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Type synthesis of primitive Schoenflies-motion generators

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ABSTRACT

A noteworthy type of motion called Schoenflies motion and often termed *X*-motion for brevity is presented. A specified set of *X*-motions is endowed with the algebraic structure of a four-dimensional (4D) Lie group. This 4D displacement Lie subgroup includes any translation and any rotation provided that the axis of rotation is parallel to a given direction. In the paper, some preliminary fundamentals about the Lie group of displacements are recalled; the 4D Lie subgroup of *X*-motion is emphasized. Then serial concatenations of one-dof Reuleaux pairs and hinged parallelograms lead to the enumeration of all possible general architectures of mechanical generators for a given *X* subgroup. Meanwhile, their corresponding embodiments are graphically displayed for a future use in the structural synthesis of parallel manipulators. These generators are sorted into four classes based on the number of prismatic pairs. In total, forty-three distinct mechanical generators of *X*-motion are revealed and eighty-two ones having at least one parallelogram are also derived from them. Some chains that are defective generators of *X*-motion are also identified through an approach based on the group dependency.

1. Introduction

In modern non-relativistic geometry, the set of geometrical points has the algebraic properties of a three-dimensional (3D) affine space. Any ordered couple of points determines a bound vector. Any equivalence class of equipollent bound vectors is called a free vector. The set of free vectors has the algebraic properties of a 3D vector space. This vector space can be endowed with the Euclidean metrics and thus constitutes an Euclidean vector space. The set of points is called 3D Euclidean affine space, while the set of vectors is 3D Euclidean vector space. Point transformations that maintain the length of any vector and the oriented angle between any couple of vectors (or orientation-preserving isometries) are usually called displacements and represent rigid-body motion. Alternatively, matrices acting on point coordinates can represent displacements in a given Cartesian frame of reference. In the general theory of matrix Lie groups, this corresponding matrix set is usually called SE(3) standing for special Euclidean group acting on coordinate arrays, which can be identified with elements of R^3. However, matrix notation provides no information on the frame that is required for defining a geometric operation. Therefore, direct geometrical reasoning is impossible by means of notations ignoring the chosen frames of reference.

The screw theory, which was introduced by Ball [1] in the nineteenth century, reflects the differential aspect of the displacement Lie group $\{\boldsymbol{D}\}$ [2–4]. It is very useful for studying the instantaneous kinematic property of mechanism. Nevertheless, the displacement Lie subgroup approach benefits by dealing with the finite motion, directly. The algebraic properties of the displacement set, which is a 6D Lie group, play a key role in the understanding of mechanism mobility [5,6]. In any given Cartesian frame of reference, $\{\boldsymbol{D}\}$ is represented by the matrix group SE(3). However, in order to allow truly geometrical

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reasoning, an intrinsic formulation of the displacement transformations has to be employed [7] and the Lie algebra of the twist velocity fields needs too [4,8]. The comprehensive list of all geometrical Lie subgroups of {$\boldsymbol{D}$} is given by Hervé [4–8], while matrix Lie subgroups of SE(3) can be found in Selig [9,10]. Arthur Schönflies studied a special time-dependent motion that is somehow related to a Lie subgroup of {$\boldsymbol{D}$}; Schönflies is spelt Schoenflies in the original relevant publications [11,12]; this is why, in this paper, the spelling "Schoenflies" is preferred. Schoenflies worked first on kinematics but became best known for his work on crystallography. Actually, part of his work on kinematics is devoted to a special type of time-dependent motion defined as the case of instantaneous motions, axodes of which are cylinders or prismatic surfaces. Oddly, the Schoenflies contribution on kinematics does not focus on the concept of algebraic continuous group. In the chapter "Special motions" of their book "Theoretical Kinematics" [13], Bottema and Roth reported this special motion type as "Schoenflies motion" in a manner that is also independent of group theory. This motion type considered as being a set of point transformations will be simply termed X-motion for the sake of conciseness, whose designation was also used in the already published works [4,8,21–23]. A specified X-motion is a 4D displacement Lie subgroup, which contains any translation and any rotation provided that the axis of rotation is parallel to a given direction. This kind of motion has multiple mechanical generators and plays a vital role in the type synthesis of the lower-mobility parallel manipulator [14].

In the following work, we begin recalling some fundamentals of Lie group theory and defining the Schoenflies subgroups of displacements. Then, we systematically enumerate all possible mechanical generators of a Schoenflies subgroup, which are made of serial arrays of one-dof Reuleaux pairs [15] or hinged parallelograms. With the help of 3D computer graphics, their corresponding architectures are classified and graphically displayed. Finally, using the group dependency we also identify chains that are defective for the generation of X-motion.

2. The Schoenflies displacement subgroup

2.1. Kinematics and group theory

A group is a non-empty set endowed with a closed product operation and this operation satisfies definition conditions [2,3], which are the associativity, the existence of one identity element and the existence of one inverse for any element. In a Lie group, the group has also the algebraic properties of a smooth manifold and the algebraic structure of manifold is consistent with the algebraic properties of group. Therefore, the group multiplication (product of two elements and inverse of an element) is a smooth mapping. The set of rigid-body motions or displacements that is denoted {$\boldsymbol{D}$} is a 6D Lie group of transformations, which act on the points of the 3D Euclidean affine space. The algebraic structure of a 6D Lie group of the set of Euclidean displacements {$\boldsymbol{D}$} is a fundamental tool in the analysis of qualitative properties of mechanisms.

There are 12 categories of Lie subgroups of the group {$\boldsymbol{D}$} [4] including two improper subgroups that are the identity subgroup {$\boldsymbol{E}$} and the displacement group {$\boldsymbol{D}$} itself. Using intrinsic geometrical entities instead of coordinates and components in a given frame of reference, the proper subgroups of {$\boldsymbol{D}$} can be expressed as {$\boldsymbol{R}(N, \mathbf{u})$}, {$\boldsymbol{T}(\mathbf{v})$}, {$\boldsymbol{H}(N, \mathbf{u}, p)$}, {$\boldsymbol{C}(N, \mathbf{u})$}, {$\boldsymbol{T}(Pl)$}, {$\boldsymbol{G}(\mathbf{w})$}, {$\boldsymbol{S}(N)$}, {$\boldsymbol{T}$}, {$\boldsymbol{Y}(\mathbf{u}, p)$} and {$\boldsymbol{X}(\mathbf{u})$}. The curly brackets are a conventional way for denoting displacement sets. The capital characters $\boldsymbol{R}$, $\boldsymbol{T}$, $\boldsymbol{H}$, $\boldsymbol{C}$, $\boldsymbol{S}$, $\boldsymbol{G}$, $\boldsymbol{Y}$, $\boldsymbol{X}$ designate a type of motion, more precisely a class of conjugacy employing the terminology of group theory. {$\boldsymbol{R}(N, \mathrm{u})$} means rotations around the axis determined by (N, $\mathbf{u}$) where N is a point belonging to the axis and $\mathbf{u}$ is a unit vector parallel to the axis. {$\boldsymbol{T}(\mathbf{v})$} indicates the set of translations parallel to the given unit vector $\mathbf{v}$. {$\boldsymbol{H}(N, \mathbf{u}, p)$} holds for the helical displacements of pitch p around the axis (N, $\mathbf{u}$). {$\boldsymbol{C}(N, \mathbf{u})$} is a set of cylindrical motions around the axis (N, $\mathbf{u}$). {$\boldsymbol{T}(Pl)$} is a set of planar translations that are parallel to the plane Pl. {$\boldsymbol{G}(\mathbf{w})$} means planar gliding parallel to the plane Pl determined by $\mathbf{w}$ where $\mathbf{w}$ is a unit vector perpendicular to the plane Pl. {$\boldsymbol{S}(N)$} represents the spherical motions (or spherical rotations) around the point N. {$\boldsymbol{T}$} is the set of spatial 3-dof translations. {$\boldsymbol{Y}(\mathbf{u}, p)$} stands for "pseudo-planar" 3-dof displacements where $\mathbf{u}$ is a given unit vector perpendicular to the plane of the pseudo-planar motion and p is the given pitch of the feasible helical displacements [16]. {$\boldsymbol{X}(\mathbf{u})$} represents a 4-dof motion, which includes the 3-dof translations and all the rotations around any axis that is parallel to the given vector $\mathbf{u}$. This motion type is called Schoenflies motion [13].

In a given kinematic chain, the set of feasible relative displacements of a rigid body with respect to a second body is called a mechanical or kinematic bond. A kinematic bond is a mathematical entity; it is a subset of the set of displacements. Generally, a kinematic bond is a submanifold of {$\boldsymbol{D}$}, which has a dimension n, $0 \leqslant n \leqslant 6$. The integer n is called dimension or degree of freedom of the bond or also connectivity between the considered bodies. However, in very special kinematic chains, a kinematic bond has a bifurcation and is not a manifold [17]. In these singular cases, the degree of freedom is not well defined. A kinematic chain generating a given bond will be named mechanical generator of the bond. A given bond generally has several generators that can be considered as kinematic equivalencies. To clarify that a group is not only a set of several objects or entities, it is worth further recalling the closure of the product in a subgroup, that is, the product of two elements of a given subgroup belongs to the same subgroup. That property plays a key role for proving the kinematic equivalencies.

2.2. The Schoenflies group

A precisely specified Lie subgroup of Schoenflies displacements is denoted {$\boldsymbol{X}(\mathbf{w})$}, $\boldsymbol{X}$ stands for Schoenflies (or Schönflies) motion type, and $\mathbf{w}$ is a given unit vector characterizing the axes of the feasible rotations. Assume that (O, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) is an orthonormal frame of reference and M is any point of the three-dimensional Euclidean affine space. A displacement $\boldsymbol{X}(\mathbf{w}; a, b, c, \theta)$ belonging to $\{\boldsymbol{X}(\mathbf{w})\}$ transforms M into the point M' according to the formula

$$M \to M' = O + a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + exp(\theta\mathbf{w}\times)(\mathbf{OM}) \tag{1}$$

where a, b, c, and θ are four canonical parameters of the 4D Lie group, $\mathbf{OM} = M - O$ is the vector obtained from the ordered couple of points O and M, and $\mathbf{w}\times$ is the skew-symmetric linear operator of the vector product by $\mathbf{w}$. It is worth noting that a displacement is a change of rigid-body position that is also called a motion. The absence of motion is mathematically modeled by the identity transformation. All the subsets of displacements contain the absence of motion from the home position.

Eq. (1) can also be expressed as

$$(\mathbf{OM}') = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + exp(\theta\mathbf{w}\times)(\mathbf{OM}) \tag{2}$$

$$\text{or} \begin{pmatrix} \mathbf{OM}' \\ 1 \end{pmatrix} = \begin{pmatrix} exp(\theta\mathbf{w}\times) & a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{OM} \\ 1 \end{pmatrix} \tag{3}$$

in which $exp(\theta\mathbf{w}\times)(\mathbf{OM})$ is obtained by the action on $(\mathbf{OM})$ of the exponential series of the linear operator $\theta\mathbf{w}\times$. By using $(\mathbf{w}\times)^3 = -\mathbf{w}\times$, this series further becomes

$$\begin{aligned} exp(\theta\mathbf{w}\times)(\mathbf{OM}) &= \mathbf{OM} + \theta\mathbf{w} \times (\mathbf{OM}) + (1/2!)\theta^2\mathbf{w} \times [\mathbf{w} \times (\mathbf{OM})] + \ldots + (1/n!)\theta^n(\mathbf{w}\times)^n(\mathbf{OM}) + \ldots \\ &= (\mathbf{OM}) + \sin\theta\mathbf{w} \times (\mathbf{OM}) + (1 - \cos\theta)\mathbf{w} \times [\mathbf{w} \times (\mathbf{OM})] \end{aligned} \tag{4}$$

The formula in Eq. (4) is the vector expression of the Rodrigues formula [18], and represents the rotation of an angle θ of vector $\mathbf{OM}$ around the axis $(O, \mathbf{w})$.

If the four canonical parameters of $\{\boldsymbol{X}(\mathbf{w})\}$ have infinitesimal amplitudes, namely if a, b, c, θ become da, db, dc, dθ, then we obtain an infinitesimal displacement in $\{\boldsymbol{X}(\mathbf{w})\}$. Any point M becomes $M' = M + \mathbf{dM}$ and the infinitesimal vector $\mathbf{dM}$ can readily be calculated. For the infinitesimal motion, Eq. (4) yields $exp(\mathrm{d}\theta\mathbf{w}\times)(\mathbf{OM}) = \mathbf{OM} + \mathrm{d}\theta\mathbf{w}\times(\mathbf{OM})$ and Eq. (2) yields

$$(\mathbf{OM}') = (\mathbf{OM}) + \mathbf{dM} = \mathrm{d}a\mathbf{u} + \mathrm{d}b\mathbf{v} + \mathrm{d}c\mathbf{w} + exp(\mathrm{d}\theta\mathbf{w}\times)(\mathbf{OM}) = \mathrm{d}a\mathbf{u} + \mathrm{d}b\mathbf{v} + \mathrm{d}c\mathbf{w} + (\mathbf{OM}) + \mathrm{d}\theta\mathbf{w} \times (\mathbf{OM}) \tag{5}$$

Hence,

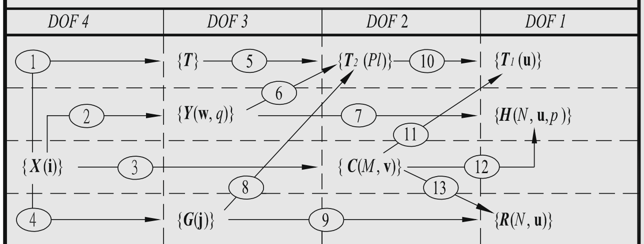
$$\mathbf{dM} = \mathrm{d}a\mathbf{u} + \mathrm{d}b\mathbf{v} + \mathrm{d}c\mathbf{w} + \mathrm{d}\theta\mathbf{w} \times (\mathbf{OM}) \tag{6}$$

$$\text{or} \begin{pmatrix} \mathbf{dM} \\ 0 \end{pmatrix} = \begin{pmatrix} d\theta\mathbf{w}\times & da\mathbf{u} + db\mathbf{v} + dc\mathbf{w} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{OM} \\ 1 \end{pmatrix} \tag{7}$$

The linear operator $\$(\mathbf{w}; \mathrm{d}\theta; \mathrm{d}a, \mathrm{d}b, \mathrm{d}c) = \begin{pmatrix} d\theta w\times & dau + db\mathbf{v} + dc\mathbf{w} \\ 0 & 0 \end{pmatrix}$ is the twist of a $\{\boldsymbol{X}(\mathbf{w})\}$ motion. The set of these twists is a 4D vector space called a four-system of screws when employing the Ball terminology [1]. The canonical vector base of the 6D vector space of all the screws (twists and wrenches) is derived from the Cartesian frame of reference

Table 1
Graph of subgroups of Schoenflies motion.

Graph of binary relation of "to be a Lie subgroup of a Lie subgroup $\{X(\mathbf{i})\}$"



Geometrical conditions

1	$\forall \mathbf{i}$	6	$Pl \perp \mathbf{w}$	11	$\mathrm{u} = \pm \mathrm{v}$
2	$\mathrm{w} = \pm\mathrm{i}$, $\forall q$	7	$\mathrm{u} = \pm \mathrm{w}, p = \pm q \,\forall N$	12	$N \in$ line (M, v), $\mathrm{u} = \pm \mathrm{v}, \forall p$
3	$\mathrm{v} = \pm\mathrm{i}, \forall M$	8	$Pl \perp \mathrm{j}$	13	$N \in$ line (M, v), $\mathrm{u} = \pm \mathrm{v}$
4	$\mathrm{j} = \pm \mathrm{i}$	9	$\mathrm{u} = \pm\mathrm{j}, \forall N$		
5	$\forall Pl$	10	$\mathrm{u} \,//\, Pl$		

Notations:
N, M are points; $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{i}, \mathbf{j}$ are unit vectors; Pl a plane direction or vector plane;
p, q are real number; // means parallel and $\perp$ perpendicular;
line$(N, \mathbf{u})$ is the straight line, a reference frame of which is $(N, \mathbf{u})$.

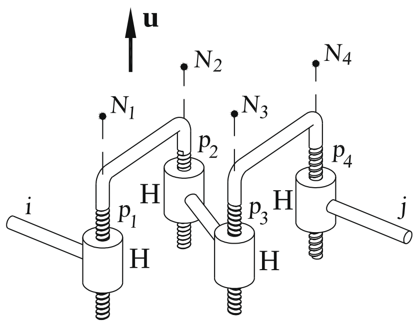


Fig. 1. A general generator of *X*-motion.

Table 2
Generators of Schoenflies motion – $\{\boldsymbol{X}(\mathbf{u})\}$.

Class	$\{\boldsymbol{X}(\mathbf{u})\} = \{\boldsymbol{H}(N_1, \mathbf{u}, p_1)\}\{\boldsymbol{H}(N_2, \mathbf{u}, p_2)\}\{\boldsymbol{H}(N_3, \mathbf{u}, p_3)\}\{\boldsymbol{H}(N_4, \mathbf{u}, p_4)\}$				
I (no P)	HHHH	HHHR	HHRH	HHRR	RHHR
	HRRH	HRHR	HRRR	RHRR	
II (one P)	PHHH	HHPH	PHHR	PHRH	PRHH
	HHPR	HRPH	RHPH	PHRR	PRHR
	PRRH	RRPH	RHPR	HRPR	PRRR
	RRPR				
III (two Ps)	HHPP	HPPH	HPHP	PHHP	HRPP
	RHPP	RPHP	RPPH	HPRP	PRHP
	RRPP	RPPR	RPRP	PRRP	
IV (three Ps)	PPPH	PPPR	PPHP	PPRP	

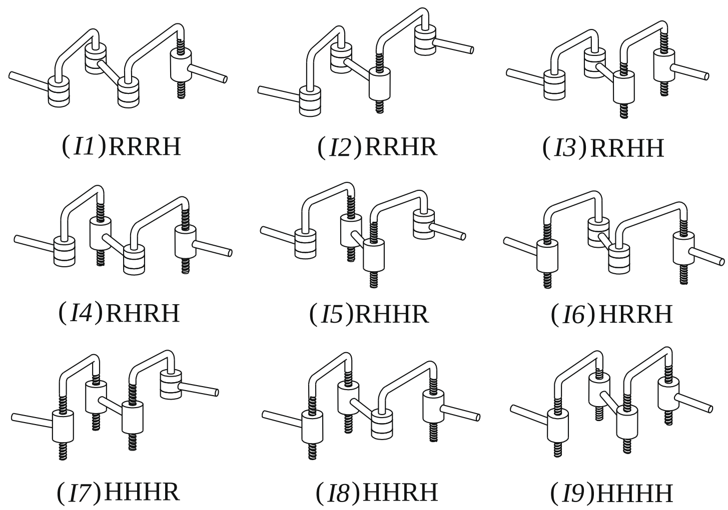


Fig. 2. Class *I* of *X*-motion generators.

$(O, \mathbf{u}, \mathbf{v}, \mathbf{w})$ and is $\begin{pmatrix} \mathbf{u}\times & \mathbf{0} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{v}\times & \mathbf{0} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{w}\times & \mathbf{0} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{u} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{v} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{0} & \mathbf{w} \\ 0 & 0 \end{pmatrix}$. The components in that canonical vector base of a twist representing an infinitesimal $\{\boldsymbol{X}(\mathbf{w})\}$ motion are $(0, 0, \mathrm{d}\theta; \mathrm{d}a, \mathrm{d}b, \mathrm{d}c)$. The set of these twists can be denoted $\{\boldsymbol{x}(\mathbf{w})\}$ with a small $\boldsymbol{x}$ rather than a capital $\boldsymbol{X}$.

In the general 6D vector space of screws, the Lie bracket or commutator of two twists $\$_1$ and $\$_2$ is defined by $[\$_1, \$_2] = \$_1\$_2 - \$_2\, \$_1$. This operation is a kind of product between the screws and one can readily verify that this product provides a screw [8,10]. Hence, the screw product defined by the Lie bracket is a closed operation in the vector space of screws. In general algebra, such a property defines a particular algebraic structure that is called algebra. The algebra of screws is antisymmetric (or skew-symmetric) and is called the Lie algebra of screws. The Lie algebra of screws is the Lie algebra $\{\boldsymbol{d}\}$ of the Lie group $\{\boldsymbol{D}\}$ of displacements. It is straightforward [8,10] to verify

$$\$_1 \in \{\boldsymbol{x}(\mathbf{w})\} \quad \text{and } \$_1 \in \{\boldsymbol{x}(\mathbf{w})\} \Rightarrow [\$_1, \$_2] \in \{\boldsymbol{x}(\mathbf{w})\}. \tag{8}$$

The closure in $\{\boldsymbol{x}(\mathbf{w})\}$ of the screw product characterizes a Lie subalgebra of screws. From Lie's theory of continuous groups of transformations, there is a one-to-one correspondence between the 4D Lie subgroup $\{\boldsymbol{X}(\mathbf{w})\}$ and its 4D Lie subalgebra $\{\boldsymbol{x}(\mathbf{w})\}$. The Lie subalgebras of twists were used in [8,10] for proving the completeness of the list of displacement Lie subgroups in [5,6].

The $\{\boldsymbol{X}(\mathbf{w})\}$ group has eight categories of proper Lie subgroups. They are:

(a) $\{\boldsymbol{T}(\mathbf{s})\}$: set of rectilinear translations parallel to any given vector $\mathbf{s}$, $\forall\mathbf{s}$;
(b) $\{\boldsymbol{R}(N, \mathbf{w})\}$: set of rotations around any given axis $(N, \mathbf{w})$, $\forall$ axis $(N, \mathbf{w})$;
(c) $\{\boldsymbol{H}(N, \mathbf{w}, p)\}$: helical motions of given axis $(N, \mathbf{w})$ with the pitch p, $\forall\ (N, \mathbf{w})$, $\forall p$;
(d) $\{\boldsymbol{T}(Pl)\}$: set of planar translations parallel to the given Pl-plane, $\forall Pl$;
(e) $\{\boldsymbol{C}(N, \mathbf{w})\}$: set of cylindrical movements about any given axis $(N, \mathbf{w})$, $\forall\ (N, \mathbf{w})$;
(f) $\{\boldsymbol{T}\}$: set of the spatial translations;
(g) $\{\boldsymbol{G}(\mathbf{w})\}$: set of the planar gliding motions perpendicular to $\mathbf{w}$;
(h) $\{\boldsymbol{Y}(\mathbf{w}, p)\}$: the pseudo-planar motions perpendicular to $\mathbf{w}$ and with any given pitch p.

The improper subgroups of $\{\boldsymbol{X}(\mathbf{w})\}$ are $\{\boldsymbol{E}\}$, and the group $\{\boldsymbol{X}(\mathbf{w})\}$ itself. More generally the inclusion of a displacement Lie subgroup inside of another displacement Lie subgroup is a binary relation of partial order when employing the terminology of abstract algebra. The graph of that relation is shown in Table 1.

3. Enumeration of primitive *X*-motion generators

In what follows, we will study the generation of *X* kinematic bond and enumerate all possible architectures of its corresponding mechanical generators in detail.

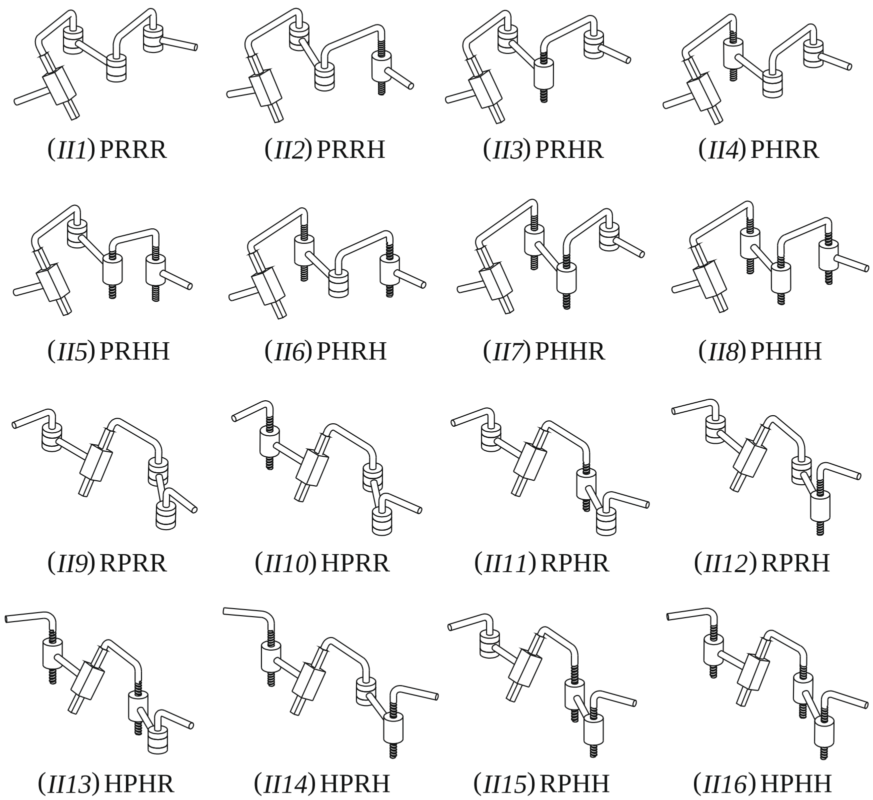


Fig. 3. Class *II* of *X*-motion generators.

3.1. Generators with Reuleaux lower pairs

The serial setting of two kinematic pairs produces a kinematic bond between the distal bodies and this bond is the product of the pair bonds. In any group, the product of sets is the set of products. Generally, the product of two subgroups is not a subgroup. Nevertheless, the main useful fact is the closure of the product of two elements in any subgroup. A subgroup of a group is a subset of the group, which is also an algebraic group for the same product. Let G be a subgroup of a given group H. If $A \subset G$ and $B \subset G$, that is, A and B are subsets of G, then, from the closure of product in the group, it implies $AB \subseteq G$. When the group H has a dimension and the subsets also have a dimension, $dim\ AB = dim$ G leads to $AB = G$. The equality is generally valid only in a neighborhood of the identity. Consequently, the Schoenflies subgroup can be generated by many serial arrays of kinematic pairs. The following set equality can be regarded as a generic expression of the decomposition of the 4D subgroup $\{\boldsymbol{X}(\mathbf{u})\}$ into a product of four 1D subgroups, which are generated by the 1-dof Reuleaux pairs [15], as shown in Fig. 1.

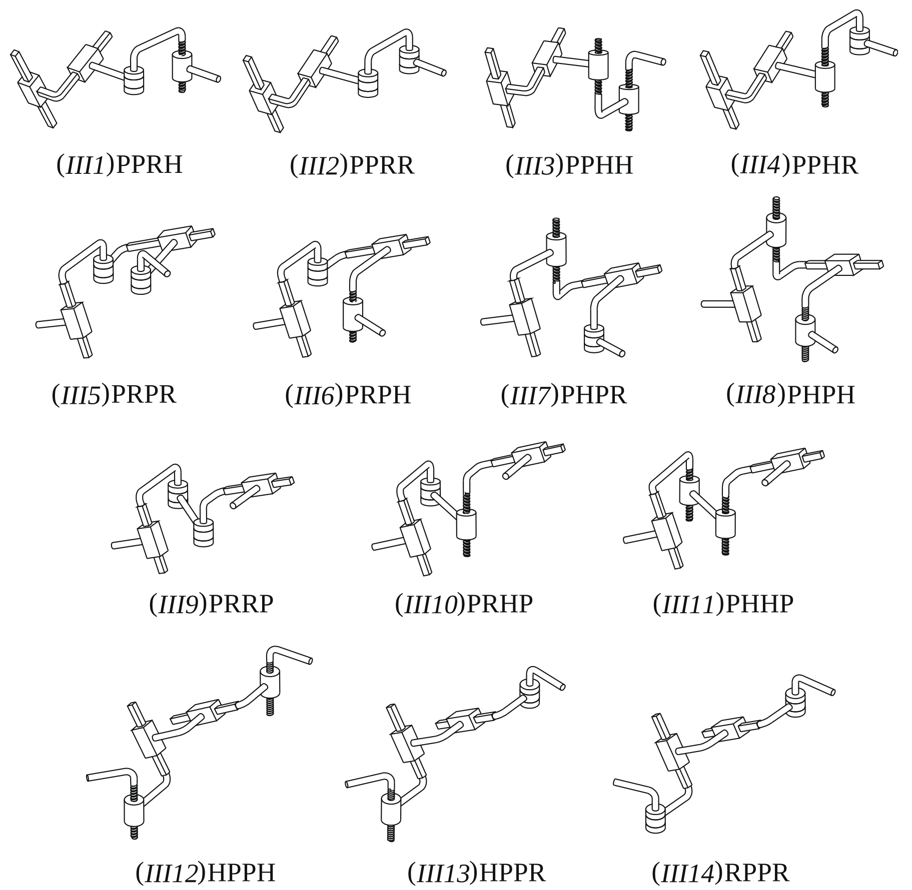


Fig. 4. Class *III* of *X*-motion generators.

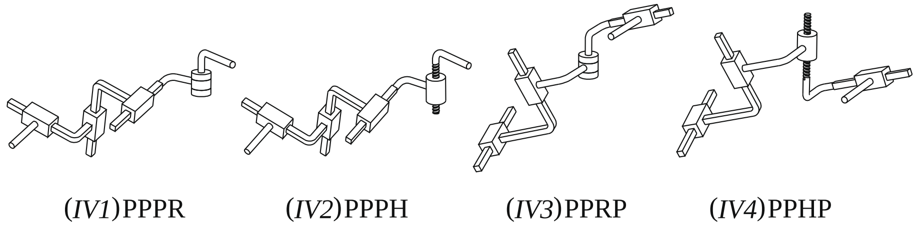


Fig. 5. Class *IV* of *X*-motion generators.

$$\{\boldsymbol{X}(\mathbf{u})\} = \{\boldsymbol{H}(N_1,\mathbf{u},p_1)\}\{\boldsymbol{H}(N_2,\mathbf{u},p_2)\}\{\boldsymbol{H}(N_3,\mathbf{u},p_3)\}\{\boldsymbol{H}(N_4,\mathbf{u},p_4)\} \tag{9}$$

In set Eq. (9), one, two or three of the factors $\{\boldsymbol{H}(N_i, \mathbf{u}, p_i)\}$ may be replaced by a 1D subgroup $\{\boldsymbol{T}(\mathbf{s}_i)\}$ of translation parallel to the unit vector $\mathbf{s}_i$, provided that the vectors $\mathbf{s}_i$ are linearly independent. One, two or three pitches may also be equal to zero. Obviously, set Eq. (9) is valid in a neighborhood of the identity $\boldsymbol{E}$ if and only if (iff) the product in its right side is a 4D manifold; else the product as well as the open chain of Fig. 1 are singular. The detection of transitory singularity is out of the scope of this paper.

Actual achievements of $\{\boldsymbol{X}(\mathbf{u})\}$ mechanical generators can be obtained by placing in series kinematic pairs represented by subgroups of $\{\boldsymbol{X}(\mathbf{u})\}$; a serial arrangement of four 1-dof kinematic pairs without intermediate link having passive motion makes up a mechanical generator of the subgroup $\{\boldsymbol{X}(\mathbf{u})\}$. What has to be noticed is that a 4-R chain is obviously defective for generating $\{\boldsymbol{X}(\mathbf{u})\}$ because such a chain has a redundant internal mobility and generates $\{\boldsymbol{G}(\mathbf{u})\} \subset \{\mathbf{X}(\mathbf{u})\}$. More defective generators of $\{\boldsymbol{X}(\mathbf{u})\}$ will be identified in the next section. The comprehensive list of all possible combinations of 1-dof kinematic pairs generating the Schoenflies motion is shown in Table 2. These combinations are sorted into four classes based on the number of prismatic pairs. There are forty-three general-type architectures of *X*-motion generators. The most general generator of an *X*-group is HHHH where the screws H have parallel axes and four pitches must not be equal. A P pair is a limit case of an H pair either with a pitch becoming infinite or with an axis going at infinity, or a combination of both previous situations. An R pair is an H with a zero pitch. All the corresponding kinematic chains that generate *X*-motion are also graphically displayed in Figs. 2–5. It is worth noticing that reversing the order of joints in any serial arrangement of Table 2 also yields a mechanical generator of $\{\boldsymbol{X}(\mathbf{u})\}$.

3.2. Generators with hinged parallelograms

The famous Delta robot [20] implements successfully hinged parallelograms in three limb chains that are generators of three *X*-motions. Various potential applications of *X*-motion generators with parallelograms are going to be elucidated in further works. In fact, circular translation and rectilinear translation are not the same motion. The opposite bars of a 1-dof hinged parallelogram can move while remaining parallel. Hence, the coupling between two opposite bars generates relative 1-dof circular translation that is a 1D manifold contained in the 2D subgroup of planar translation; the plane is the one of the parallelogram. Consequently, for a small motion, a hinged parallelogram is equivalent to a prismatic pair.

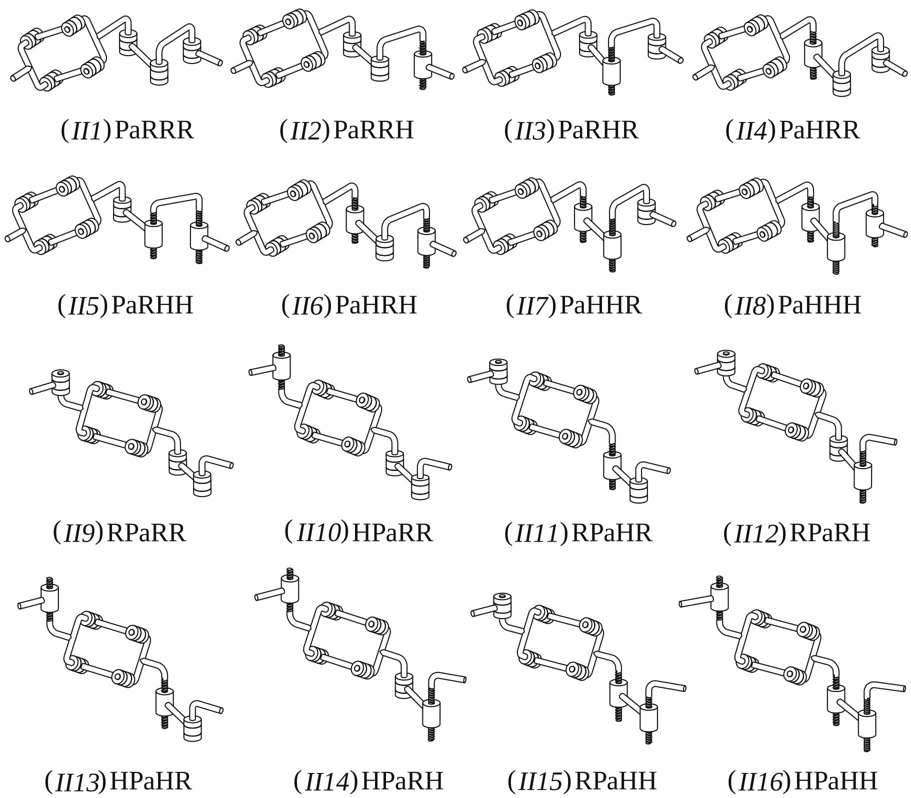


Fig. 6. Class *II* of *X*-motion generators with one hinged parallelogram.

Replacing all P pairs by hinged parallelograms, we obtain all possible *X*-motion generators including hinged parallelograms; these generators are shown in Figs. 6–8. Flattened parallelograms are singular and must be avoided. When only one P pair for each of the generators shown in Fig. 4 is replaced by one hinged parallelogram, Figs. 9 and 10 are readily obtained. Here, we must notice that the four generators, (*III9*) PRRPa, (*III11*) PHHPa, (*III12*) HPPaH and (*III14*) RPPaR in Fig. 10 are cancelled out because they have architectures that are equivalent by kinematic inversions to chains shown in Fig. 9.

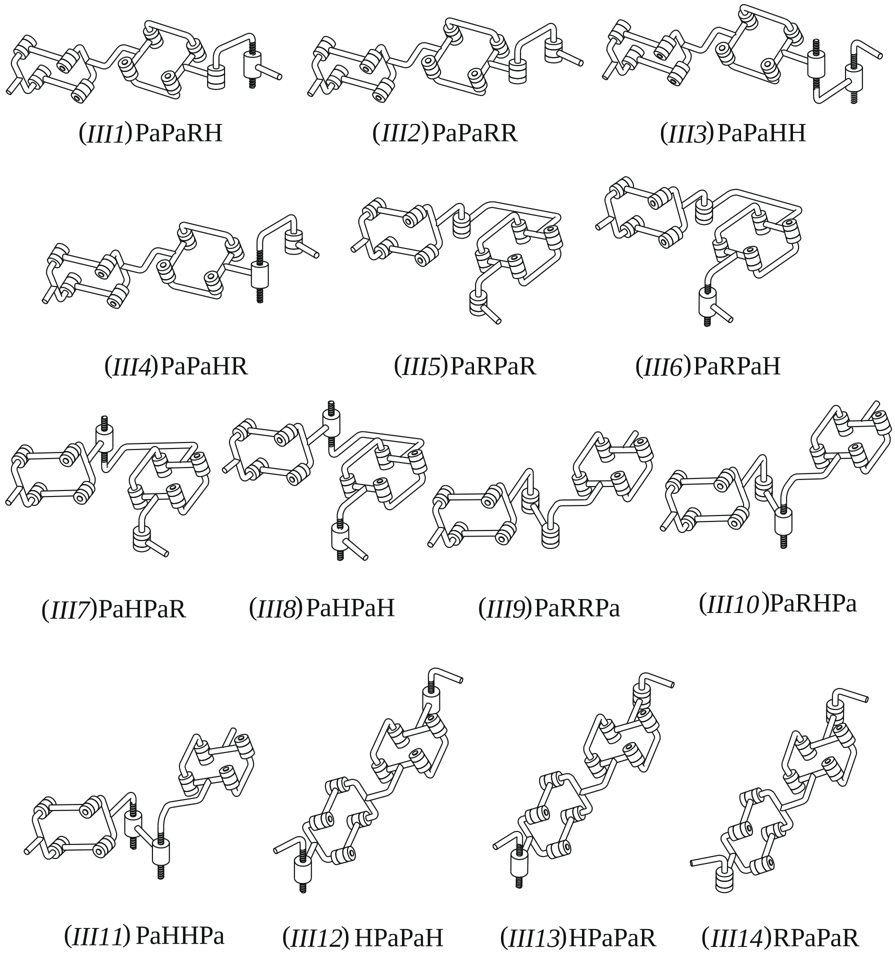


Fig. 7. Class *III* of *X*-motion generators with two hinged parallelograms.

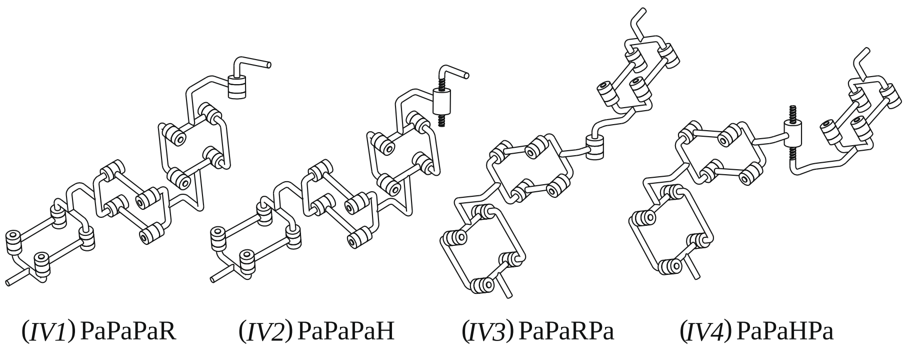


Fig. 8. Class *IV* of *X*-motion generators with three hinged parallelograms.

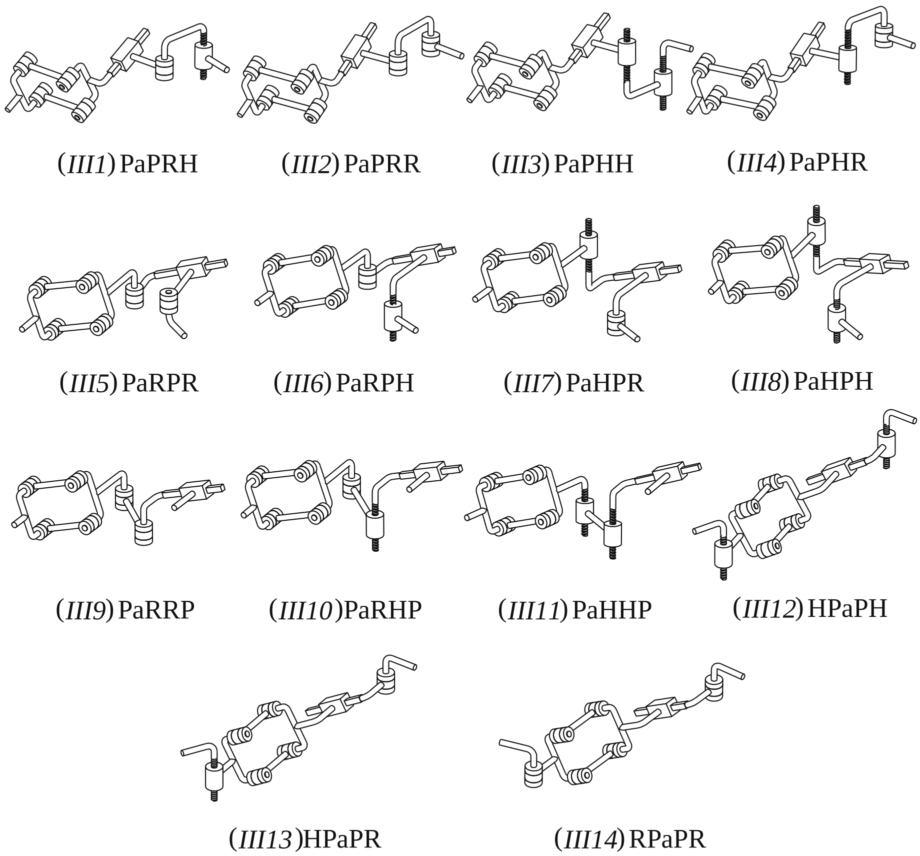


Fig. 9. Class *III* of *X*-motion generators with one hinged parallelogram – I.

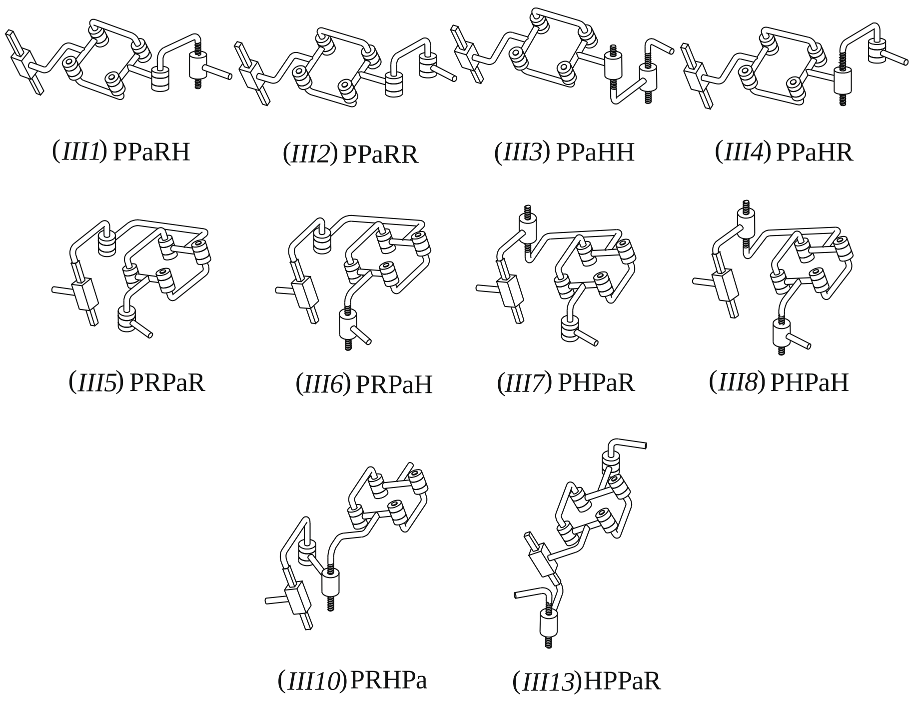
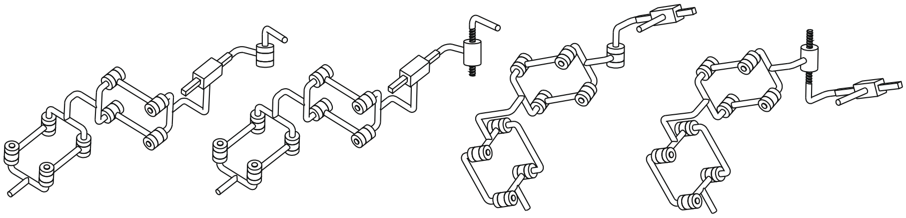
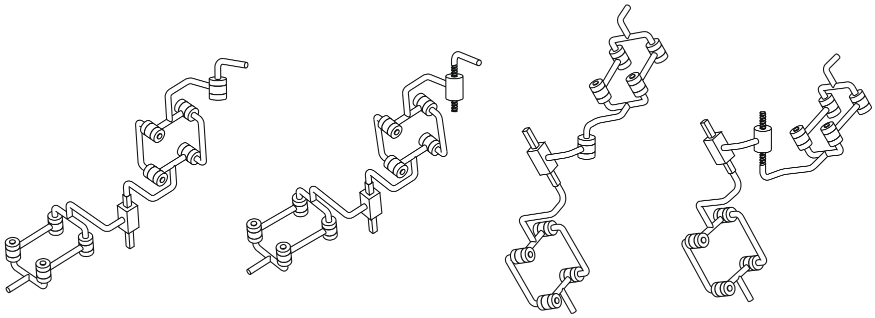


Fig. 10. Class *III* of *X*-motion generators with one hinged parallelogram – II.



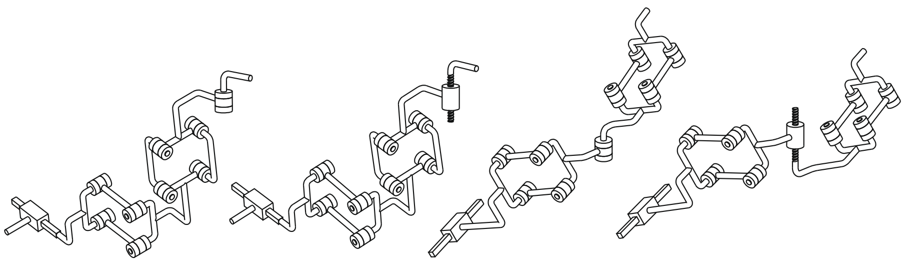
$(IV1)$PaPaPR $(IV2)$PaPaPH $(IV3)$PaPaRP $(IV4)$PaPaHP

Fig. 11. Class *IV* of *X*-motion generators with two hinged parallelograms – I.



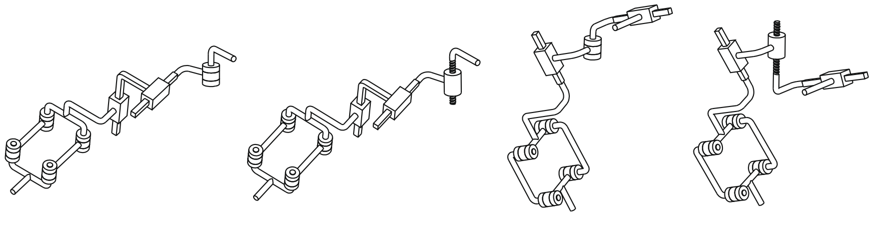
$(IV1)$PaPPaR $(IV2)$PaPPaH $(IV3)$PaPRPa $(IV4)$PaPHPa

Fig. 12. Class *IV* of *X*-motion generators with two hinged parallelograms – II.



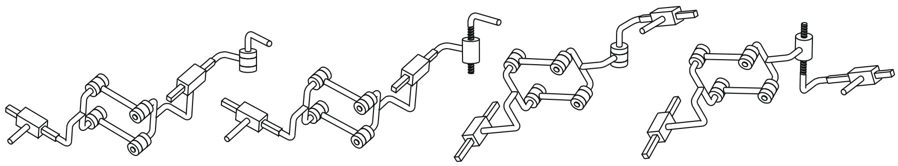
$(IV1)$PPaPaR $(IV2)$PPaPaH $(IV3)$PPaRPa $(IV4)$PPaHPa

Fig. 13. Class *IV* of *X*-motion generators with two hinged parallelograms – III.



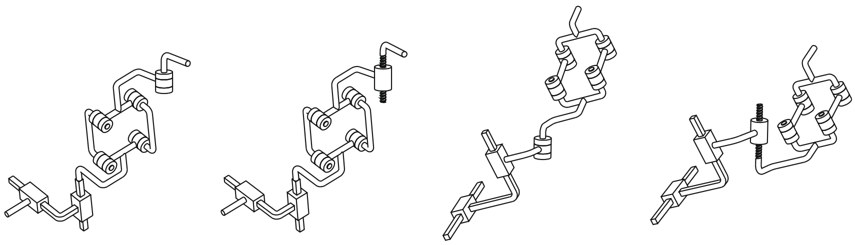
$(IV1)$PaPPR $(IV2)$PaPPH $(IV3)$PaPRP $(IV4)$PaPHP

Fig. 14. Class *IV* of *X*-motion generators with one hinged parallelogram – I.



(*IV1*)PPaPR (*IV2*)PPaPH (*IV3*)PPaRP (*IV4*)PPaHP

Fig. 15. Class *IV* of *X*-motion generators with one hinged parallelogram – II.



(*IV1*)PPPaR (*IV2*)PPPaH (*IV3*)PPRPa (*IV4*)PPHPa

Fig. 16. Class *IV* of *X*-motion generators with one hinged parallelogram – III.

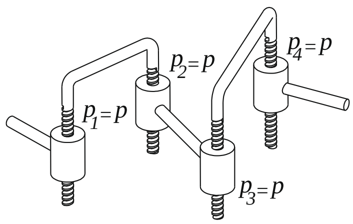
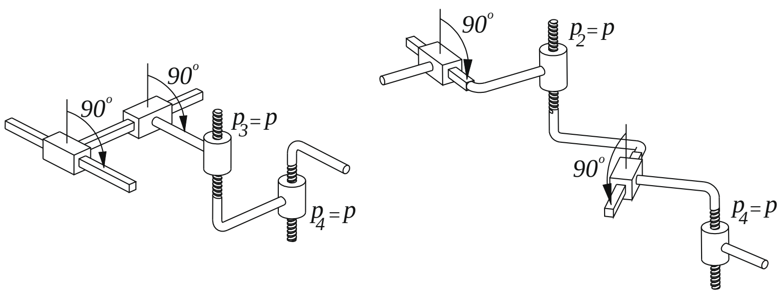


Fig. 17. The defective generator with four identical pitches.



(a) *PP*HH (b) *P*H*P*H

Fig. 18. Defective generators with two identical pitches and two P pairs.

Figs. 11–13 are generators of *X*-motion obtained by the replacement of the two P pairs in chains of Fig. 5 by two hinged parallelograms. Likewise, Figs. 14–16 are *X*-motion generators derived by replacing only one P pair in each generator of Fig. 5 with one hinged parallelogram. That way, we obtain a total of eighty-two chains having at least one parallelogram, noticing that the kinematic inversion of each of these foregoing chains is also an adequate chain for generating *X*-motion.

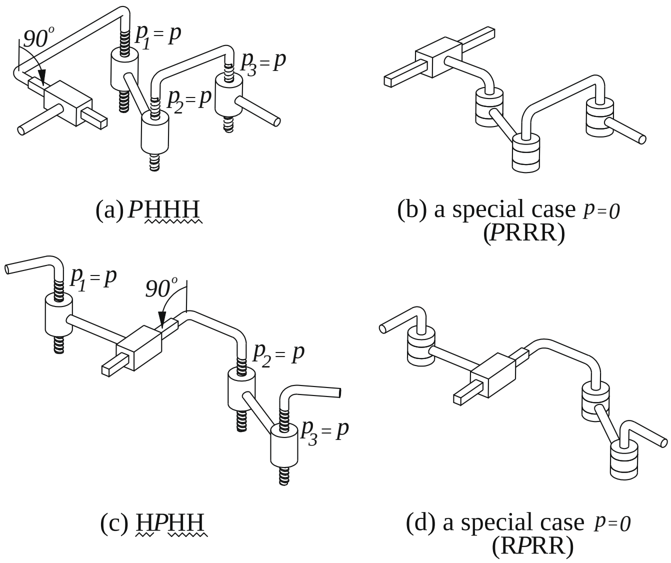


Fig. 19. Defective generators with three identical pitches and one P pair.

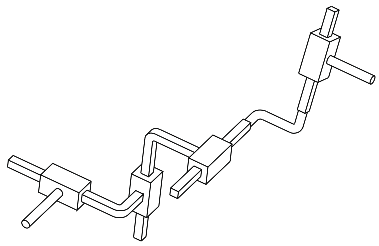


Fig. 20. A defective generator with four prismatic pairs.

4. Defective *X*-motion generators

A defective chain for generating *X*-motion arises from the permanent singularity of the chain. Then the chain does never generate the desired *X*-motion. Such a phenomenon is not properly a singularity. As a matter of fact, singular means specific of special poses of the chain. However, such an abuse of language has some practical interest because the same geometric condition may yield transitory or permanent failure in the generation of *X*-motion. Clearly, open chains obtained from the trivial or exceptional 4-bar 1-dof closed chains with 1-dof Reuleaux pairs by splitting in two parts for any one link are defective *X*-motion generators. Using group dependency, we can derive all possible cases of defective chains for the generation of Schoenflies motion. In general, the singularity happens iff the following set equation

$$\{\boldsymbol{H}(N_1,\mathbf{u},p_1)\}\{\boldsymbol{H}(N_2,\mathbf{u},p_2)\}\{\boldsymbol{H}(N_3,\mathbf{u},p_3)\}\{\boldsymbol{H}(N_4,\mathbf{u},p_4)\} = \{\boldsymbol{E}\} \tag{10}$$

does not imply the set equations

$$\{\boldsymbol{H}(N_1,\mathbf{u},p_1)\} = \{\boldsymbol{H}(N_2,\mathbf{u},p_2)\} = \{\boldsymbol{H}(N_3,\mathbf{u},p_3)\} = \{\boldsymbol{H}(N_4,\mathbf{u},p_4)\} = \{\boldsymbol{E}\}. \tag{11}$$

which are solved iff the helical motion angles are equal to zero. Here, the subset of displacements represents variations of position from the home posture. The absence of displacement necessarily belongs to the set of feasible displacements.

Set Eq. (10) is the mathematical model of a mechanism obtained from the open chain pictured in Fig. 1 by welding the distal bodies *i* and *j* on a fixed frame. Such a closed-loop mechanism generally cannot move and, then, the open chain of Fig. 1 effectively generates *X*-motion. If a link in the closed mechanism can move, then the generator of *X* bond is defective or permanently singular. Two kinds of singularities may happen; the undesired motion either has only infinitesimal amplitude or can have finite amplitude. The detection of undesired infinitesimal motion is done through the study of a possible linear dependency of the four twists. This topic will be studied in another work. On the other hand, group theory is a fruitful tool for

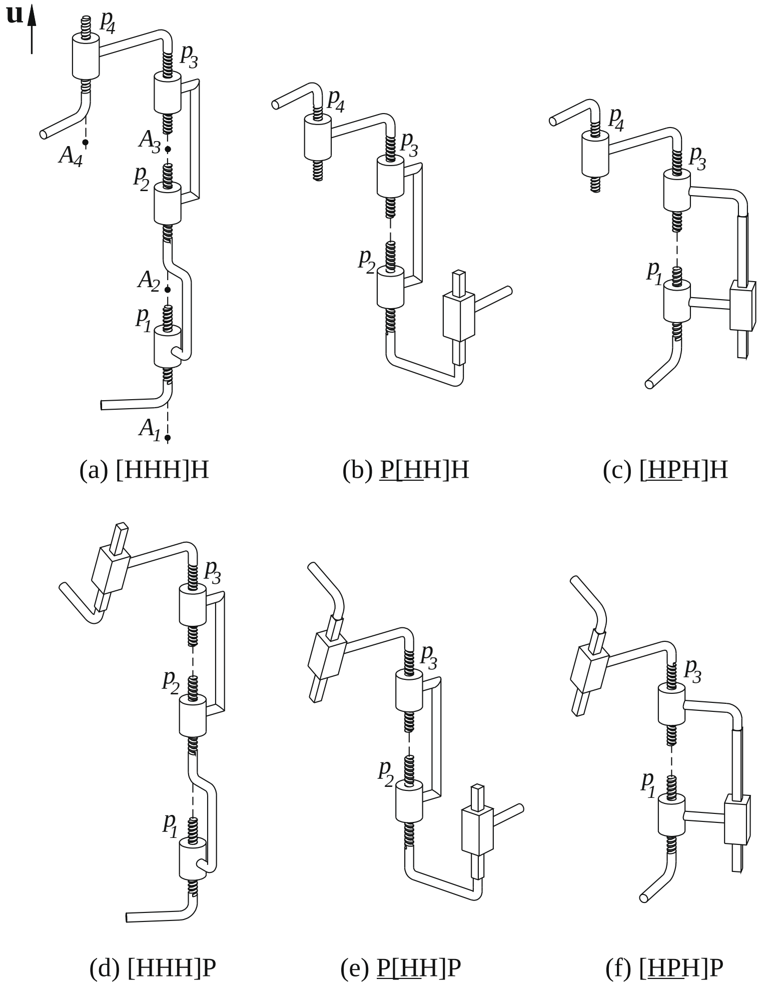


Fig. 21. Defective generators with three coaxial H pairs.

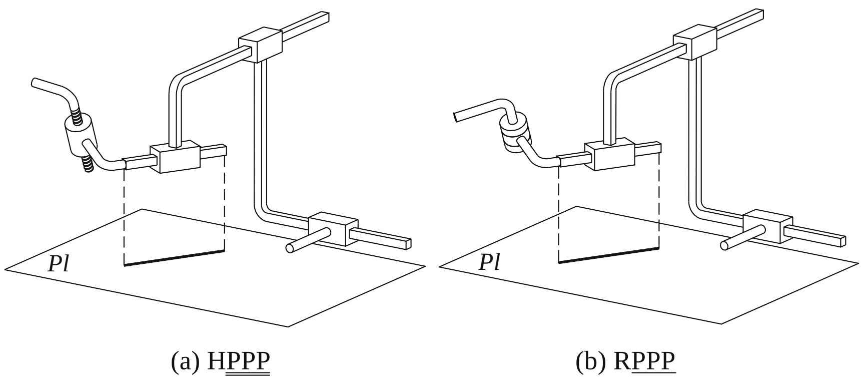


Fig. 22. Defective generators with three coplanar P pairs.

the characterization of finite motion. Beyond the trivial and exceptional cases that are explained through the group dependency of displacement subsets, only four paradoxical cases were definitely established by Delassus [19]. Myard's work [24] is also devoted to the study of paradoxical closed chains with five or six revolute pairs, which are beyond the subject of our paper. In spite that special exceptional chains have been misled to be paradoxical ones in [25], the paradoxical mobility still

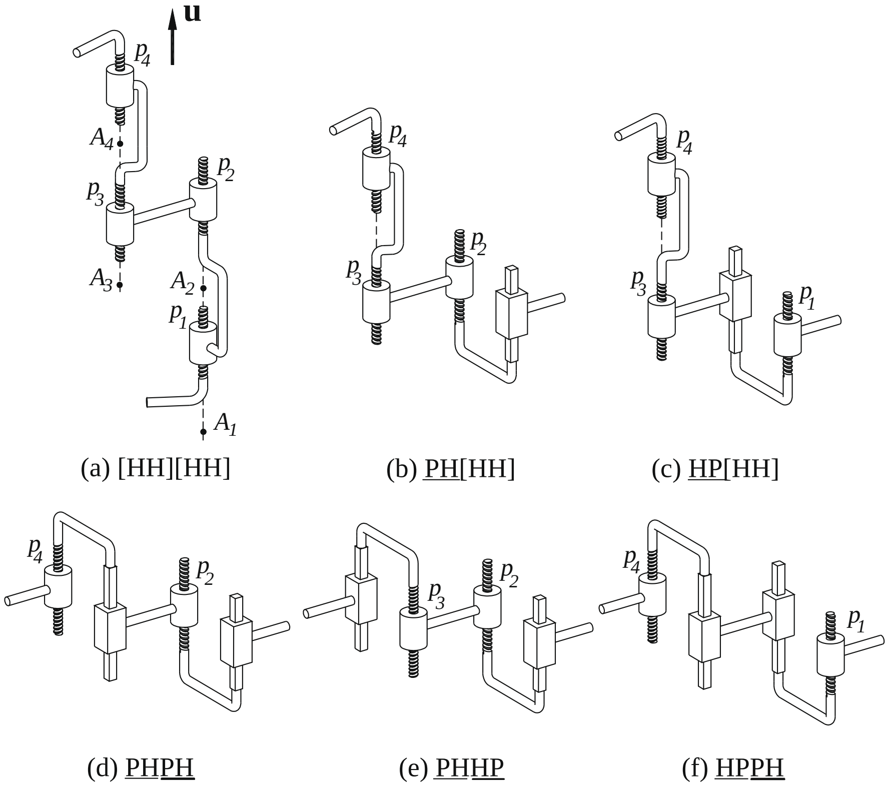


Fig. 23. Defective generators with an exceptional mobility.

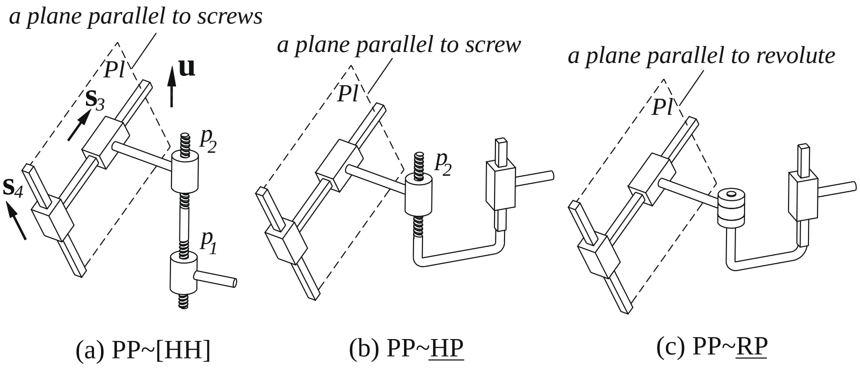


Fig. 24. Defective generators with a passive exceptional mobility.

cannot be explained only by the group dependency, which does not require the use of the Euclidean metrics. The paradoxical chains of Delassus can yield passive motion with finite amplitude. This kind of singularity will be confirmed in further work. Neglecting the paradoxical mobility, which is transitory in an open chain, a link of the previous mechanism can move permanently iff two open sub-chains generate two dependent kinematic bonds, the intersection of which is not $\{\boldsymbol{E}\}$. In order to avoid the defective generators, the following cases must be considered:

Case A. In set Eq. (10), a product of three factors is equal to a 3D subgroup of $\{X(\mathbf{u})\}$ and the fourth 1D factor is included in this subgroup.

Referring to Fig. 17, if the four pitches are equal, then $[\{\boldsymbol{H}(A_1,\mathbf{u},p)\}\{\boldsymbol{H}(A_2,\mathbf{u},p)\}\{biH(A_3,\mathbf{u},p)\}] = \{\boldsymbol{Y}(\mathbf{u},p)\}$ and $\{\boldsymbol{H}(A_4,\mathbf{u},p)\} \subset \{\boldsymbol{Y}(\mathbf{u},p)\}$ implies $[\{\boldsymbol{H}(A_1,\mathbf{u},p)\}\{\boldsymbol{H}(A_2,\mathbf{u},p)\}\{\boldsymbol{H}(A_3,\mathbf{u},p)\}]\{\boldsymbol{H}(A_4,\mathbf{u},p)\} = \{\boldsymbol{Y}(\mathbf{u},p)\}\{\boldsymbol{H}(A_4,\mathbf{u},p)\} = \{\boldsymbol{Y}(\mathbf{u},p)\} \neq \{\boldsymbol{X}(\mathbf{u})\}$. Hence, this chain fails in generating Schoenflies motion for any pose and, in other words, it is a defective chain for the

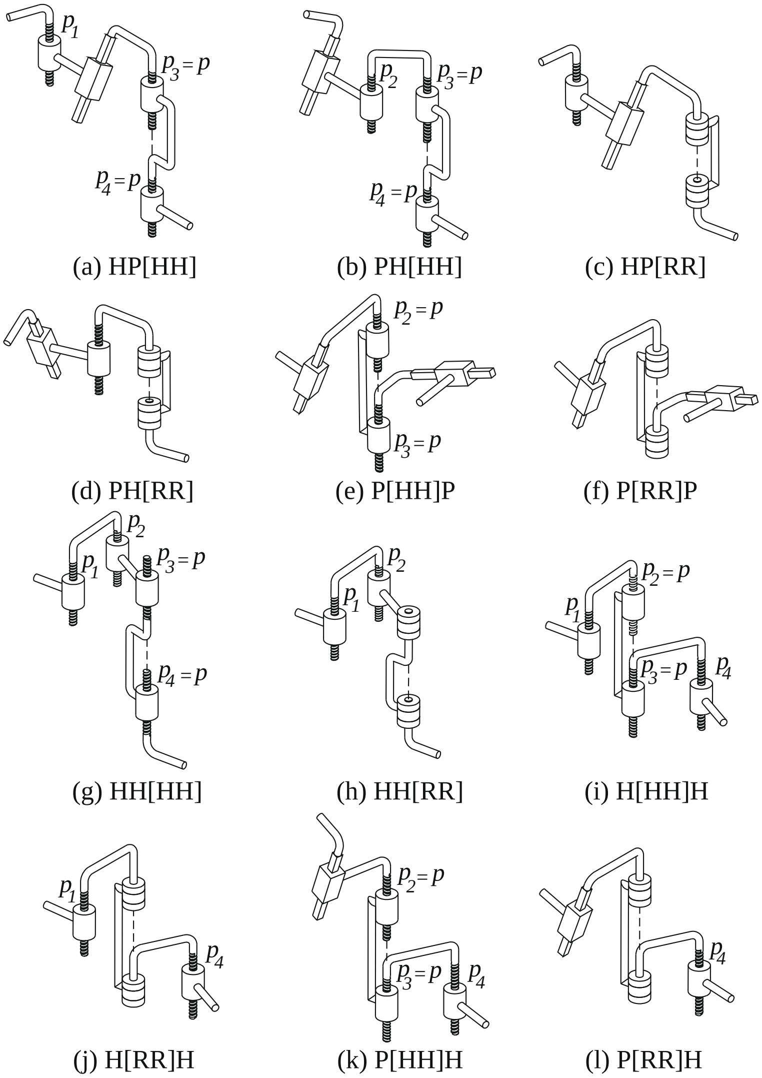


Fig. 25. Defective generators with two adjacent coaxial H or R pairs.

generation of *X*-motion. The four pitches must not be all equal. Pitches may be equal to zero but not all zeros. When four pitches are zeros, the chain generates the planar gliding motion, $\{\boldsymbol{Y}(\mathbf{u}, 0)\} = \{\boldsymbol{G}(\mathbf{u})\}$.

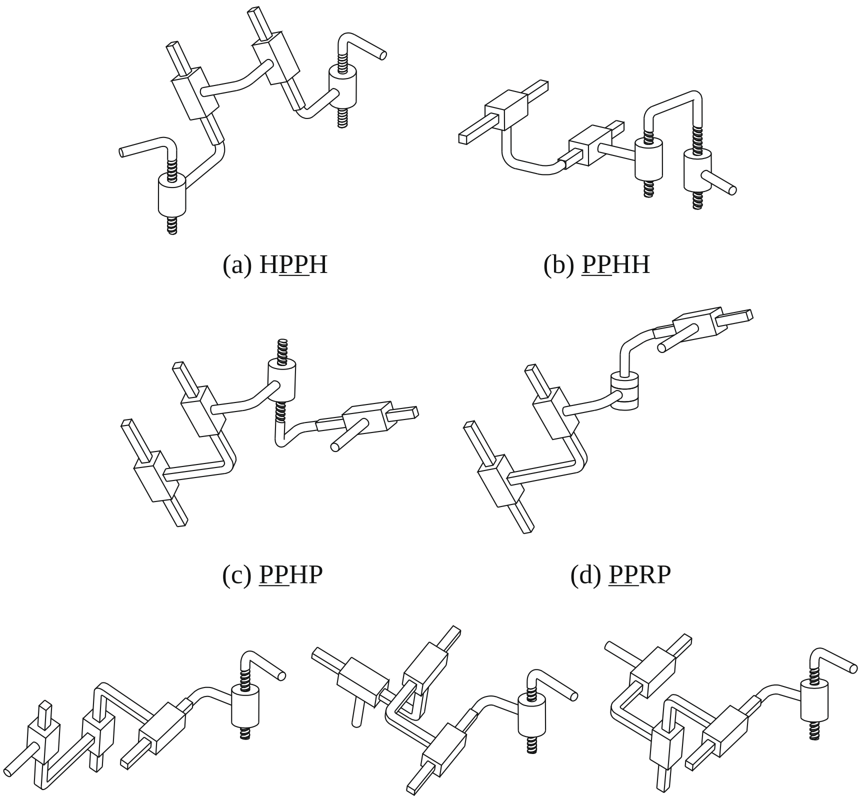
By the same token, one can demonstrate that if two screw pitches are equal, then two P pairs must not be perpendicular to $\mathbf{u}$. For instance, two chains of Fig. 18 actually generate the 3-dof pseudo-planar motion rather than 4-dof *X*-motion. Furthermore, if three screw pitches are equal and one P pair is perpendicular to the parallel H axes, as shown in Fig. 19, these chains are trivial chains of a subgroup of pseudo-planar motion and never generate *X*-motion.

One additional defective generator, a series of four prismatic pairs that generates $\{\boldsymbol{T}\}$ instead of $\{\boldsymbol{X}(\mathbf{u})\}$, is displayed in Fig. 20 for completeness.

Case B. A product of two factors is a 2D subgroup and one among the other two factors is included into this subgroup.

For example, $p_1 \neq p_2, A_2 \in \text{line}(A_1, \mathbf{u})\text{or}(\mathbf{A}_1\mathbf{A}_2) \times \mathbf{u} = \mathbf{0} \Rightarrow \{\boldsymbol{H}(A_1, \mathbf{u}, p_1)\}\{\boldsymbol{H}(A_2, \mathbf{u}, p_2)\} = \{\boldsymbol{C}(A_1, \mathbf{u})\}$; $A_3 \in \text{line}(A_1, \mathbf{u}) \Rightarrow \{\boldsymbol{H}(A_3, \mathbf{u}, p_3)\} \subset \{\mathbf{C}(A_1, \mathbf{u})\} \Rightarrow [\{\boldsymbol{H}(A_1, \mathbf{u}, p_1)\}\{\boldsymbol{H}(A_2, \mathbf{u}, p_2)\}]\ \{\boldsymbol{H}(A_3, \mathbf{u}, p_3)\}\{\boldsymbol{H}(A_4, \mathbf{u}, p_4)\} = \{\boldsymbol{C}(A_1, \mathbf{u})\}\{\boldsymbol{H}(A_3, \mathbf{u}, p_3)\}\{\boldsymbol{H}(A_4, \mathbf{u}, p_4)\} = \{\boldsymbol{C}(A_1, \mathbf{u})\}\{\boldsymbol{H}(A_4, \mathbf{u}, p_4)\} \neq \{\boldsymbol{X}(\mathbf{u})\}$. Hence, three axes must not be coaxial. Fig. 21a shows such a defective chain with three coaxial H pairs. The subgroup $\{\boldsymbol{C}(A_1, \mathbf{u})\}$ can also be generated by PH or HP arrays (PR or RP when the pitch of H is zero) if the P is parallel to the H axis (R axis). Fig. 21b–f shows other defective *X*-motion generators being in this situation, in which the replacement of any screw H by revolute R yields a defective *X*-generator chain, too.

In Fig. 22, the cases with three prismatic pairs that are parallel to a plane are defective generators of *X*-motion and must also be avoided.



(a) HPPH (b) PPHH

(c) PPHP (d) PPRP

(e) PPPH (f) PPPH (g) PPPH

Fig. 26. Defective generators with two parallel P pairs.

Case C. A product of two factors is a 2D subgroup and the product of the other two factors is another subgroup, which is dependent with respect to the first subgroup.

In other words, the intersection of the two 2D subgroups is a 1D subgroup. From the list of products of dependent subgroups [5], we obtain only two possible situations, namely,

C1. $[\{\boldsymbol{H}(A_1,\mathbf{u},p_1)\}\{\boldsymbol{H}(A_2,\mathbf{u},p_2)\}][\{\boldsymbol{H}(A_3,\mathbf{u},p_3)\}\{\boldsymbol{H}(A_4,\mathbf{u},p_4)\}] = \{\boldsymbol{C}(A_1,\mathbf{u})\}\{\boldsymbol{C}(A_3,\mathbf{u})\}$ if $A_2 \in$ line $(A_1, \mathbf{u})$ and $A_4 \in$ line $(A_3, \mathbf{u})$. We have $\{\boldsymbol{C}(A_1,\mathbf{u})\} \cap \{\boldsymbol{C}(A_3,\mathbf{u})\} = \{\boldsymbol{T}(\mathbf{u})\}$. Hence, if two axes are collinear, then, the other two axes must not be collinear. For instance, the open chain of Fig. 23a is a defective chain for the generation of *X*-motion. The subgroups $\{\boldsymbol{C}(A_i,\mathbf{u})\}$ with either (i= 1 or 3) or (i= 1 and 3),can also be generated by PH or HP arrays (PR or RP when the pitch is zero) if the P is parallel to the H axis (R axis). These defective generators are shown in Fig. 23b–f. It is noteworthy that a defective *X* generator happens when a revolute pair arbitrarily replaces any screw in these generators.

C2. $[\{\boldsymbol{H}(A_1,\mathbf{u},p_1)\}\{\boldsymbol{H}(A_2,\mathbf{u},p_2)\}][\{\boldsymbol{H}(A_3,\mathbf{u},p_3)\}\{\boldsymbol{H}(A_4,\mathbf{u},p_4)\}]$ can be equated to $\{\boldsymbol{C}(A_1,\mathbf{u})\}\{\boldsymbol{T}(Pl)\}$ with $\{\boldsymbol{C}(A_1,\mathbf{u})\} \cap \{\boldsymbol{T}(Pl)\} = \{\boldsymbol{T}(\mathbf{u})\}$; in this case, the plane Pl of vectors $\mathbf{s_3}$, $\mathbf{s_4}$ is parallel to $\mathbf{u}$. Consequently, if two screws are coaxial, then the plane of two P pairs must not be parallel to the screw axis. The chain in Fig. 24a shows this kind of defective generator. It is a defective chain with a passive exceptional mobility. Once more, the subgroup $\{\boldsymbol{C}(A_1,\mathbf{u})\}$ can also be generated by PH or HP arrays (PR or RP when the pitch is zero) when the P is parallel to the H axis (R axis), as shown in Figs. 24b and c. Here, special cases of Fig. 22 are discarded for simplicity.

Case D. If two adjacent pairs generate the same 1D subgroup, then, obviously, the open serial chain generates a 3D manifold included in the 4D subgroup $\{X(\mathbf{u})\}$. The required four DOFs of a generator of *X*-motion are not achieved.

Hence, two adjacent H or R pairs must not be coaxial with the same pitch and two adjacent P pairs must not be parallel. Moreover, in a PPP subchain two non-adjacent P pairs that are parallel remain parallel, what must be avoided, such as Fig. 26g. Chains belonging to this case are shown in Figs. 25 and 26, in which R pairs can replace H pairs.

To sum up, the defective *X*-motion generators are briefly tabulated in Table 3. These open chains have passive internal 1-dof mobility: the connectivity is 3 instead of 4. Moreover, their inversions are also defective chains for generating *X*-motion.

Table 3
Defective *X*-motion generators.

Type		*Defective X-motion generators*	
Case A Trivial chains associated to one 3D subgroup of {X(**u**)}		HHHH *P*HHH H*P*HH *PP*HH *P*H*P*H H*PP*H	pseudoplanar chains
		RRRR *P*RRR R*P*RR *PP*RR *P*R*P*R R*PP*R	planar chains (p= 0)
		PPPP	spatial translation
Case B A subchain has an internal 1-dof mobility of a trivial chain associated to a 2D subgroup of {X(**u**)}		[HHH]H P[HH]H [HPH]H [HH]PH [HHH]P P[HH]P [HPH]P [HH]PP [HHH]R P[HH]R [HPH]R [HH]PR	internal 1-dof mobility in an cylindrical subchain with three pairs
		[RHH]H [HRH]H [HHR]H P[RH]H P[HR]H [RPH]H [HPR]H [HR]PH [RH]PH	
		[RHH]P [HRH]P [HHR]P P[RH]P [RPH]P P[HR]P [HPR]H [HR]PP [RH]PP	
		[RHH]R [HRH]R [HHR]R P[RH]R [RPH]R P[HR]R [HPR]R [HR]PR [RH]PR	
		PHPH PHPR PRPH PRPR	
		HPPH HPPR RPPH RPPR (belongs also to Case D)	
		HPPP RPPP	PPP planar translation subchain
Case C The chain embodies the product of two dependent 2D subgroups of {X(**u**)}	**C1**	[HH][HH] PH[HH] HP[HH] PHPH PHHP [RH][HH] [HR][HH] [HR][RH] PR[HH] RP[HH] [RH][HR] PR[HR] RP[HR] [RH][RH] PR[RH] RP[RH] PRPH PRHP PHPR PRPR PRRP	two dependent cylindrical motions
		HPPH HPPR RPPR /two contiguous parallel Ps/ (belongs also to Case D)	
	C2	PP~[HH] PP~HP PP~RP	dependent planar translation and cylindrical motion
		PP~PH PP~PR (belongs also to Case B: PPPH PPPR)	
Case D Two pairs generate the same 1-dof motion		H[HH]H HH[HH] HP[HH] PH[HH] P[HH]H P[HH]P	contiguous coaxial H pairs with equal pitches
		H[RR]H HH[RR] HP[RR] PH[RR] P[RR]H P[RR]P R[RR]H HR[RR] RH[RR] RP[RR] PR[RR] P[RR]R	contiguous coaxial R pairs
		PPHH PPHP PPPH HPPH PPPH PPHR PPRH PPRP PPPR RPPH PPPR PPRR RPPR	contiguous parallel P
		PPPH PPPR	Noncontiguous parallel P

Symbolic notations of geometric conditions:

P means that the P is perpendicular to the parallel H (or R) axes

PH or HP : the P is parallel to the H (or R) axis; PP : the P pairs are parallel

[H…H] : 2 or 3 coaxial H (or R) pairs

H…H : the H pairs have equal pitches (& necessarily they have parallel axes)

PPP : three contiguous P pairs are parallel to a plane

PP~[HH] : the common axis of [HH] is parallel to the plane of PP

5. Conclusions

The paper is devoted to the primitive generators of Schoenflies motion, also called *X*-motion for conciseness. A set {$\boldsymbol{X}(\mathbf{u})$} of *X*-motions with a given vector **u** orienting the axes of its feasible rotations is endowed with the algebraic structure of a four-dimensional Lie group. In this paper, the kinematics and the Lie group algebraic properties of a Schoenflies displacement set are explained. The type synthesis method that is based on the closure of the product in an *X* subgroup leads to a comprehensive enumeration of all possible general architectures of *X*-motion generators, which generators are made of serial concatenation of Reuleaux lower pairs or hinged parallelograms. The special cases of chains that are permanently defective in the generation of X-motion are also detected by using group dependency. The transitory failure of a chain in the generation of X-motion will be addressed in another paper.

Acknowledgements

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