MECH 314
Dynamics of Mechanisms

April 16, 2011

Planetary Gear Train Velocities

1 Planetary Gears and Pitch Point Gear Velocities

Figure 1: Internal Ring-Gear, Three Planets on Spider-Arm and the Sun Gear as Semi-Pictorial

Fig. 1 shows a planetary train composed of spur-gears. It may be regarded as a set of four rigid bodies –because two of the three planet gears are kinematically redundant– that transmit motion to one another via rolling-without-slipping joints, represented by mating gear teeth, and revolute, R-joints between planets and arm, as well as, possibly, elsewhere to support shafts rigidly connected to ring or arm. One configuration of the train might be to have the the ring immobile as the fixed frame FF. This case produces a schematic like that shown in a two view schematic, Fig. 2. Notwithstanding CGK, wherein four rigid bodies are counted, and five 1dof joints are counted, *i.e.*, the arm R-bearing, the sun shaft R-bearing, the planet-ring RWS interface, the planet-arm R-bearing and the sun-planet RWS interface, that yields, for planar motion,

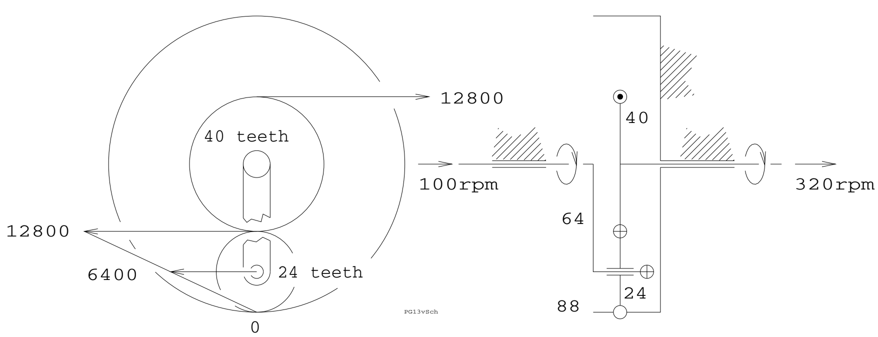

Figure 2: Stationary Internal Ring-Gear, Single Planets on Arm and the Sun Gear in Schematic

$$3(4-1) - 2 \times 5 = -1$$

this is a 1dof mechanism. Single-view schematics, showing only the central elevation section view on the right will be used later because it will be assumed that the reader can visualize the pitch point velocity distributions, shown in the plan-view on the left. Here two views are used to help in making the transition from a picture of the actual machinery, like Fig. 1, to the essential section with pitch line velocity cues, *i.e.*, $\bigcirc$, $\bigoplus$, $\odot$ that represent a stationary, approaching and receding point on the plane of the section, respectively. In this example we are given the number of gear teeth as 40 for the sun and 24 for the planet(s). Input angular velocity of the arm shaft is given as 100rpm. This is sufficient information to deduce all pitch point "velocities" (or at least numbers in proportion to these magnitudes), other radii (in terms of tooth numbers) and angular velocities. The arm radius is $40 + 24 = 64$teeth. The internally toothed ring gear radius is $40+24+24 = 88$teeth. All points on the surface of the cylinder embedded in the rigid body of the arm at a radius of 64 are moving so as to recede on the point shown at a pseudo-velocity of $100 \times 64 = 6400$. Since the point on the planet at radius 88 is stationary, the point at radius 40 is receding at twice this speed, *i.e.*, 12800. Imagine a wheel rolling on a surface. If the point of contact at the bottom is stationary, the point at the top is moving at twice the speed of the axle centre and in the same direction. As an exercise on the side, map a few point velocities on planet in the plan view to show that these do not violate the rigidity criterion. Since the bottom of the sun is also receding at 12800, its top is approaching at the same speed while its shaft centreline is stationary. To get the angular speed of the sun shaft one merely reverses to multiplication process whereby we arrived at a pitch point speed of 12800 and divide this by the radius of the sun gear to get $12800/40 = 320$rpm in the same sense as the input. To master this little problem, satisfy yourself the the 24tooth gear is rotating at $-366\frac{2}{3}$rpm relative to the arm.

2 Problem 1, 08-04-24 & 09-12-17

Consider the planetary gear train shown in Fig. 3. It appeared twice on a final exam without so much as a single change in specification.

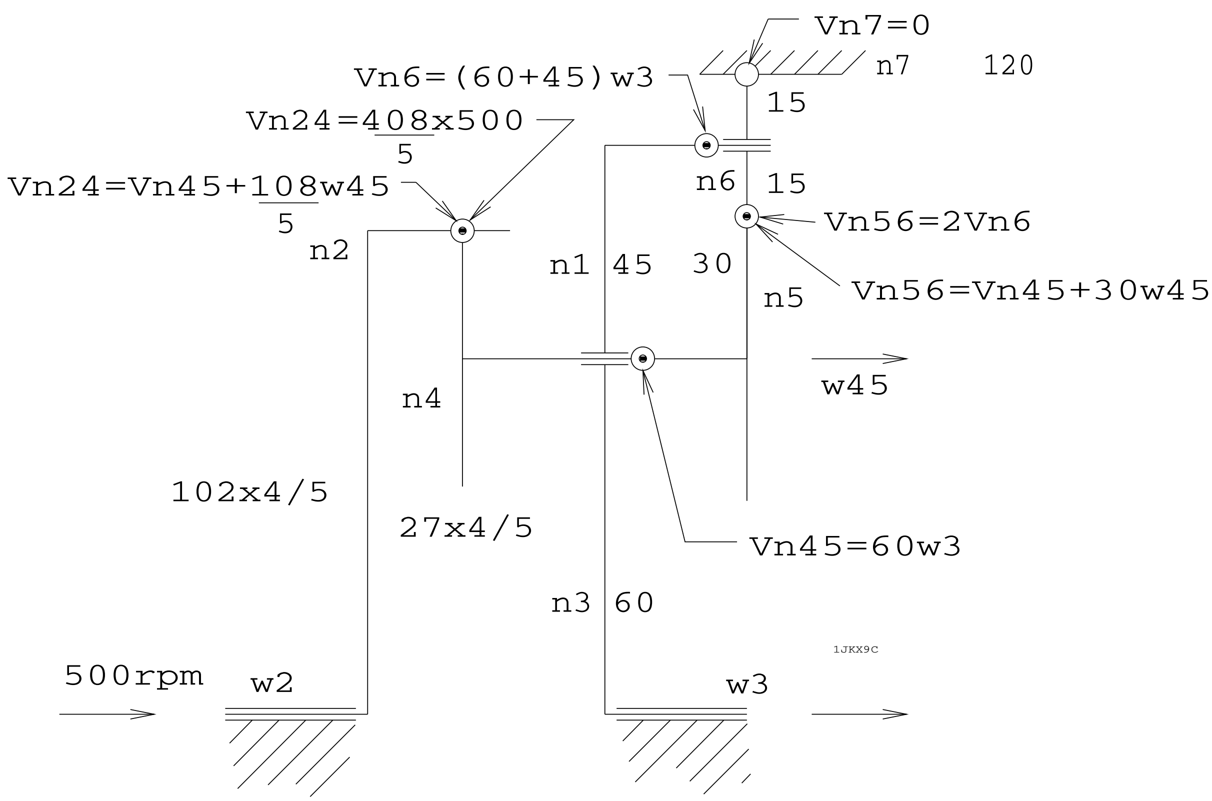

Figure 3: A Planetary Gear Train

2.1 Description

- This is a five-rigid-body system. To decode the labels in Fig. 3 with respect to the subsequent text, consider the following.

  1. `ni`$=N_i$ are the individual gears and their radius expressed in numbers of teeth.
  2. `Vnij` are expressions of pitch point velocity between meshing gears. *E.g.*, `Vn24` is the velocity of meshing gear teeth on `n2`$=N_2$ and `n4`$=N_4$. The circles with the centred dot mean a velocity vector arrow "coming at you". `Vn7` is stationary hence no dot.
  3. `w45`$=\omega_{45}$ the angular speed, in rpm, of gears `n4`$= N_4$ and `n5`$= N_5$, solidly mounted on a common shaft, with respect to the arm represented as [`n1`$=N_1$]+[`n3`$=N_3$].
  4. `w2`$= \omega_2$ and `w3`$= \omega_3$ are the respective angular speeds of input and output shafts.

- Input, on the left, is an internally toothed gear with $N_2 = 102$ teeth rotating clockwise, when viewed from the left, at $\omega_2 = 500$rpm.

- Output, on the right, is a rotating arm supported on a shaft coaxial with N_2 and rotating at ω_3, to be determined.

- Part way up this arm is a shaft on which two gears are fixed. The shaft and gears are in a bearing and free to rotate with respect to the arm. The left gear with $N_4 = 27$ teeth meshes with N_2. The gear on the right is $N_5 = 30$ teeth.

- At the top of the arm and free to rotate on it is a gear $N_6 = 15$ teeth.

- The fifth body is stationary. It is an internal gear $N_7 = 120$ teeth. It meshes with N_6 which in turn meshes with N_5 below it.

- The second question asks you to find ω_{61} the angular speed of N_6 with respect to the arm.

2.2 Compatibility

- There are two aspects of compatibility to be considered.

- For shafts ω_2 and ω_3 to be coaxial

$$k(N_2 - N_4) - (N_7 - 2N_6 - N_5) = k(102 - 27) - (120 - 2 \times 15 - 30) = 0$$

  Therefore the gear pair N_2, N_4 on the left cannot be of the same diametral pitch as the triplet N_7, N_6, N_5 on the right, *i.e.*, $k \neq 1$. To use the proposed *pitch point velocity method*, that uses tooth number as gear pitch radius, a factor $k = 4/5$ is introduced to satisfy the equation.

- For shafts ω_2 and ω_3 to be coaxial the arm radius must be assigned two radial increments. N_1 is the lower portion from shaft ω_3 to the shaft fixed to N_4 and N_5 and rotating at ω_{45} with respect to the arm. N_3 is the upper extending from shaft ω_{45} to the shaft of N_6. N_1 and N_3 are also measured in "numbers or teeth". These considerations produce the following results for N_1 and N_3.

$$k(N_2 - N_4) - (N_7 - N_6 - N_1) = 0, \ \ (N_1 + N_3) - (N_7 - N_6) = 0$$

  so $N_1 = 45$ teeth and $N_3 = 60$ teeth.

2.3 Output Shaft Angular Speed

Looking at the two expressions for `Vn24` in Fig. 3 one may write

$$N_3\omega_3 + kN_4\omega_{45} - kN_2\omega_2 = 0$$

Similarly for `Vn56`

$$2(N_1 + N_3)\omega_3 - N_3\omega_3 - N_5\omega_{45} = 0$$

With given values

$$k = \frac{4}{5},\ N_1 = 45,\ N_2 = 102,\ N_3 = 60,\ N_4 = 27, \omega_2 = 500$$

the unknowns are calculated as

$$\omega_3 = \frac{1224000}{5040} \backsimeq \mathbf{242.9}\text{rpm},\ \omega_{45} = \frac{6120000}{5040} \backsimeq 1214.3\text{rpm}$$

2.4 Relative Angular Speed ω_{61} of Gear N_6 on Arm $N_1 + N_3$

To get ω_{61}, the answer to the second question, compute the absolute angular speed of N_6, *i.e.*, ω_6 by finding the midpoint velocity `Vn6` of N_6 and noting that that its instant centre is a stationary point on N_7 and then subtracting ω_3 which is now known. `n6`$= N_6$ is taken as negative downward while $N_1 + N_3$ is upward.

$$\omega_{61} = \frac{(N_1 + N_3)\omega_3}{(-N_6)} - \omega_3$$

With given values

$$N_1 = 45,\ N_3 = 60,\ N_6 = 15,\ \omega_3 = 242.9$$

one obtains $\omega_{61} = -\mathbf{1942.9}$rpm.

3 Humpage's Reduction Gear

On p.395 of [1] in Fig. 9.11 there appears a planetary gear train. This is shown in Fig. 4.

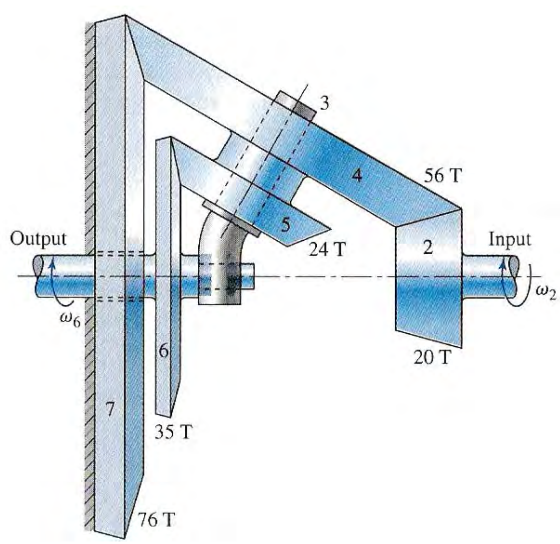

Figure 4: Semi-Pictorial of Humpage's Reduction Gear

Below in Fig. 5 it is reproduced as a schematic using the same notation as was introduced in the previous problem. Furthermore the system was constructed to scale using the two radii $2N_4$ and $2N_5$ and the coaxial offset distance N_2. Here the number of teeth represent the gear pitch radii properly with $k = 1$. Again, the pitch point velocity method will be used. The circles with crosses in them represent vector arrows going away so as to be compatible with ω_2 assumed to be given and in the sense shown in the text book. Note that N_7 is fixed so V7$= 0$. The problem will be to find ω_6. Using the method of pitch point velocities one may write the equations for V4, V6 and ω_6.

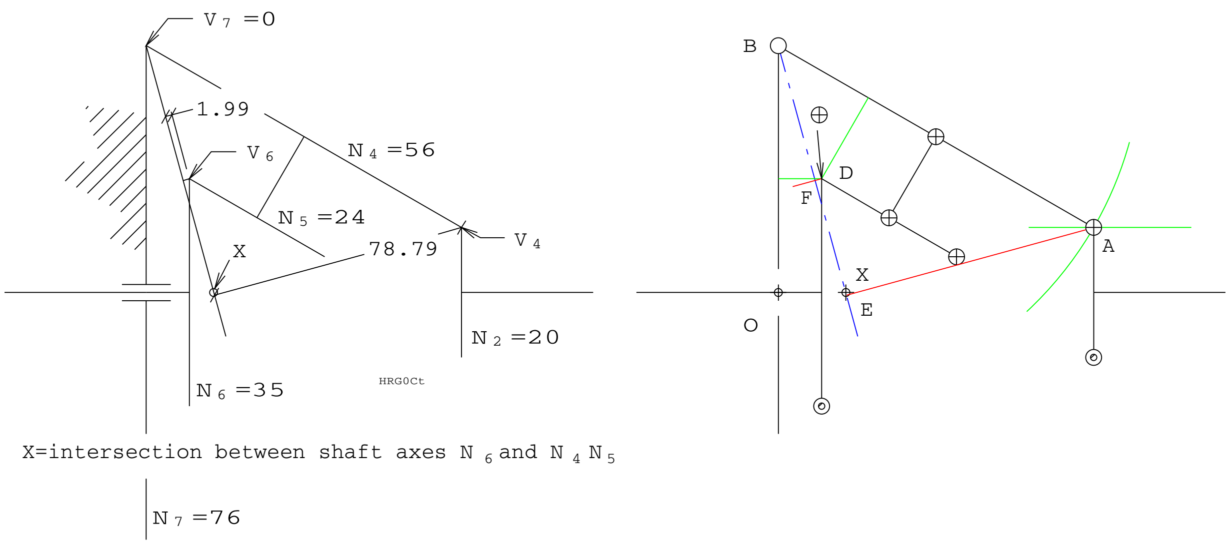

Figure 5: Humpage's Planetary Gear Train

$$\text{V4} = N_2\omega_2,\ \text{V6} = \frac{|\mathbf{r}_{FD}|}{|\mathbf{r}_{EA}|}\text{V4},\ \omega_6 = \frac{\text{V6}}{N_6}$$

Since points B and X on the rigid body of connected gears N_5N_6 are stationary at this instant the line BX is the instantaneous axis or rotation of the body and its angular velocity vector is $\boldsymbol{\omega} = k_\omega \mathbf{r}_{BX}$. By taking gear radii as their respective tooth numbers one may easily determine that $\angle OBA = 60°$. Then coordinates of points B and A can be determined as $B(x_B, y_B) = (0, 76)$ and $A(x_A, y_A) = (56\sqrt{3}, 20)$. Similarly, D is on a line normal to AB on a point $N_4 - N_5 = 32$ from B so $D(x_D, y_D) = (23\sqrt{3}/3, 35)$. Since X is on the intersection of a normal to AB, half-way along, and the line $y = 0$, $X(x_X, y_X) = (12\sqrt{3}, 0)$. Note that $\mathbf{r}_{BX} = [(x_X - x_B)\ (y_X - y_B)\ 0]^\top$. Points E and F can be found on lines normal to BX and on A and D, respectively. The two pairs of equations of lines that intersect on E and F are

$$\begin{aligned}
x_By_X - x_Xy_B + (y_B - y_X)x_E + (x_X - x_B)y_E &= 0\\
-(x_X - x_B)x_A - (y_X - y_B)y_A + x_X - x_B)x_E + (y_X - y_B)y_E &= 0\\
x_By_X - x_Xy_B + (y_B - y_X)x_F + (x_X - x_B)y_F &= 0\\
-(x_X - x_B)x_D - (y_X - y_B)y_D + x_X - x_B)x_F + (y_X - y_B)y_F &= 0
\end{aligned}$$

Therefore

$$E(x_E, y_E) = \left(\frac{1176\sqrt{3}}{97}, -\frac{76}{97}\right),\quad F(x_F, y_F) = \left(\frac{636\sqrt{3}}{97}, \frac{3344}{97}\right)$$

- First we find $\boldsymbol{\omega}$ by finding the scalar k_ω with

$$\boldsymbol{\omega} \times \mathbf{r}_{XA} = k_\omega \mathbf{r}_{BX} \times \mathbf{r}_{XA} = \text{V4} = N_2\omega_2$$

where $\mathbf{r}_{XA} = [(x_A - x_X)\ (y_A - y_X)\ 0]^\top$.

- Then we find V6 as

$$\mathtt{V6} = \boldsymbol{\omega} \times \mathbf{r}_{FD} = \boldsymbol{\omega} \times \begin{bmatrix} x_D - x_F \\ y_D - y_F \\ 0 \end{bmatrix}$$

- Finally, with $\omega_2 = 2000$rpm, $N_2 = 20$ and $N_6 = 35$, $\omega_6 = 4250/147 \simeq 28.91156$rpm.
- Now one may appreciate why rote methods, like that used in the text to solve this very problem, were developed. After examining Fig. 5 don't you think plotting the gear train section to scale and simply measuring the perpendicular distances EA and FD directly is the clearest way to understand what is going on? The fact that EA and FD extend on the same (right) side of BX immediately confirms that ω_2 and ω_6 trurn in the same sense.

4 Ferguson's Paradox

On p.407 of [1] in Fig. 9.16 there appears a planetary gear train. This is shown in Fig. 6.

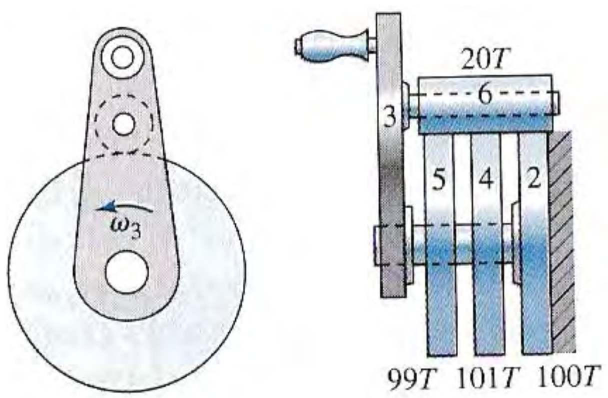

Figure 6: Semi-Pictorial of "Ferguson's Paradox" Planetary Gear Set

Below in Fig. 7 it is reproduced as a schematic using the same notation as was introduced initially. The problem is to find ω_4 and ω_5 in terms of the given crank arm input angular speed ω_3. The paradox arises from the observation that ω_4 turns slowly, with respect to ω_3, in the same direction while ω_5 turns slowly in the opposite sense. Once more pitch point velocities and diametral pitch factors like k are used to explain and produce a solution. To maintain the axis on N_6 parallel with that on the gear set N_2, N_4, N_5 we impose a metric based on N_2, N_6. This gives

$$N_3 = N_2 + N_6 = k'(N_4 + N_{6'}) = k''(N_5 + N_{6''}) = 120,\ k' = \frac{120}{121},\ k'' = \frac{120}{119}$$

The velocities V_3 and V_6 can be computed by observing that $V_2 = 0$ at the instant centre on the stationary gear N_2.

$$V_3 = (N_2 + N_6)\omega_3 = N_3\omega_3,\ V_6 = (N_2 + 2N_6)\omega_3$$

Due to the adjustment in "virtual diametral pitch" the following pitch point velocities and angular speed can be obtained.

$$V_{6'} = \frac{N_6 - k'N_{6'}}{N_6}V_3,\ \omega_4 = \frac{V_{6'}}{k'N_4},\ V_{6''} = \frac{N_6 - k''N_{6''}}{N_6}V_3,\ \omega_5 = \frac{V_{6''}}{k''N_5}$$

Entering the numerical values

$$N_4 = 101,\ N_5 = 99,\ N_6 = N_{6'} = N_{6''} = 20,\ k' = \frac{120}{121},\ k'' = \frac{120}{119}$$

produces the two angular speeds.

$$\omega_4 = \frac{N_6 - k'N_{6'}}{k'N_4N_6}(N_2 + N_6)\omega_3 = \frac{+\omega_3}{101},\ \omega_5 = \frac{N_6 - k''N_{6''}}{k''N_5N_6}(N_2 + N_6)\omega_3 = \frac{-\omega_3}{99}$$

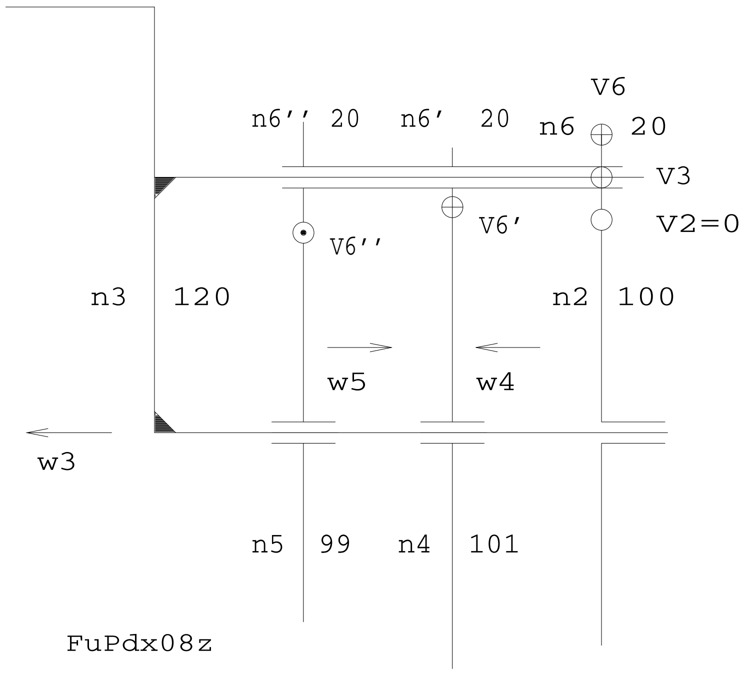

Figure 7: A High Reduction Ratio Dual Differential

Fig. 7 is drawn to scale except for the radii of gears $N_4, N_5, N_{6'}, N_{6''}$. Changes in these radii are exaggerated to illustrate why $V_{6'}$ is receding from the observer while $V_{6''}$ approaches. Compare the positions of these pitch point velocities to the stationary instant centre V_2.

5 A Non-Trivial Compound Gear Train

Fig. 8 shows a compound reduction gear to handle the minute and hour reductions from the second shaft of a pendulum clock. It is problem 9.20 on p.417 of [1]. The train illustrated in the problem statement of Fig. 8 is

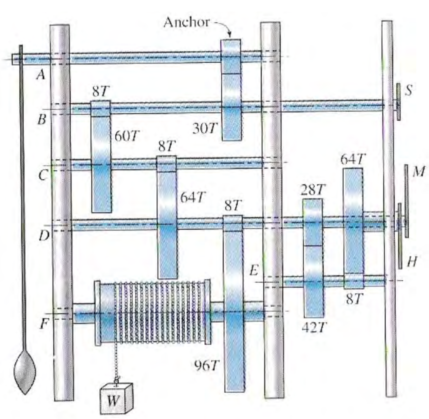

Figure 8: Speed reduction Gear Train for a Clock

reproduced below as Fig. 9 using the pitch point velocity convention adopted for planetary gear analysis. It may be simpler to follow here, in a slightly more complicated situation than that introduced in Fig. 2 but without the additional concept of bodies rotating upon bodies. The answers sought in this problem were

- How many teeth must the upper (anchor) gear have if the pendulum releases one tooth every two seconds?
- What is the rotational speed of the drum on which the cord, supporting the weight W, is wound?
- Show that the minute and hour hands rotate at the angular speeds and sense required.

In addition opportunity is taken to explain the adopted concept of *pseudo-tooth number* to maintain dimensional compatibility when using tooth numbers as pitch radii.

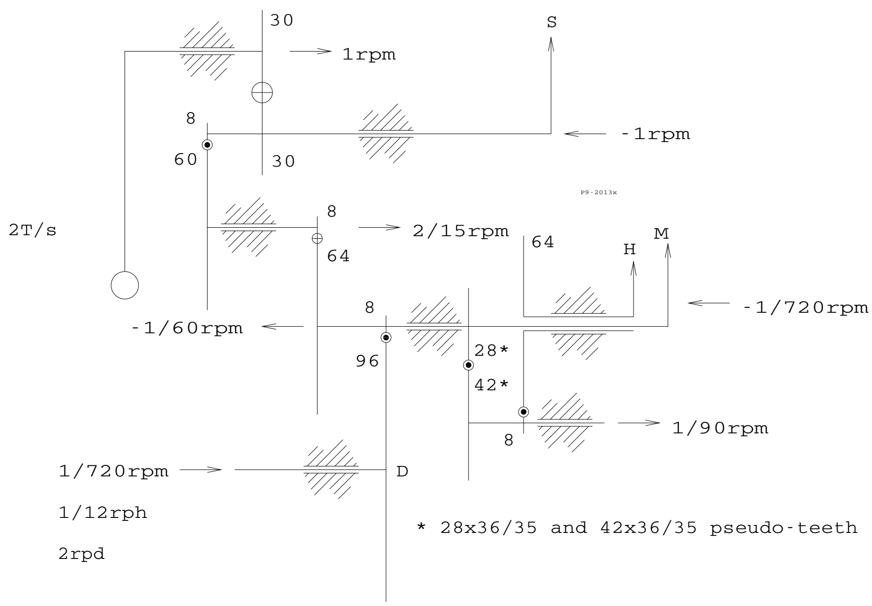

Figure 9: Schematic of Gear Train for a Pendulum Clock

6 Three-Speed Bicycle Hub

You figure out how the mechanism shown in Fig. 10 works. By zooming in one can see reasonably well how the various components fit together and operate. There are quite a few, *e.g.*, the pawls and ratchet to implement the free-wheel feature one expects to find on all, except "sprint" bicycles, and all these make it more difficult for those unexperienced to deduce the principle of operation of the gear changer itself which is the whole point in presenting this illustration.

Figure 10: A Sturmey-Archer Hub

References

[1] J.J. Uicker, Jr., G.R. Pennock and J.E. Shigley (2011) *Theory of Mechanisms and Machines, 4th ed.*, Oxford, ISBN 9780-19-537123-9.