Distance from Conic to Point, Plane or Line

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Abstract. To avoid close proximity between building envelope and a nearby power line, geometric methods to compute normal distances from spatial point, plane and line to a parabola, approximating the catenary, are developed using projection onto ideal planes. Then line geometry is applied for the first time to reveal a unified approach.

Key words: stationary distance, conics, transformation, line geometry

1 Introduction

"Flat" catenaries can be satisfactorily approximated by parabolæ specified by two given supporting points P,Q and only the height t_2 of the lowest point or vertex T of the curve. First the problem will be reduced to a planar model by taking the

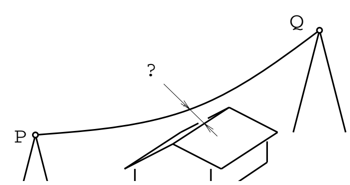

Fig. 1 Minimum Distance from Ridge Pole to Power Line

point or line (expressed in point view) on the origin of a Cartesian frame while the parabola –or its subsequent projection on the plane taken normal to the line– initially in standard form is translated, after possible rotation, to assume its required position relative to the origin. Planar point or line –The line may represent an edge or line view of an intruding plane.– to parabola distances are computed using products of homogeneous planar point or line coordinate vectors and matrices representing conic coefficients, rotation, translation and orthogonal projection. Line geometry is introduced. All solution cases are treated as a product of a 5×6 homogeneous matrix of constraint equation coefficients and a Plücker coordinate vector.

1.1 Origin to Translated Parabola

The missing coefficient a and coordinate t_1 are computed given $P(p_1, p_2), Q(q_1, q_2)$ and t_2 of $T(t_1, t_2)$. The solution with the closest point on the left appears in Fig. 2. Constraint equations, $\left(\mathbf{p}^\top, \mathbf{q}^\top\right) \mathbf{T}\mathbf{M}_{sp}\mathbf{T}^\top (\mathbf{p}, \mathbf{q}) = 0$, Eqs. 1, are set up as follows.

$$\begin{bmatrix} 1 & p,q_1 & p,q_2 \end{bmatrix} \begin{bmatrix} 1 & -t_1 & -t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & a & 0 \\ -1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -t_1 & 1 & 0 \\ -t_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ p,q_1 \\ p,q_2 \end{bmatrix}$$

$$= a(p_1 - t_1)^2 + t_2 - p_2 = 0, \; a(q_1 - t_1)^2 + t_2 - q_2 = 0 \tag{1}$$

Eliminating a produces, Eq. 2, a quadratic in t_1, its values being given for the numerical example shown in Fig. 2.

$$(p_2 - q_2)t_1^2 + 2[(q_2 - t_2)p_1 + (t_2 - p_2)q_1]t_1 + (t_2 - q_2)p_1^2 + (p_2 - t_2)q_1^2 = 0 \tag{2}$$

The parabola size coefficient a is obtained with either one of Eq. 1, linear in a, having chosen the positive root of t_1. The other root places P, Q both in the right hand branch of the parabola. Eq. 3 is the equation of the displaced parabola, like either of Eq. 1 but in terms of an arbitrary point (x, y).

$$t_2 - y + a(x - t_1)^2 = 0 \tag{3}$$

Forming the squared distance from origin to parabola, $x^2 + y^2$, and taking the derivative set to zero of this with respect to x, having eliminated y with Eq. 3, yields a cubic, Eq. 4, in x-coordinates of stationary points.

$$2a^2x^3 - 6a^2t_1x^2 + [2a(at_1^2 + t_2) + 1]x - 2at_1(at_1^2 + t_2) = 0 \tag{4}$$

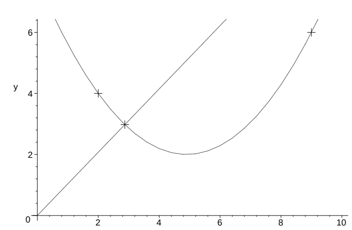

Fig. 2 A Numerical Example, with $P(2,4)$, $Q(9,6)$, $T(-5 + 7\sqrt{2}, 2)$, Closest Point at $(2.8722, 2.9777)$ with $a = 2/[49(3 - 2\sqrt{2})]$

1.2 Distance between Parabola and Line

The same numerical parameters will be used together with the line given by Eq. 5.

$$W + Xx + Yy = 0 \tag{5}$$

One finds a tangent line on the parabola that is parallel to the given one and measures the length of the normal line segment on the point of tangency that spans to the given line as shown in Fig. 3. The tangent line and the normal line are given by Eqs. 6.

$$U + Xx + Yy = 0 \text{ and } V - Yx + Xy = 0 \tag{6}$$

Coincident point on the parabola and tangent line is expressed by Eq. 7.

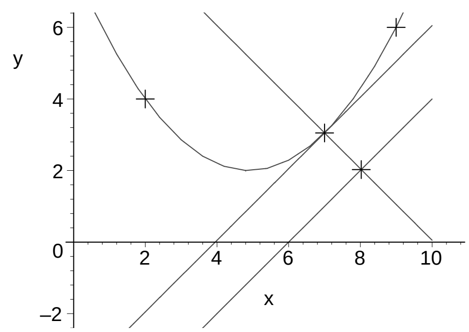

Fig. 3 Minimum Distance from Sloping Roof Plane, Line $60 - 10x + 10y = 0$ to Power Line, Eq. 4. The Length of the Segment, Given the Original Parabola Values, Is 1.4993.

$$\begin{bmatrix} t_2 + at_1^2 & -at_1 & -1/2 \\ -at_1 & a & 0 \\ -1/2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 2a \\ 0 & -1 & 2at_1 \\ 2a & 2at_1 & 4at_2 \end{bmatrix} \begin{bmatrix} U \\ X \\ Y \end{bmatrix} = \begin{bmatrix} 2aY \\ 2at_1Y - X \\ 2aU + 2at_1X + 4at_2Y \end{bmatrix} \tag{7}$$

Substituting the dehomogenized point vector above, the last term in Eq. 7, into the parabola equation, either of Eq. 1 with P or Q taken as point variables, provides a linear equation in U. This symbolic value of U completely defines the tangent line, Eq. 8.

$$X^2 - 4aY(t_1X + t_2Y) + 4aY(Xx + Yy) = 0 \tag{8}$$

Similarly by substituting the point coordinates into the normal line equation the value of V is determined to yield the equation of the normal line, Eq. 9.

$$2at_1(Y^2 - X^2) - 2aX(U + 2t_2Y) - XY + 2aY(Xy - Yx) = 0 \tag{9}$$

Intersecting the normal line and the edge view of the roof plane yields

$$[V \ -Y \ X]^\top \times [W \ X \ Y]^\top = Q\{q_0 : q_1 : q_2\}$$

the other end point of the line segment between given and tangent line.

1.3 Orthogonal Projection

Because the plane of the catenary is not generally normal to the ridge line an image of the parabola, projected orthogonally on to a plane normal to the ridge line or building wall or roof, is usually required. Operations proceed as follows.

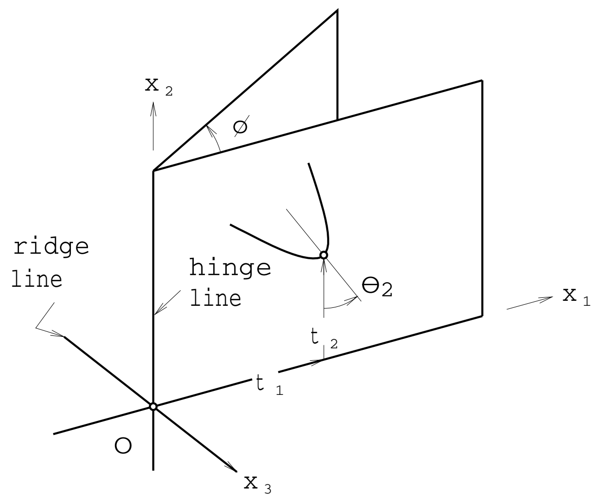

Fig. 4 Roof Ridge and Hinge Lines,

- The standard form parabola coefficient matrix $\mathbf{M}_{sp}$, the rotation operator $\mathbf{R}_\theta$, the translation operator $\mathbf{T}$ and the orthogonal projection operator $\mathbf{M}_o$ are constructed.

$$\mathbf{M}_{sp} = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -a & 0 \\ 1/2 & 0 & 0 \end{bmatrix}, \ \mathbf{R}_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & -t_1 & -t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{M}_o = \begin{bmatrix} \cos\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\phi \end{bmatrix}$$

- Forming the product $\mathbf{M}_o\mathbf{T}\mathbf{R}_\theta\mathbf{M}_{sp}\mathbf{R}_\theta^\top\mathbf{T}^\top\mathbf{M}_o^\top$ produces the coefficient matrix $\mathbf{M}_{op}$ of the rotated, translated parabola rotated by ϕ out of the fixed plane and projected orthogonally onto it. Let $\mathbf{M}_{op}$ and the orthogonally projected parabola image be represented as

$$\mathbf{M}_{op} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \to a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0$$

where

$$a_{00} = \cos^2\phi\,\left[t_1\sin\theta - t_2\cos\theta - a(t_1\cos\theta + t_2\sin\theta)^2\right]$$
$$a_{01} = \cos\phi\,[2a\cos\theta(t_1\cos\theta + t_2\sin\theta) - \sin\theta]/2$$
$$a_{02} = \cos^2\phi\,[2a\cos\theta(t_1\sin\theta + t_2\sin\theta) + \cos\theta + 2at_2]/2$$
$$a_{11} = -\cos^2\theta,\ \ a_{12} = -(a\cos\phi\cos\sin\theta)/2,\ \ a_{22} = -a\cos^2\phi\sin^2\theta$$

- Fig. 5 shows the standard form parabola, its rotated and translated image and its projection from its natural frame orthogonally onto the fixed frame normal to the ridge line. The narrower image with its vertex farther to the left is the required projection.

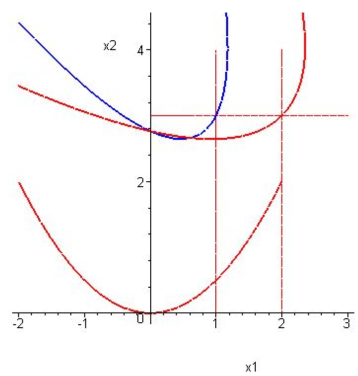

Fig. 5 Image of Three Parabolæ on Plane $x_3 = 0,\ a = 1/2,\ t_1 = 2,\ t_2 = 3,\ \theta = \pi/4,\ \phi = \pi/3$

2 Line Congruence Pairs

Confining analysis to a planar model has its drawbacks. *E.g*, cases where the ridge line is parallel to the plane of the parabola cannot be accommodated. Imagine an approach that seeks to find all lines common to two line *congruences*. Without loss in generality one contains all lines normal to tangents on a standard form parabola while the other contains those normal to an arbitrary axial line that represents the ridge. Alternately one may choose the second congruence to be all lines normal to a given plane so as to represent a building wall or roof surface or the tip of a lightening rod represented by a point. All these are shown together in Fig. 6. Consider the parabola, an arbitrary point $P(p_1, p_2)$ upon it bearing a tangent line and its normal on P. Taking an auxiliary view normal to the tangent shows the

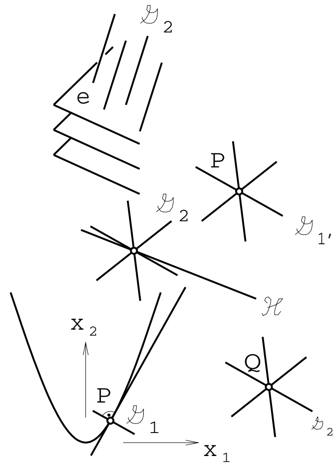

Fig. 6 Four Line Congruences

normal pencil of lines $\mathscr{G}_1'$ on P where $\mathscr{G}_1$ is the line congruence on *all* P. The line one seeks has radial Plücker coordinates $\mathscr{G}_r\{g_{01}:g_{02}:g_{03}:g_{23}:g_{31}:g_{12}\}$. $\mathscr{G}_2$ is shown thrice. $Q\in\mathscr{G}_2$ depicts the congruence on given point $Q\{1:q_1:q_2:q_3\}$. $\mathscr{G}_2\perp e$ shows parallel lines normal to all given planes $e\{E_0:E_1:E_2:E_3\}$ in a parallel pencil. This situation is equivalent to $Q\{0:q_1:q_2:q_3\}$ an absolute point where $q_1=E_1,q_2=E_2,q_3=E_3$. This example will not be detailed below because it is just a special case of a congruence on a given point. Finally $\mathscr{G}_2\perp,\cap\mathscr{H}$ where line $\mathscr{H}_a\{H_{01}:H_{02}:H_{03}:H_{23}:H_{31}:H_{12}\}$ is given by its axial coordinates.

2.1 Line Geometry

What follows makes use of elementary line geometry. Recall points $P(p_1,p_2)$ on the standard form parabola on the plane $x_3=0$.

$$p_2-ap_1^2=0 \tag{10}$$

Tangent lines on this curve have slope $dp_2/dp_1=2ap_1$. Spatial direction vectors of lines $\mathscr{G}$ in the congruence normal to tangents on the parabola can be expressed as

$$[g_{01}\ \ g_{02}\ \ g_{03}]^\top\cdot[1\ \ 2ap_1\ \ 0]^\top=0. \tag{11}$$

The parabola is taken in standard form while $Q,e,\mathscr{H}$ are given as expressed in this frame. The case where $dp_2/dp_1\to\infty$ can be safely ignored because it occurs indefinitely far up the branches of the parabola. Therefore all radial lines $\mathscr{G}\{g_{01}:$

$g_{02} : g_{03} : g_{23} : g_{31} : g_{12}\}$ must satisfy the normality condition Eq. 12,

$$g_{01} + 2ap_1 g_{02} = 0. \tag{12}$$

In addition the condition $P \in \mathscr{G}$, recalling that spatially $P\{p_0 : p_1 : p_2 : p_3\},\ \ p_0 = 1,\ \ p_3 = 0$, provides the two middle equations chosen from the doubly singular set, expressed in *axial* coordinates of $\mathscr{G}$ such that $G_{ii} = 0$, is given by Eqs. 13. In synopsis, $P \in \mathscr{G}$. If $\mathscr{G} \cap P = e$ then $\sum_{j=0}^{3} G_{ij} p_j = E_i \neq 0,\ G_{ji} = -G_{ij},\ G_{ii} = 0$. If $\mathscr{G} \cap e = P$ then $\sum_{j=0}^{3} g_{ij} P_j = p_i \neq 0$.

$$\begin{array}{c} G_{00}p_0 + G_{01}p_1 + G_{02}p_2 + G_{03}p_3 = 0,\ \ -G_{01}p_0 + G_{11}p_1 + G_{12}p_2 - G_{31}p_3 = 0 \\ -G_{02}p_0 - G_{12}p_1 + G_{22}p_2 + G_{23}p_3 = 0,\ \ -G_{03}p_0 + G_{31}p_1 - G_{23}p_2 + G_{33}p_3 = 0 \end{array} \tag{13}$$

which lead to Eqs. 14

$$-g_{23} + g_{03}p_2 = 0,\ \ -g_{31} - g_{03}p_1 = 0 \tag{14}$$

because of the term by term proportional equivalence, Eq. 15.

$$\{g_{01} : g_{02} : g_{03} : g_{23} : g_{31} : g_{12}\} \propto \{G_{23} : G_{31} : G_{12} : G_{01} : G_{02} : G_{03}\} \tag{15}$$

The orthogonality between the first and second vector element triads of line coordinates, Eq. 16, called the Plücker condition or quadric, is used to get p_1, hence p_2, to yield coordinates of point P where $\mathscr{G}$ intersects the parabola.

$$g_{01}g_{23} + g_{02}g_{31} + g_{03}g_{12} = 0 \tag{16}$$

2.2 *Parabola to Line, Point and Plane*

Lines $\mathscr{G}$ that are normal to and intersect the ridge line $\mathscr{H}$ constitute the second congruence. These provide the following two necessary constraint equations Eqs. 17.

$$H_{23}g_{01} + H_{31}g_{02} + H_{12}g_{03} = 0,\ \ H_{01}g_{01} + H_{02}g_{02} + H_{03}g_{03} + H_{23}g_{23} + H_{31}g_{31} + H_{12}g_{12} = 0 \tag{17}$$

All this can be arranged in Eq. 18, a system of five homogeneous linear equations in g_{ij}, in detached coefficients form. Recall that the first three rows represent Eqs. 12 and 14 while the last two are, respectively, normality of directions $\mathscr{H} \perp \mathscr{G}$ and intersection $\exists\ \mathscr{H} \cap \mathscr{G}$, the two equations Eqs. 17.

$$\begin{bmatrix} 1 & 2ap_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ap_1^2 & -1 & 0 & 0 \\ 0 & 0 & -p_1 & 0 & -1 & 0 \\ H_{23} & H_{31} & H_{12} & 0 & 0 & 0 \\ H_{01} & H_{02} & H_{03} & H_{23} & H_{31} & H_{12} \end{bmatrix} \begin{bmatrix} g_{01} \\ g_{02} \\ g_{03} \\ g_{23} \\ g_{31} \\ g_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

Solving homogeneously yields all six $g_{ij} = g_{ij}(p_1)$. Inserting these into the Plücker condition results in a cubic, Eq. 19, in p_1, as was Eq. 4, and a trivial solution (factor).

$$H_{12}(2aH_{23}p_1 - H_{31})(c_3p_1^3 + c_2p_1^2 + c_1p_1 + c_0) = 0 \quad (19)$$

Large coefficients $c_k(a, H_{ij})$ are omitted above. Using the first and second equations from the set Eqs. 13, with Q replacing P, to form the last two rows in the 5×6 matrix (or alternately with e replacing P) one obtains Eq. 20.

$$\begin{bmatrix} 0 & 0 & 0 & q_1, E_1 & q_2, E_2 & q_3, E_3 \\ 0 & -q_3, -E_3 & q_2, E_2 & -1, 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{01} & g_{02} & g_{03} & g_{23} & g_{31} & g_{12} \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top \quad (20)$$

Solving homogeneously yields all six $g_{ij} = g_{ij}(p_1)$. Inserting these into the Plücker condition produces a simple cubic with Q and a linear equation with e, Eqs. 21, in p_1. Both have trivial factors.

$$aq_3^3p_1^2[2a^2p_1^3 - (2aq_2 - 1)p_1 - q_1] = 0, \;\; aE_3^3p_1^2(2aE_2p_1 + E_1) = 0 \quad (21)$$

3 Conclusion

This paper grew from difficulties experienced by a colleague in Innsbruck who ran afoul of municipal authorities while building her house. Although it is unconventional to put references here, rather than at the beginning, one may see relevance to wider application by realizing that [3] was written to help in dynamic simulation of a falling chain. Klien's little book [1] covers many of the projective geometry transformations used but I could not find material describing conics (and quadrics) in terms of their (symmetric) coefficient matrices. This was acquired by osmosis from exposure to my many Austrian geometer friends. Pottmann and Wallner's line geometry text [2] treats that subject thoroughly while [4] is helpful in grasping the basics. Though I was asked to consider removing the approach and analysis connected with the first three problems I decided to retain it in the interest of illustrating the geometric thinking that leads to selection of ideal frames in problem formulation. This was carried through to conceive Fig. 6 which explains the novel application of line geometry in the unification of these types of problem.

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