MECH 314
Dynamics of Mechanisms

March 15, 2011

Inertia Force Reactions: Three Models

1 Overview

This pertains to planar statics. Simplified force transfer conventions in planar multi-body dynamics will be reviewed. Then two elementary approximation models of the equilibrium of three concentrated forces on a rigid body –or the replacement of the combined weight support and mass acceleration force vectors acting at the body mass centre, by two, acting at pin-joints (revolute (R)-joints), so as to maintain dynamic equilibrium with two adjacent bodies in the kinematic chain– are introduced. Although both are firmly based on first principles of structural mechanics these two models produce entirely different results. A reconciliation is proposed. It relies on what is believed to be a novel concept, *viz.*, choosing the point on the line of action (LoA) of the combined inertia force that minimizes the sum of squares of the two replacement forces. Application of the sound engineering design practice of choosing key dimensionless parameters yields a unified result that parametrically expresses the optimum point on the LoA in terms a point on the line of the R-joint centres and the direction of the instantaneous inertia force.

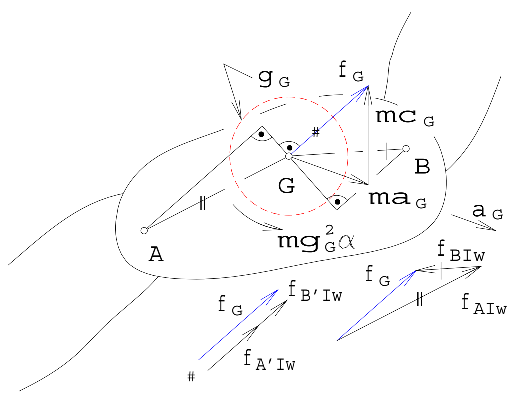

Figure 1: Dynamic Effects on a Planar Rigid Body in a Kinematic Chain

2 Introduction to Planar Multi-Body Dynamics

Consider Fig. 1. It shows a rigid body link connected to two adjacent ones by R-joints at A and B. Assuming no other significant sources of load on this link, forces $\mathbf{f}_A$ and $\mathbf{f}_B$

$$\mathbf{f}_A = \mathbf{f}_{AIw} + \mathbf{f}_{A*},\ \ \mathbf{f}_{AIw} = \mathbf{f}_{AI} + \mathbf{f}_{Aw} \text{ and } \ \mathbf{f}_B = \mathbf{f}_{BIw} + \mathbf{f}_{B*},\ \ \mathbf{f}_{BIw} = \mathbf{f}_{BI} + \mathbf{f}_{Bw} \ \text{ such that } \ \mathbf{f}_{A*} = -\mathbf{f}_{B*}$$

must act along a line on A and B, respectively, so as to sustain the inertia force $\mathbf{f}_{GI}$, the weight $\mathbf{f}_{Gw}$ (Note that $\mathbf{f}_G = \mathbf{f}_{GI} + \mathbf{f}_{Gw}$) and the inertia torque $\mathbf{\Gamma}$ given by

$$\mathbf{f}_{GI} = m\mathbf{a}_G,\ \ \mathbf{f}_{Gw} = m\mathbf{c}_G,\ \ \mathbf{\Gamma} = mg_G^2\boldsymbol{\alpha} \ \text{ so } \ \mathbf{f}_{GI} = \mathbf{f}_{AI} + \mathbf{f}_{BI},\ \ \mathbf{f}_{Gw} = \mathbf{f}_{Aw} + \mathbf{f}_{Bw},\ \ \mathbf{\Gamma} = \mathbf{r}_{BA} \times \mathbf{f}_{A*} = \mathbf{r}_{AB} \times \mathbf{f}_{B*}$$

Combining forces and moments like these, that arise due to diverse effect and act together, is called *superposition* [2]. Parameters that appear above are defined as follows.

- m is the mass of the link,

- $\mathbf{a}_G$ is the acceleration of the link mass centre,
- $\mathbf{c}_G$ is the *reverse* gravity acceleration vector whose magnitude is usually approximated by the well known constant 9.81m/s^2,
- g_G is the radius of gyration; a convenient way to lump the integral

$$|\mathbf{I}| = \int\int \rho(x^2 + y^2)dxdy = mg_G^2$$

  where x, y are taken with respect to G and the product $dxdy = dA$ represents an infinitesimal element of link area while $\rho(x, y)$ is the mass per unit area of the laminar link (g_G is just the square root of the second moment of area about the mass centre.),

- $\boldsymbol{\alpha}$ is the instantaneous angular acceleration of the link and
- $\mathbf{r}_{AB} = -\mathbf{r}_{BA}$ are the position vectors of B relative to A and *vice-versa*, respectively while $\mathbf{f}_{A*}$ and $\mathbf{f}_{B*}$ form an *ambiguous* couple on A and B to furnish $\boldsymbol{\Gamma}$.

All issues reviewed here are dealt with in detail in any "first course" text on engineering mechanics like Beer & Johnston [1].

2.1 Approach Rationale

Before examining how $\mathbf{f}_G$ may be reasonably distributed so as to be included in the resultant (vector sum) $\mathbf{f}_A + \mathbf{f}_B$ consider how the inertia torque $\boldsymbol{\Gamma} = mg_G^2\boldsymbol{\alpha}$, necessary to provide the instantaneous angular acceleration $\boldsymbol{\alpha}$, is to be furnished. Look at Fig. 2. The moment vector field of a torque is uniform over the infinite expanse of a planar rigid

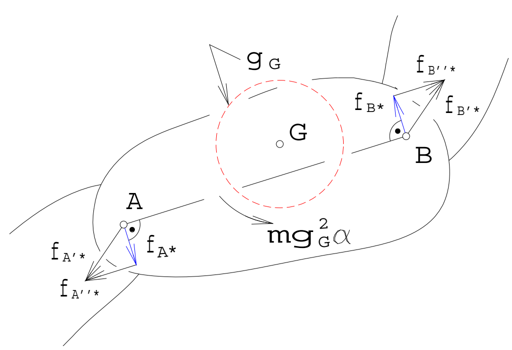

Figure 2: Inertia Torque and Applied Couple

body. It may be injected by any couple or pair of equal and opposite forces of appropriate magnitude. In the case of the link shown these are the forces $\mathbf{f}_{A*}$ and $\mathbf{f}_{B*}$ which, taken together, generate a moment $\boldsymbol{\Gamma}$ about *all* points on the link. Notice that a couple pair like $\mathbf{f}_{A'*}$ and $\mathbf{f}_{B'*}$ would do likewise. This ambiguity is resolved by choosing a pair of *minimum* magnitude. There is no reason to assume existence of components like $\mathbf{f}_{A''*}$ and $\mathbf{f}_{B''*}$, that provide no dynamic effect. This *principle of relaxation* is a ploy commonly applied in engineering mechanics. It will be used in what follows to distribute $\mathbf{f}_G$. It is of interest to observe that spatial line geometry was developed, at least in part, due to its relevance to *statics*; the study of the action and equilibrium of forces and their moments. One may regard homogeneous Plücker coordinates, *i.e.*, a *line*, as the model for a concentrated force of arbitrary magnitude and the moment it exerts about some convenient point, the chosen origin. A torque or couple falls neatly into this model as a force of infinitesimal magnitude acting at an infinite distance so as to generate a torque vector in the appropriate direction. These notions are pursued in [3] and the history of statics is extensively documented in [4].

3 Distribution and Ambiguity

One way to arrive at $\mathbf{f}_{AIw}$ and $\mathbf{f}_{BIw}$ is shown in Fig. 3. It is based on the fact that the pair satisfies force equivalence $\mathbf{f}_{AIw} + \mathbf{f}_{BIw} = \mathbf{f}_G$ and since their LoAs are along AG and BG, respectively, moment equivalence is also satisfied. This model follows the idea conveyed by what is called "method of joints" [1] used to analyze axial force members of a planar pin-jointed truss composed of triangular panels. In this example the sum of squares is almost three times

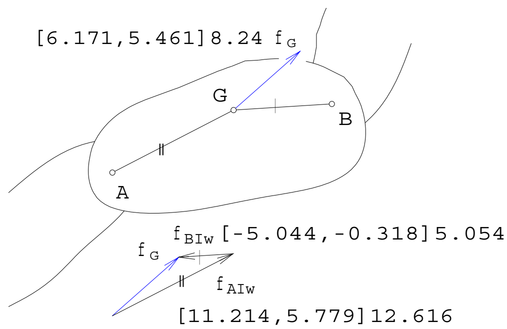

Figure 3: Method of Joints Model

the square of $\mathbf{f}_G$.

$$12.616^2 + 5.054^2 = 187.71 >> 8.24^2 = 67.9$$

Another way to establish this equivalence is shown in Fig. 4. In this case the sum of squares is identical $\mathbf{f}_G^2$. One can

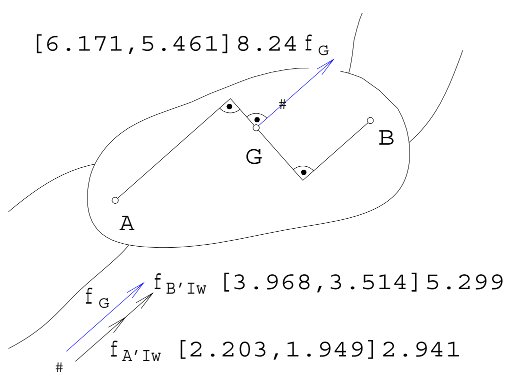

Figure 4: Simple Beam Model

do no better and thus might naïvely assume that choosing LoAs of $\mathbf{f}_{AIw}$ and $\mathbf{f}_{BIw}$ both parallel to $\mathbf{f}_G$, *i.e.*, so all three concur on the absolute line, is the optimal procedure. Alas, this is not so and a moments reflection will reveal that if LoA $\mathbf{f}_G$ intersects segment AB outside the interval then $\mathbf{f}_{AIw}^2 + \mathbf{f}_{BIw}^2 > \mathbf{f}_G^2$.

3.1 Disposition Ambiguity

Fig. 5 shows a counterexample where the concurrent joint force force model provides a lesser sum of squares of replacement forces at A and B, albeit still greater than $\mathbf{f}_G^2$, than does the parallel force beam model. The only thing one may say at this point is that if the angle between $\mathbf{f}_G$ and the segment AB is small then distribution according to concurrent joint forces is preferable. Can some unifying compromise be reached concerning these two models? How can the problem be parameterized so that the (beam) method, that relies only on the direction of $\mathbf{f}_G$, be treated together with the other (joint) method, that relies on this direction *and* the relative position of points A, B and G?

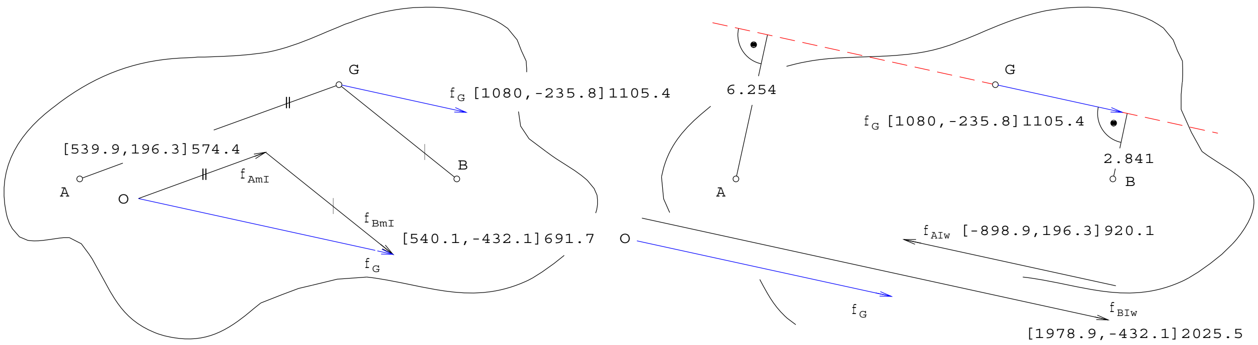

Figure 5: A Counterexample

4 Parametrization and Optimization

There is no loss of generality in adopting the scheme illustrated in Fig. 6, *i.e.*, choosing the interval between A and B to be of unit length and unit scaling the force $|\mathbf{f}_G| = 1$, too. Then the LoA of $\mathbf{f}_G$ is defined by its intercept with $A(0,0,), B(1,0)$ at $G'(g,0)$ and $\cos\theta$ where θ is the positive (CCW) angle that the LoA makes with AB. Now G

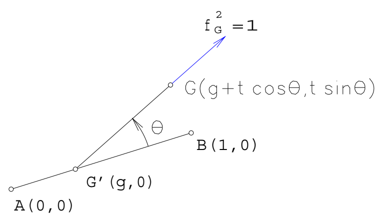

Figure 6: Parameterization

is located at some distance t from G' along the line defined by intercept distance g and angle θ and therefore can accommodate both of the models introduced above. With the joint model $\mathbf{f}_G$ and G are given and t and θ can be deduced therefrom. In the case of the beam model G can be deduced and obviously $t \to \infty$. What is more important however is that any point G along the LoA of $\mathbf{f}_G$ can be represented in terms of parameter t as can the sum of squares of distributed force magnitudes $\mathbf{f}_{AIw}^2 + \mathbf{f}_{BIw}^2$ at A and B. Minimizing this sum in terms of t will produce a good compromise. The actual point of application of $\mathbf{f}_G$ is not of concern regarding dynamic equivalence of the force system on the link. It is only necessary that it lies on the prescribed line. Applying the method of joints to a variable location $G(t)$ one obtains

$$(g + t\cos\theta)k_A + (g + t\cos\theta - 1)k_B - \cos\theta = 0, \;\; t\sin\theta k_A + t\sin\theta k_B - \sin\theta = 0 \;\to\; k_A = \frac{1-g}{t}, \;\; k_B = \frac{g}{t}$$

It is proposed to minimize the following sum of squares.

$$\{(g + t\cos\theta)k_A\}^2 + \{(g + t\cos\theta - 1)k_B\}^2 + (t\sin\theta k_A)^2 + (t\sin\theta k_B)^2]_{min}$$

Taking the derivative with respect to t, multiplying out the denominator t^3 and simplifying imposes the following necessary condition on t.

$$t = 2\frac{g(1-g)}{(2g-1)\cos\theta}$$

So $t = 0$ at $g = 0$ and $g = 1$, $\forall\,\theta$. Otherwise the expression above yields t for any other combinations $(g, \cos\theta)$.

4.1 Function and Example

To visualize $t(g, \cos\theta) = 0$ it is more convenient to show, instead of t, t^{-1} which is smooth over the complete range except for expected spikes at $g = 0$ and $g = 1$. Fig 7 displays t^{-1} plotted against $-1.5 \leq g \leq 1.5$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is not necessary to look at g far beyond because the function soon flattens out toward the plane $t^{-1} = 0$. Both model predictions tend to converge to $\frac{1}{t} \to 0$ and $t \to \infty$ at large values of $\pm g$ as AB becomes more and more like an axial force member.

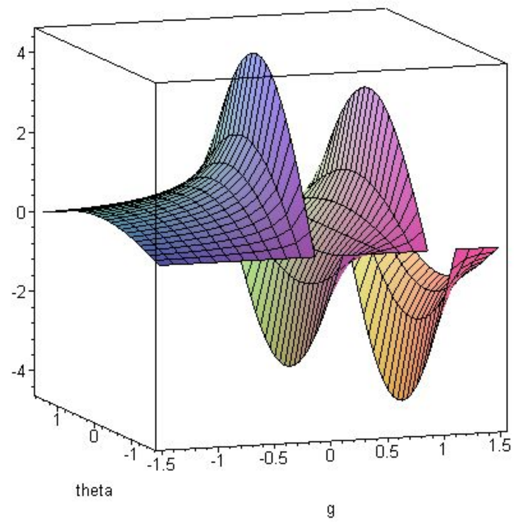

Figure 7: The Function t^{-1}

The superior joint model example shown on the left in Fig. 5 is reexamined by noting that the serendipitous choice of G and direction $\mathbf{f}_G$, which were taken arbitrarily, is confirmed to be optimum upon calculating t with the observed values shown as shown in Fig. 8.

$$\theta = 167.68^\circ, \quad g = 1.8326 \;\; \text{yields} \;\; t = 1.1719$$

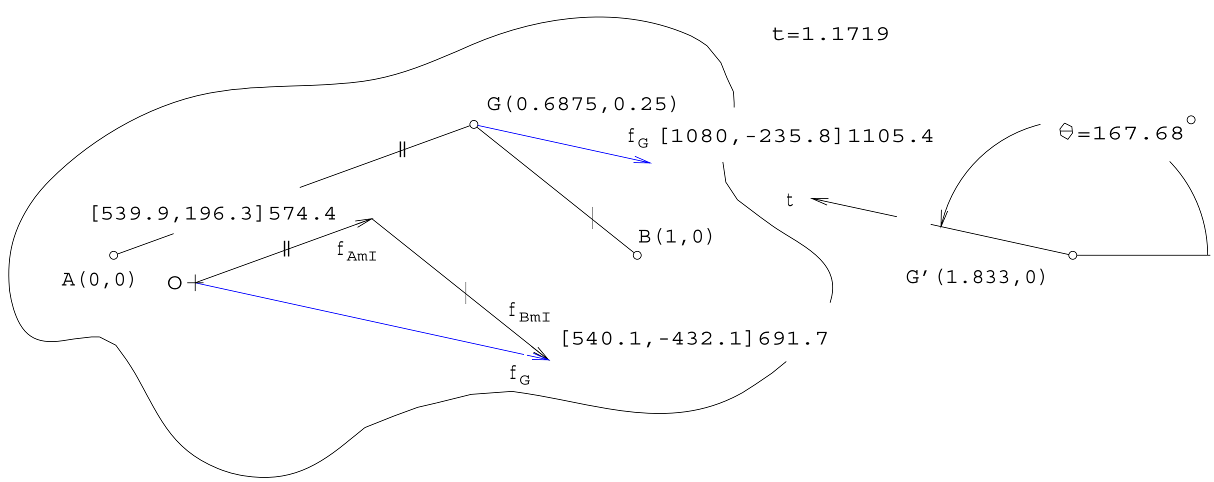

Figure 8: Confirmation of Optimum t with an Example

5 Conclusion

Although this simple exercise does not exactly make use of line geometry in its full-blown spatial glory it does treat the notion of concentrated forces as lines. Moreover it relates this to elementary structural mechanics and rigid body dynamics in the plane and introduces a novel optimization metric, all in the context of simple planar geometry and elementary calculus [5]. This investigation was undertaken so that one might reasonably estimate forces $\mathbf{f}_A$ and $\mathbf{f}_B$ applied to an isolated planar rigid body in some given instantaneous state of motion without the usual artifact of embedding it as, say, the coupler or connecting rod of a 4-bar or slider-crank mechanism. Such usual textbook cases require the simultaneous solution of *three* free-body equation sets and the imposition of, arguably, equally artificial assumptions in the form of given conditions at the anchor points, usually R- or P-joints. As regards slightly extending this avenue of inquiry one might investigate optimum situations where neither $(\mathbf{f}_A \parallel \mathbf{r}_{AG}) \cap (\mathbf{f}_B \parallel \mathbf{r}_{BG})$ nor $\mathbf{f}_A \parallel \mathbf{f}_B \parallel \mathbf{f}_G$ applies. Might not the parametrization described in Fig. 6 and subsequent approach to force minimization be useful in modeling the dynamics of a 4-wire planar cable driven manipulator supported by four triplets like $\mathbf{f}_A,\ \mathbf{f}_B,\ \mathbf{f}_G$?

References

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