MECH 576
Geometry in Mechanics

September 16, 2009

Inspiration, the Essential Grassmannian and Projectivity

1 Inspiration

Four of my favourite quotes, that pertain to geometric thinking, are written below. The second is from a modern book on classical geometry and, despite the title, applies equally to projective geometry, in general. The third is about *geometric* abstraction and its rewards. Like orgasm, its sensation is difficult to describe and is best conveyed by experience. The fourth bears witness to the notion that if you know geometric fundamentals thoroughly, you can use these to solve many apparently unrelated mathematical problems.

- "All geometry is projective geometry." -Arthur Cayley, 1821-1895.
- "Although projective geometry is, from the abstract viewpoint, nothing but linear algebra in disguise, it is a geometric language which allows a unified approach to such different things as Euclidean geometry of points, the differential geometry of ruled surfaces, or spherical kinematics."
  Pottmann, H. and Wallner, J. (2001) *Computational Line Geometry*, Springer, ISBN 3-540-42058-4, p.1.
- "The significant problems we face cannot be solved at the same level of thinking we were at when we created the." -Albert Einstein, 1879-1955.
- "How many proofs of Pythagoras' theorem do you know? I know 37!" -Pál Erdös, 1913-1996.

2 Essential Grassmannian

Fig. 1 shows a geometric interpretation of non-zero and zero valued determinants of square matrices that contain n rows of n point coordinates in a homogeneous, n-dimensional vector space that represents an $(n-1)$-dimensional projective space. B stands for "bulk", times a constant, a generalization that becomes "area" in two dimensions and "volume" in three. The non-zero determinant is multiplied by reciprocal of the factorial of the dimension of the projective space. Things become interesting when the top row (given) coordinates are replaced by those of a *variable* point and the determinant assumes zero value. As the area or volume becomes zero by the migration of the variable point into linear dependency the determinant assumes the form of a linear (scalar) equation in two or three variables. Notice how the line in 3D space is represented by *two* variable point rows and the Grassmannian matrix becomes doubly rank deficient.

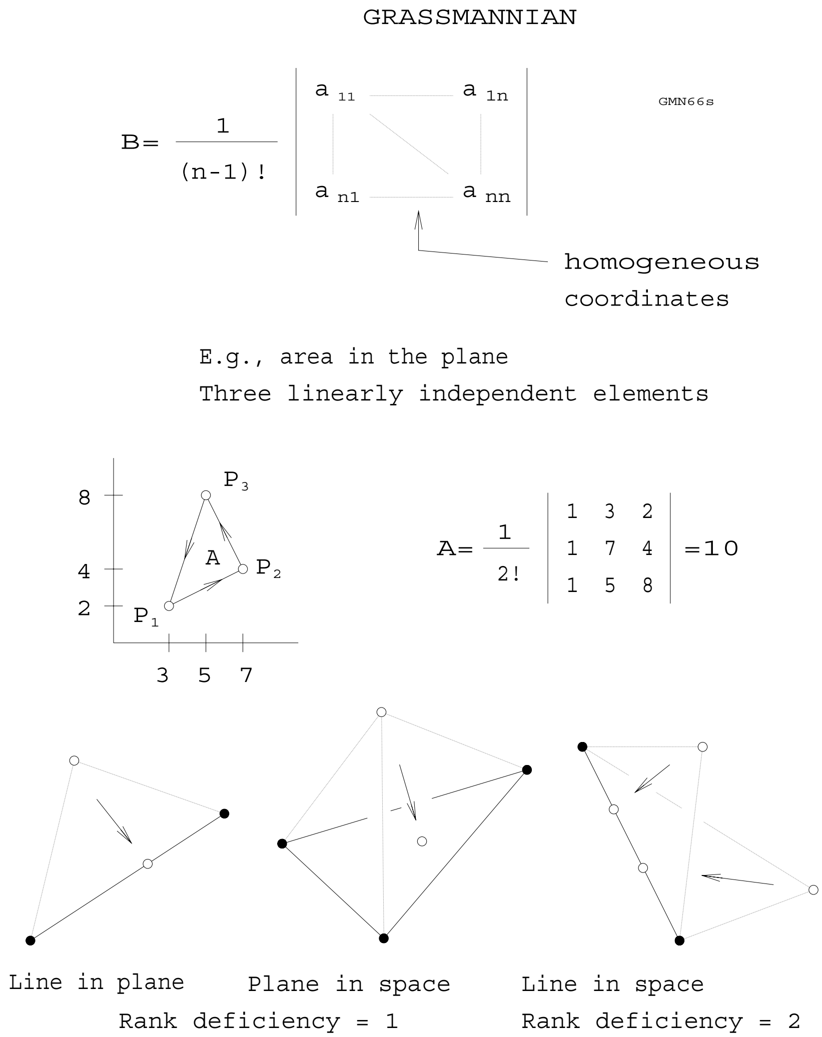

Figure 1: Zero and Non-Zero Grassmannian Determinants

3 Projectivity

Fig. 2 illustrates the geometric interpretation of a *projective collineation* or simply *projectivity.* In this case a mapping of point P, an element on line h, on plane $\mathcal{E}$, via a central perspectivity (point) projection on C, to plane Π. This produces images, line h' and point P' that in turn are mapped to h'' and P'' on Λ via central point C'. This double perspective mapping, except in special cases, like with lines g and f, represents a general *homogeneous transformation matrix operator*, used to transform vectors and other matrices. Some of these will be encountered and explained later. As regards f and g, because $C \cap f$ is parallel to $\mathcal{E}$, f has no *pre-image* on $\mathcal{E}$. Because $C \cap g$ is parallel to $\mathcal{E}$, g has no *image* on $\mathcal{E}$. The projective transformation is not (entirely) *bijective.* In simple, multi-view engineering drawings one encounters lines like g and h such that their images in adjacent views appear collinear, *e.g.*, a due north bearing line requires a side view in order to unambiguously map points between adjacent (conjugate) views; a third (x-)coordinate is missing. Then a third view or perspectivity is needed to completely represent that line's mapping.

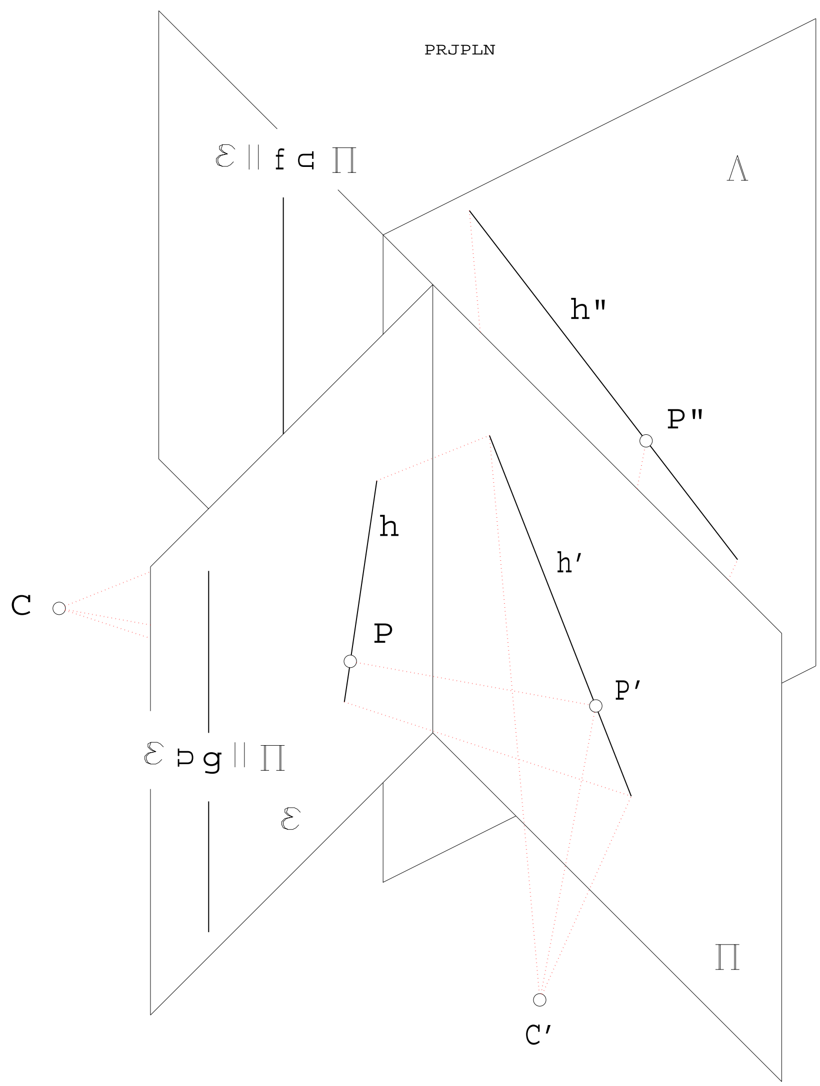

Figure 2: Projectivity and Perspectivity