MECH 314
Dynamics of Mechanisms

January 12, 2011

Introduction and Basic Concepts

1 A Rigid Body

Below in Fig. 1 the concept of a planar or spatial rigid body is illustrated. The two blobs or "space potatoes" as I sometimes call them are made of a rigid solid material that constitutes the the material out of which mechanism or machine "links" are imagined to be made. They don't expand, contract or distort in any other way that affects real material. This idea allows us to study pure motion without interference of reality in the form of elasticity, thermal expansion, viscosity or other phenomenæ that detract from an ideal situation. The two examples shown are intended to represent a planar and a spatial rigid body. A rigid body is characterized by two essential properties.

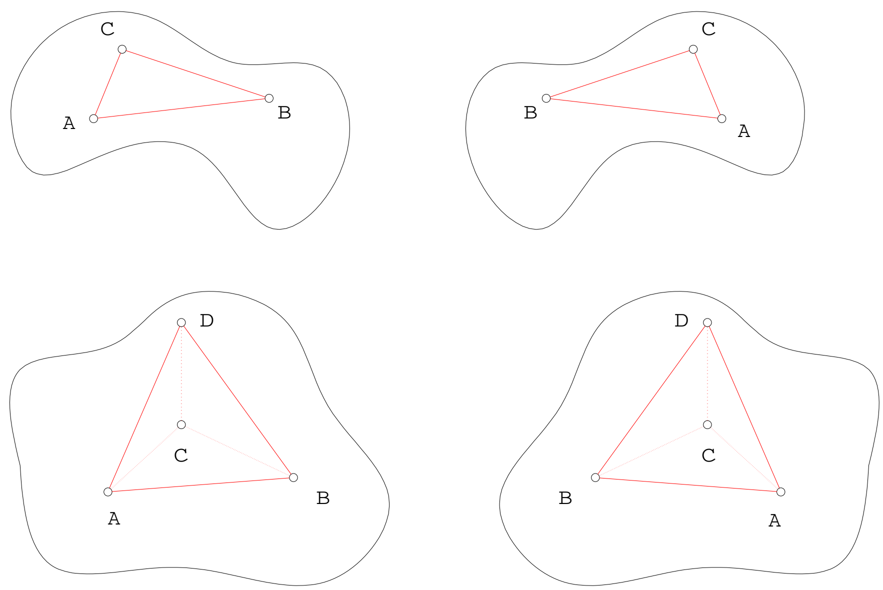


Figure 1: Rigid Body Concept

- The distance between any two points, so as to move with it, fixed to the body remains constant, *i.e.*, does not change. *E.g.*, this might be the distance between points A and B on either body. This distance is, in the respective cases

$$\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \text{ (planar case) } \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \text{ (spatial case)}$$

- Reflections such as the turning over of the planar triangle, shown above as $ABC \rightarrow BAC$, and the turning inside-out of the spatial tetrahedron, shown above as $ABCD \rightarrow BACD$, are not allowed regardless of distances between points being preserved.

2 Joints

Mechanism and machine parts or *links* nevertheless move with respect to one another because adjacent links are connected by articulations called *joints*. Fig. 2 shows a number of these. The only two that can occur in planar mechanisms, the main topic in this course, are *revolute* joints (R-joints) and translational *prismatic* joints (P-joints). These are one degree of freedom joints (1dof). Since planar motion enjoys only three degrees of freedom (3dof) with two of translations, say, in x and y, and one of rotation, say, about axes with directions parallel to z, P-joints and R-joints both *remove* 2dof between any pair of planar rigid bodies they connect. In the spatial case with 6dof, P- and R-joints remove 5dof.

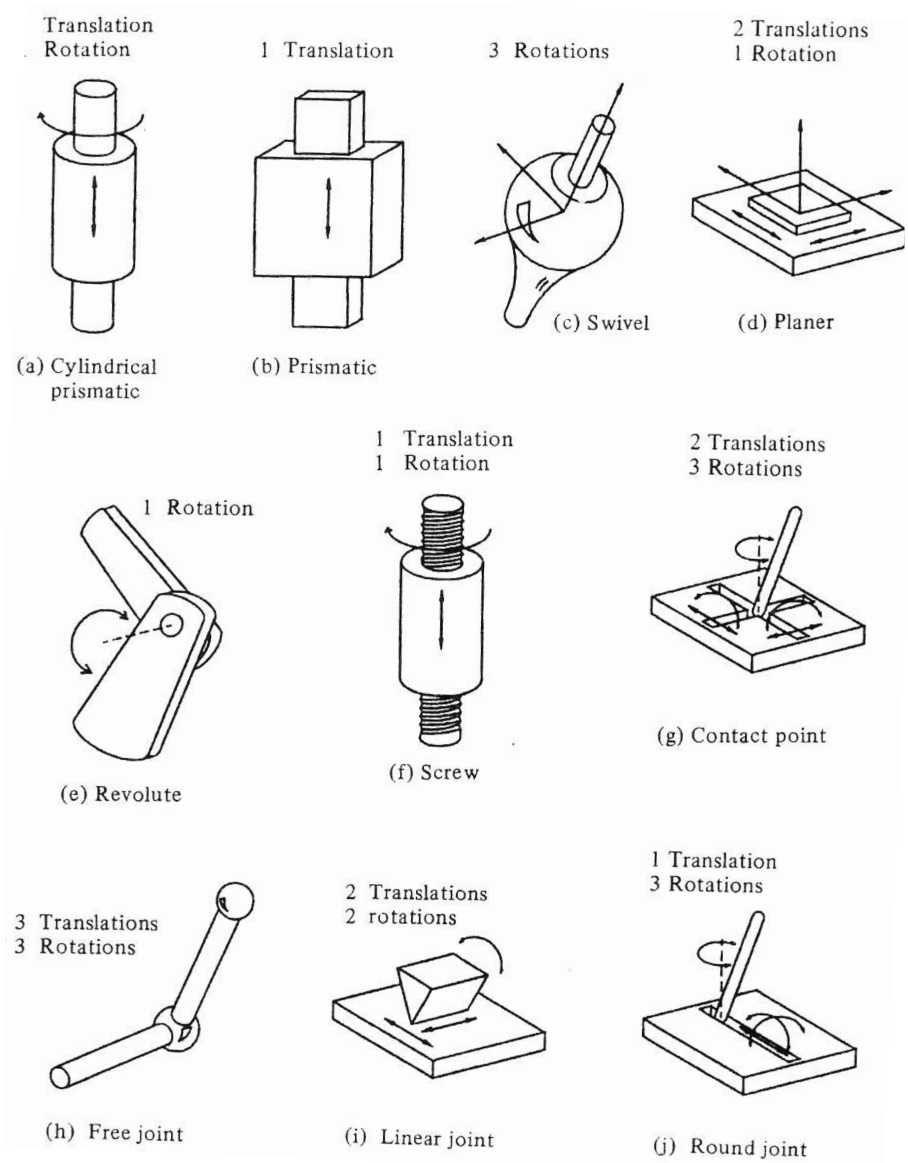


Figure 2: Some Mechanical Joint Schematics (Taken from [1])

1. *Lower pair* joints are those where relative motion occurs on *surface* rather than line or point contact. Which of the above are lower pairs? The Chebychev-Grübler-Kutzbach relation (CGK), to be introduced next, applies only to mechanisms containing only lower pairs. Is contact between a pair of cylinders, that roll without slipping on each other, a lower pair contact?

2. Why cannot a cylindrical joint be used in planar mechanisms? After all planar motion admits 2dof of translation and 1dof of rotation and a cylindrical joint admits 1dof of each.

3 The Chebychev-Grübler-Kutzbach Relation (CGK)

This "magic formula", that is sometimes used to determine the mobility m –m_e stands for mobility of planar systems, m_r for 3D spatial ones–, or dof, of a kinematic chain, loop or network, doesn't always work but at least it establishes a *lower mobility bound* or the lowest dof number that a system can have. It is based on four parameters.

- The inherent number of dof, d in the space under consideration: *E.g.*, 3D Euclidean space admits d =6dof; 3 translational degrees and 3 rotational degrees. Motion in planes is confined to d =3dof; a rotation and 2dof translation.
- The number of links n, or rigid bodies, in the mechanism under consideration.
- The number j of joints, of each dof type i, that articulate the mechanism.
- The number $d - i$ of dof that any given joint type *removes* from the inherent number of dof in the space.

$$m = d(n-1) - \sum_{i=1}^{d-1}(d-i)j_i, \;\; m_e = 3(n-1) - 2j_1 - j_2, \;\; m_r = 6(n-1) - 5j_1 - 4j_2 - 3j_3 - 2j_4 - 1j_5$$

It is not strictly true that only 1dof P- and R-joints can exist in planar mechanisms. Examples of 2dof exceptions include wheels rolling *with* sliding along straight or curved paths or with respect to one another and round pins that can move along and rotate in straight or curved slots of constant width. However somehow this sort of situation can be interpreted as a local P- and R-joint combination. Let us now try the couple of examples in Fig. 3.

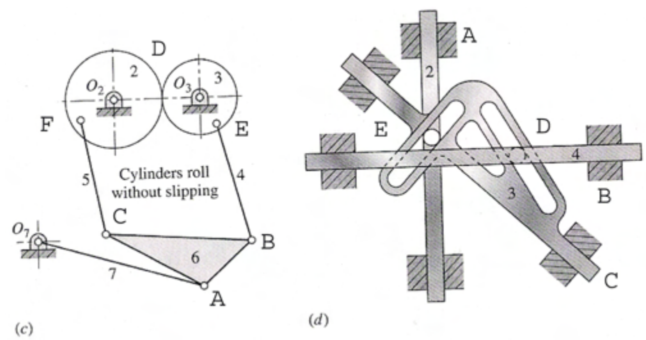


Figure 3: What is the Mobility of These Two Mechanisms? (Taken from [2])

Diagram (c) Contains $n = 7$ links, all conveniently numbered except for 1, taken as the "ground" link. There are 8 R-joints, $O_{2,3,7}, A, B, C, E, F$, some that were labeled after the fact. That is something you should do, when solving such problems, in order not to mis-count, the most common source of error in these situations. There is also a rolling-without-slipping cylinder pair in contact at D. All 9 joints are type $i = 1$ so $m_e = 3(7-1) - 9(2) = 0$. This concludes that the assembly has no mobility and will not move if one attempts to push, pull or turn any link. One may try to confirm this by disconnecting link 7 at joint A. Holding the cylinders fixed in any position, thus fixing joint positions F and E, establishes the motion of point A as that of a 4-bar mechanism coupler point. This curve is known to be of sixth degree and that certainly will not comply with the circular path of A rotating about O_7 at the radius of link 7. This seems to support the result obtained as m_e except for the fact that the distance between points E and F can be changed continuously over some range. This allows point A to "paint a patch" rather than a discrete curve. The patch will sustain an arc of the circular path centred on O_7. This is actually a 1dof mechanism! Another way to argue for 1dof is to disconnect the "geared" cylinders that leaves an n =7-link mechanism with 8 j_1 joints giving $m_e = 3(7-1) - 2(8) = 2$. Replacing the coupled motion of rolling without slipping removes an additional dof, again producing $m_e = 1$. The answer in the "back of the book" is actually wrong but technically correct when interpreted as an exercise in applying the CGK formula.

Result for example (d) is unequivocal. We are given $n = 4$. A, B, C are P-joints, $2j_1 = 2(3) = 6$, and D, E are 2dof PR-joint combinations, $1j_2 = 1(2) = 2$ so one obtains $m_e = 3(4-1) - 6 - 2 = 1$dof.

4 Four-Bar Mechanism Types and Properties

Here will be introduced and discussed *Grashof's Law*, *inversion*, *crank-rocker*, *drag link*, *double rocker*, *slider-crank*, *time ratio* and transmission angle, all in the context of four link planar 1dof mechanisms.

4.1 Grashof's Law,etc.

This equation determines whether any link in a 4-bar mechanism can turn full circle. The equation is written as

$$s + l \leq p + q$$

where s is the length of the shortest link, l is that of the longest while p and q are the lengths of the remaining two. Illustrated in Fig. 4 is such a mechanism wherein the equality holds. Examining the two half-scale images on the right one sees that this particular *inversion* is a *crank-rocker*, *i.e.*, one link turns full circle while the opposite one swings though a smaller arc.

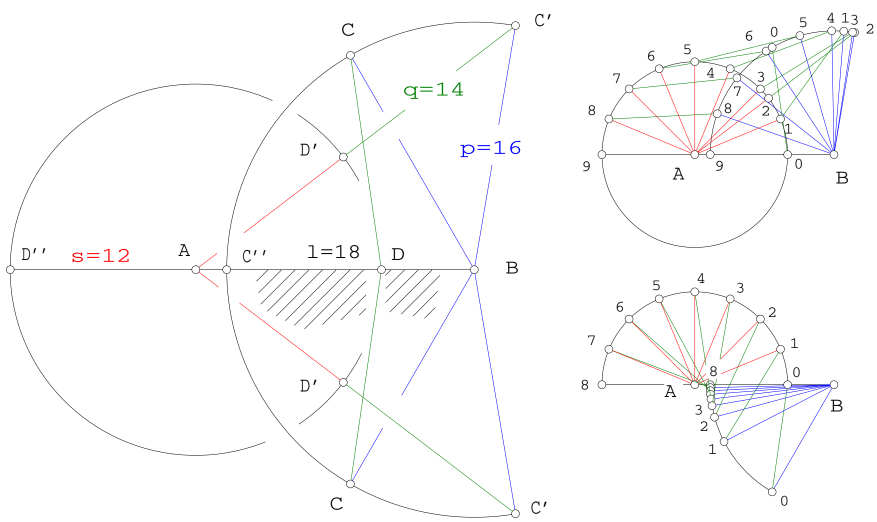


Figure 4: A Crank-Rocker Inversion and Its Two Assemblies

It would appear from Fig. 1.28 on p.36 of [2] that taking either the longest or next longest link as the base or fixed frame (FF) produces a crank-rocker. Choosing FF to be the shortest link seems to produce a drag-link mechanism where both links attached to FF can execute full turns while if the shortest link chosen to be the *coupler*, opposite to FF, the mechanism becomes a double-rocker and neither end of the coupler can go full circle. One should study the argument on pp.35-37 of [2], too. Note that "p, level" on p.37 was probably meant to be "p, lever". Every assembly sequence, there are six –press ${}_3P_3$ on your calculator– of four links has four inversions, depending on which link becomes FF. In this sense "inversions" means a specific instance of having chosen one of the four links to be FF.

4.2 Cycle Time Ratio and Transmission Angle

Crank-rocker mechanisms are useful in machine subassemblies where a link must reciprocate and be driven by a constant speed motor that turn the crank. Often the rocker (lever) must deliver more work in one direction than in the other, *i.e.*, called the working and return strokes, respectively. This means that the working stroke corresponds to a longer time period during which the rocker sweeps its entire angular range and to the greater part of the arc through which the crank rotates while always turning at the same speed. The return stroke corresponds to a shorter time interval fraction and smaller crank angle. The time ratio Q of longer time period to shorter is identical to the ratio larger α to smaller β crank angle displacement.

$$Q = \frac{\alpha}{\beta}$$

Before designing a crank-rocker, given the lever length $p = BC$, the angle δ that it must swing though and a desired value of Q, let us reexamine Fig. 4. Recall that its links fulfills the Grashof condition as an equality; a limiting case. This causes it to behave strangely as regards its time ratio. Assume that the crank is turning counter-clockwise in the initial configuration 0 shown in the in the top diagram on the right. Notice that the rocker sequence 0-1-2 is clockwise before proceeding counter-clockwise 2-3- . . . 9. As it passes position 9 after the first half turn the remaining half turn can be imagined by reflecting the top diagram about the horizontal circle diameter but proceeding from left to right backward through the number sequence. But now it is in the 0 configuration shown below and the second revolution is illustrated by the sequence illustrated below and its reflected reverse sequence for the second half turn. The full cycle of two crank revolutions sees the rocker swing through the larger fan, *i.e.*, the upper one and its reflection in the first revolution, and then, similarly, through the small fan. Notice that the mechanism is in a condition called "singularity" when any three of its R-joints are in line. There are four such positions 0, 2, the reflection of 2 and 9 this last being identical to 8 in the lower diagram. This cautionary tale is meant to

- Introduce the geometry of singularity in 4-bar mechanisms,
- Show that most probably inertia will carry the crank and coupler through a singular position in the expected manner and
- It is advisable to stay away from limiting cases when using design inequalities like the Grashof condition.

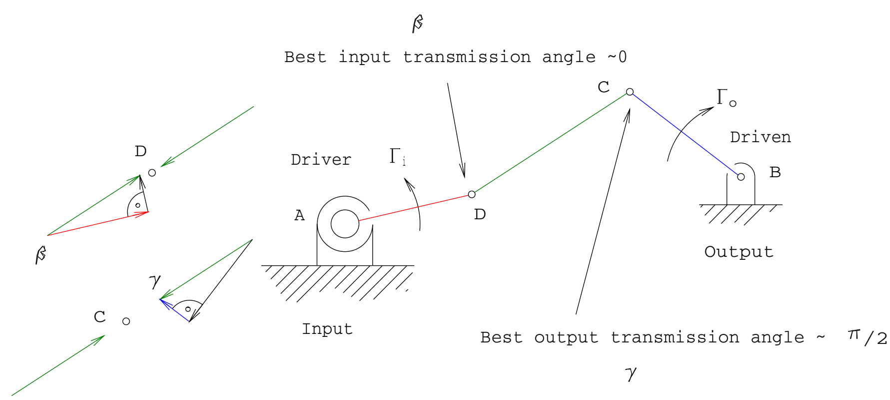


Figure 5: Input and Output Transmission Angles

Fig. 5 illustrates the notion of transmission angles. The central idea is

- That a small input transmission angle, the deflection angle β –not to be confused with β, the smaller angle and denominator of time ration Q– between crank link $s = AD$ and coupler $q = CD$ tends to
- Maximize the axial force on the coupler, given an input –motor– torque Γ_i that produces a force $F = \Gamma_i/s$ normal to the crank at D and
- That a large output transmission angle γ between coupler and rocker $p = BC$ tends to
- Maximize the torque, produced by the axial force on the coupler at point C, about the R-joint at B on FF.

This is shown by the two force equilibrium diagrams, at the left of Fig. 5, that define force equilibrium at R-joints D and C. All this can be summarized by expressing the relation between input and output torques Γ_i and Γ_o.

$$\Gamma_o = p\frac{\Gamma_i}{s} \csc\beta \sin\gamma$$

5 Designing a Crank-Rocker

Study Example 1.4. on pp.26-27 of [2] so as to apply it to the solution to Problem 1.20 on p.43. The purpose here is to provide some insight, into this and other design problems, that is not available in the text. *I.e.*, the instruction to draw the X-line in an arbitrary direction gives rise to question if not confusion. The nature of this problem is best illustrated by Fig. 6 and a little bit of "geometric thinking".

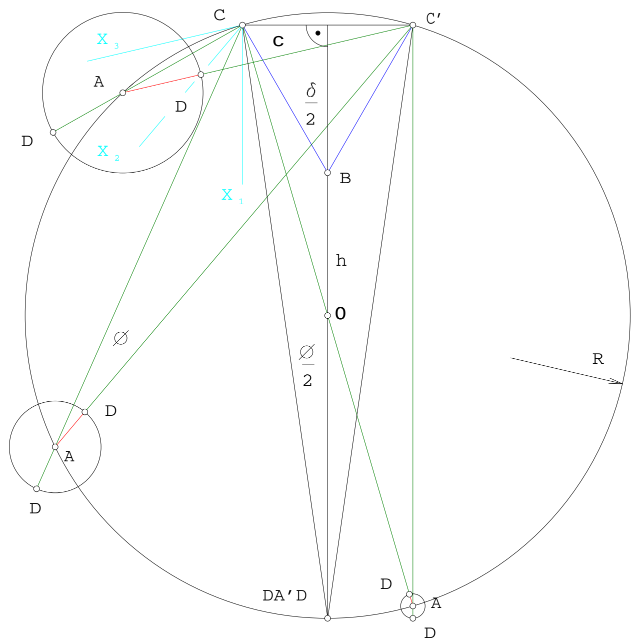


Figure 6: Designing a Crank-Rocker with $Q = 1.2$ and $\delta = \pi/3$

- Two singularities ADC' and DAC where the crank and coupler are collinear define the the time ratio angles $\alpha + \beta - 2\pi = 0$ like hands of a clock. Singularity is the hallmark of the rocker range limits shown in blue and ending on C and C'.
- Since the angle between ADC' and DAC must be ϕ

$$Q = \frac{\pi + \phi}{\pi - \phi} \rightarrow 1.2(\pi - \phi) - (\pi + \phi) = 0 \quad \therefore \ \phi = \frac{\pi}{11} \approx 16.3636 \ldots \ ^\circ$$

- Since the angle ϕ remains constant at vertex A regardless of where it is located A must lie on a circle subtended by the chord CC'.
- The next observation is that the "elbow folded" and "elbow extended" configurations of DAC and ADC', respectively, define lengths s and q in terms of the short $AC = q - s$ and long $AC' = q + s$ rays subtended from chord CC'.
- This problem has another degree of freedom. The designer is free to choose, say, s rather than the arbitrary direction of the X-line.
- Let us start by finding the centre O and radius R of the circle on which A lies.
- When $AC = AC'$ the triangle is isosceles and $s \rightarrow 0$. Now we have a third point for the circle on $CC'A'$ and we can write its equation, where $h = c \cot \frac{\phi}{2}$, $c = p \sin \frac{\delta}{2}$, using an origin midway between C and C'.

$$\begin{vmatrix} w^2 & wx & wy & x^2 + y^2 \\ 1 & -c & 0 & c^2 \\ 1 & c & 0 & c^2 \\ 1 & 0 & -h & h^2 \end{vmatrix} = x^2 + y^2 + \left[h - \frac{c^2}{h} \right] y - c^2 = 0$$

This gives the centre $O(0, y_O)$ and the square of the radius R^2 as

$$y_O = \frac{c^2 - h^2}{2h}, \quad R^2 = y_O^2 + c^2$$

Now we come to the crux of the design problem. Assume that we wish to choose the crank radius s. We specify the required unknown position of A using a polar coordinate system centred on O, Call u the tangent of the half angle that locates A on the circle or radius R.

$$A(x_A, y_A): \quad x_A = R\cos\theta, \quad y_A = R\sin\theta, \quad \cos\theta = \frac{1-u^2}{1+u^2}, \quad \sin\theta = \frac{2u}{1+u^2}$$

Using the elbow folded/elbow extended relationships for DAC and ADC' one may eliminate q and obtain what appears to be a rather ugly quartic –power four in u– relationship in s and u. Not too surprisingly, due to the symmetry evident in Fig. 6, it factors and simplifies to two quadratics, each containing a pair of dimensionless ratios ρ_1, ρ_2 in the design parameters s, c, ϕ, where $c = c(p, \delta)$.

$$u^2 - \frac{2u}{1 - 2\rho_1\rho_2/(\rho_2+1)} + 1 = 0, \quad u^2 + \frac{2u}{1 - 2\rho_1/(\rho_2+1)} + 1 = 0$$

where

$$\rho_1 = s^2/c^2, \quad \rho_2 = \cot^2\frac{\phi}{2}$$

The slider-crank design problem is obviously identical except CC' is taken as the wrist pin stroke rather than the chord of the arc traced by the tip C of the rocker that moves through the desired angle δ. If you have been paying attention you might be feeling a bit queasy about Q. What if $Q = 1$, a perfectly reasonable time ratio specification, so as to yield $\phi = 0$? No problem. Imagine a circle of infinite radius whose circumference contains CC', a horizontal line! Almost any point A on this line will constitute a feasible slider-crank or crank-rocker mechanism with $Q = 1$. Don't ask how to design one where $Q = 0$.

6 Conclusion

Examine problem P1.20 on p.43 of [2] again. Using both formulæ in u, results obtained, for a choice of $s^2/c^2 = 1$ are summarized in Fig. 7. Both equations in this limiting case yielded the same pair of polar angles, *i.e.*, $73.36°$ and $106.64°$, centred on O. Why were negative values obtained? That makes me unhappy. Can you help? Try some other values of Q to check out the validity of my results.

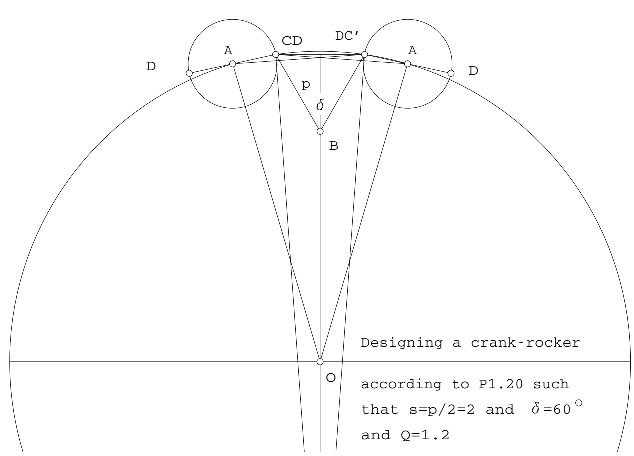


Figure 7: Designing a Crank-Rocker with $Q = 1.2$, $\delta = \pi/3$ and $s^2/c^2 = 1$

7 Reprise

For those readers who find the preceding treatment to design a crank-rocker or slider-crank "quick-return" mechanism difficult to follow, a different possibly simpler, approach will be described. Examine Fig. 8.

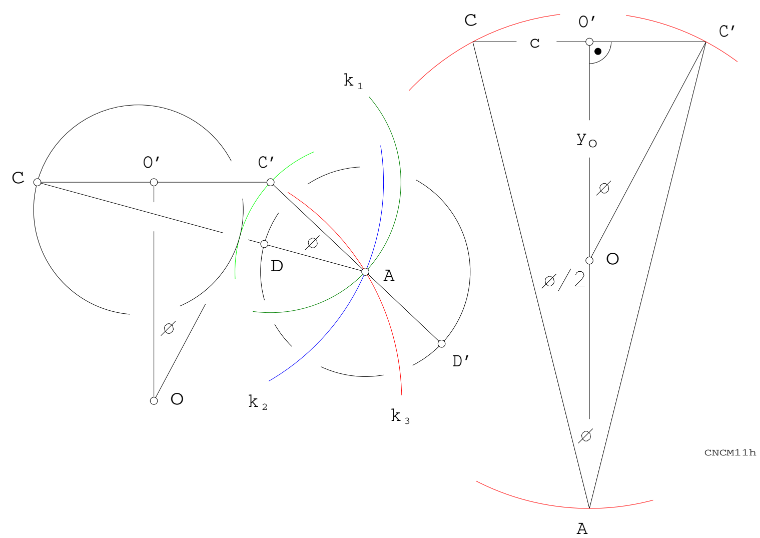


Figure 8: Designing a Crank-Rocker with a Geometric Approach Based on Intersection of Three Circles

- The two singular poses ADC and $D'AC'$ of "three-points-in-a-line" that separate the crank arcs $\alpha = \pi + \phi$ and $\beta = \pi - \phi$ show the angle ϕ to be subtended on the chord CC' of the arc traced by the oscillating rocker tip. When designing a slider-crank CC' is the gamut of the piston or wrist pin centre.
- Regardless of where the crank centre A is located the line segments AC and AC' represent the sum and difference of coupler length q and crank throw radius s, *i.e.*, $q + s$ and $q - s$, respectively.
- Therefore the isosceles triangle ACC', shown on the right of Fig. 8, represents the limiting case $s \to 0$.
- But since ϕ is the vertex angle at A, regardless of where A is situated, the locus of A must be a circumscribing circle on the three points ACC'.
- Still concerning the picture on the right, to find y_O, the distance OO' from the centre of the chord CC' to the the circumscribing circle centre O, one may invoke the following chain of logic applied to another isosceles triangle AOC' whose equal sides of length R are the segments AO and $C'O$.

$$\angle C'OO' + \angle AOC' = \pi, \quad \angle OAC' = \angle OC'A = \frac{\phi}{2}, \quad \angle OAC' + \angle OC'A + \angle AOC = \pi, \quad \therefore \ \angle C'OO' = \phi$$

- Normalizing by setting $R = 1$ reduces all distance parameters to multiples of R and leads to further mathematical simplification.
- Thus $y_O = \cos\phi$ and $c = \sin\phi$. Choosing an origin on O', all points A, a one parameter set, can be represented on the intersection of any pair of the three following circles $k_{1,2,3}$ shown on the left of Fig. 8.

$$k_1 : \ (x - c)^2 + (y - y_O)^2 - (q - s)^2 = 0 \ \text{ is the green circle, centred on } C', \text{ radius } q - s.$$
$$k_2 : \ (x + c)^2 + (y - y_O)^2 - (q + s)^2 = 0 \ \text{ is the blue circle, centred on } C, \text{ radius } q + s.$$
$$k_3 : \ x^2 + y^2 - 1 = 0 \ \text{ is the red circle, centred on } O, \text{ radius } R = 1, \text{ the circumscribing circle.}$$

- The yellow circle represents the subtraction $(q + s) - (q - s) = 2s$ and one sees immediately the diameter $2s$ of the of the circle swept by the rotation of the crank-pin at D rotating about the fixed crank-shaft axis on A.

- The analytical procedure to be described now entails the elimination of x, y, q from $k_{1,2,3}$ and making the appropriate substitutions to obtain two implicit quartics that are plotted as surfaces in ϕ, θ, s^2 and ϕ, θ, ρ where θ is the polar angle measured from the ray on O proceeding to the right and $\rho = q/s$.

First the differences $k_1 - k_3$ and $k_2 - k_3$ are formed to eliminate x^2 and y^2. Then q is eliminated from these two differences and the following substitutions are made.

$$x = \cos\theta, \;\; y = \sin\theta, \;\; c = sin\phi, \;\; y_O = \cos\phi$$

After some algebraic simplification one obtains a quadratic in s^2.

$$s^4 + 2(\sin\theta\cos\phi - 1)s^2 + \cos^2\theta\sin^2\phi = 0$$

This first design equation expresses the square s^2 of the crank radius s as a function of ϕ and the angle θ from O to a point A on the circumscribing circle. A plot of this function appears in Fig. 9. Assume that the designer prefers

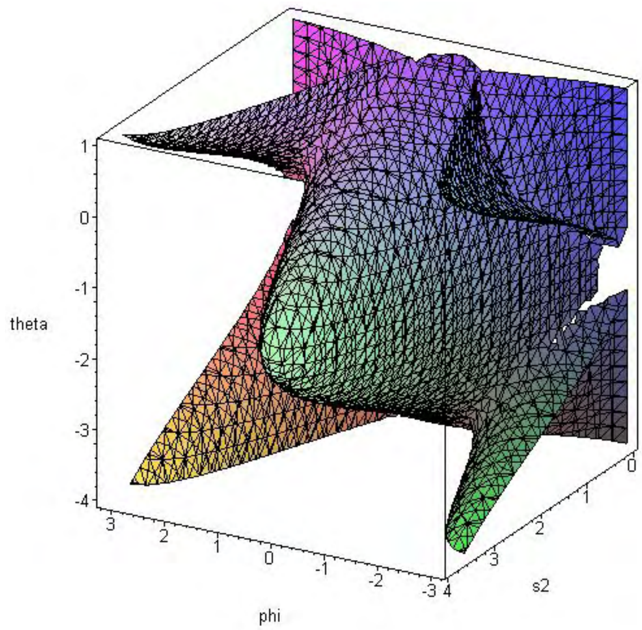


Figure 9: Crank radius s as a Function of ϕ and θ

to select, not s but the ratio $\rho = q/s$, the ratio coupler length to crank radius. Forming the difference of equations $k_1 - k_2$ yields qs. Then substitution for s^2 in terms of ρ gives us the design equation as a function of ρ instead of s..

$$qs - \cos\theta\sin\phi = 0, \;\; \therefore s^2 = \frac{\cos\theta\sin\phi}{\rho}, \;\; \cos^2\theta\sin^2\phi\rho^2 + 2(\sin\theta\cos\phi - 1)\rho + \cos^2\theta\sin^2 = 0$$

A plot of this function appears in Fig. 10. Some observations regarding these design equations are in order but these will be saved for later. Consider that the elimination process in this second method is far simpler than that in the first. Furthermore the issue of organizing dimensionless design ratios has been considerably better handles this time and the symmetry of the problem has been more fully taken advantage of.

References

[1] M. Shahinpoor (1987) *A Robot Engineering Textbook*, Harper & Row, ISBN 0-06-04-5931-X, p.84.

[2] J.J. Uicker, Jr., G.R. Pennock and J.E. Shigley (2011) *Theory of Mechanisms and Machines, 4th ed.*, Oxford, ISBN 9780-19-537123-9, p.41, P1.5 (c), (d).

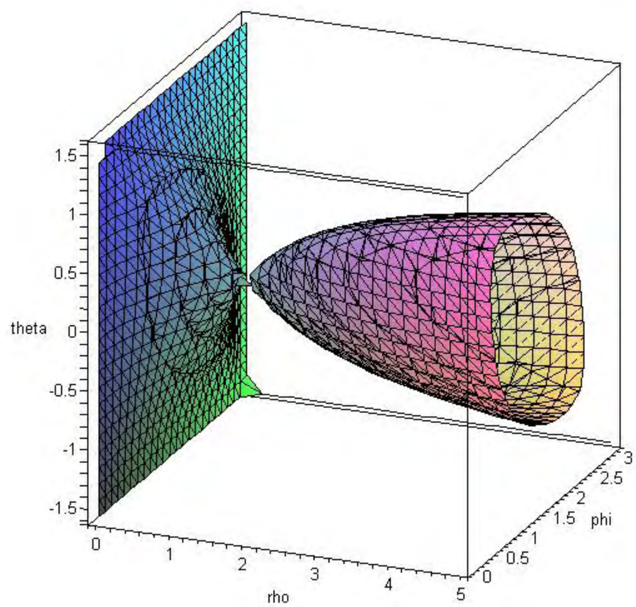
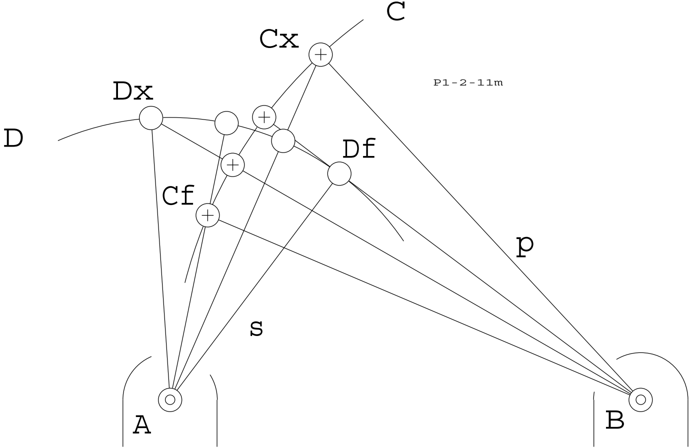


Figure 10: Coupler Length to Crank radius Ration ρ as a Function of ϕ and θ


C
Cx
P1-2-11m
Dx
D
Df
Cf
p
s
A
B