CSME Forum 2002*, CD ISBN 09730900-0-4, Kingston, 02-06, 8pp.

[2] Hayes, M.J.D. and Zsombor-Murray, P.J. (2002) "Solving the Burmester Problem Using Kinematic Mapping", *Proc. ASME-DETC2002*, MECH-34378, Montréal, 02-09, 8pp.

[3] McCarthy (2000) *Geometric Design of Linkages*, ISBN 0-387-98983-8, Ch.5, pp.86-115.

[4] Bottema, O. and Roth, B. (1979) *Theoretical Kinematics*, Dover, ISBN 0-486-66346-9, pp.393-445.