MECH 576
Geometry in Mechanics

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Hyperboloid of One Sheet on Three Given Skew Lines Revisited

1 Introduction

Consider the formulation of the implicit *point* equation $f(s_1, s_2, s_3) = 0$ of a hyperboloid of one sheet which is defined as a ruled surface swept by a radial line

$$\mathcal{S}_r\{s_{01} : s_{01} : s_{01} : s_{23} : s_{31} : s_{12}\}$$

moving so as to remain in intersection on three, given by their axial coordinates, as expressed below.

$$\begin{aligned} P_{01}s_{01} + P_{02}s_{02} + P_{03}s_{03} + P_{23}s_{23} + P_{31}s_{31} + P_{12}s_{12} &= 0 \\ Q_{01}s_{01} + Q_{02}s_{02} + Q_{03}s_{03} + Q_{23}s_{23} + Q_{31}s_{31} + Q_{12}s_{12} &= 0 \\ R_{01}s_{01} + R_{02}s_{02} + R_{03}s_{03} + R_{23}s_{23} + R_{31}s_{31} + R_{12}s_{12} &= 0 \end{aligned} \tag{1}$$

A neat way to do this is to homogeneously solve a system of five linear equations for the six radial Plücker coordinates s_{ij} of an arbitrary, variable ruling line $\mathcal{S}$ expressed in terms of the homogeneous coordinates $S\{s_0 : s_1 : s_2 : s_3\}$ of an arbitrary, variable point S, $S \in \mathcal{S}$. Inserting these four expressions into the inner product representing the Plücker quadric produces the implicit equation of the hyperboloid, in terms of point coordinates, after a degenerate quadratic factor is removed. The purpose of this article is to describe

- Details of problem formulation,
- Systematically, the ten quadratic form coefficients of the implicit equation and
- The significance of the two points implied by the removed, degenerate factor.

2 Formulation

The first three of the five equations necessary to define ruling line $\mathcal{S}$ are given by Eq. 1. Two more equations are available from the doubly rank deficient set of four equations Eq. 2.

$$\begin{array}{rrrrrr} S_0 = & & S_{01}s_1 & +S_{02}s_2 & +S_{03}s_3 & = 0 \\ S_1 = & -S_{01}s_0 & & +S_{12}s_2 & -S_{31}s_3 & = 0 \\ S_2 = & -S_{02}s_0 & -S_{12}s_1 & & +S_{23}s_3 & = 0 \\ S_3 = & -S_{03}s_0 & +S_{31}s_1 & -S_{23}s_2 & & = 0 \end{array}$$

Let us choose, arbitrarily, the second and third of these equations that state, simply, that the line $\mathcal{S}$ and the point S do not form a plane, *i.e.* S is on $\mathcal{S}$. First, however, the axial line coordinates in Eq. 2 must be converted to radial so as to be in a form compatible with Eq. 1.

$$\begin{aligned} 0s_{01} - s_3 s_{02} + s_2 s_{03} - x_0 s_{23} + 0s_{31} + 0s_{12} &= 0 \\ s_3 s_{01} + 0s_{02} - s_1 s_{03} + 0s_{23} - s_0 s_{31} + 0s_{12} &= 0 \end{aligned} \tag{2}$$

The entire system in detached coefficient form appears in Eq. 3. The implied spatial model is illustrated in Fig. 1.

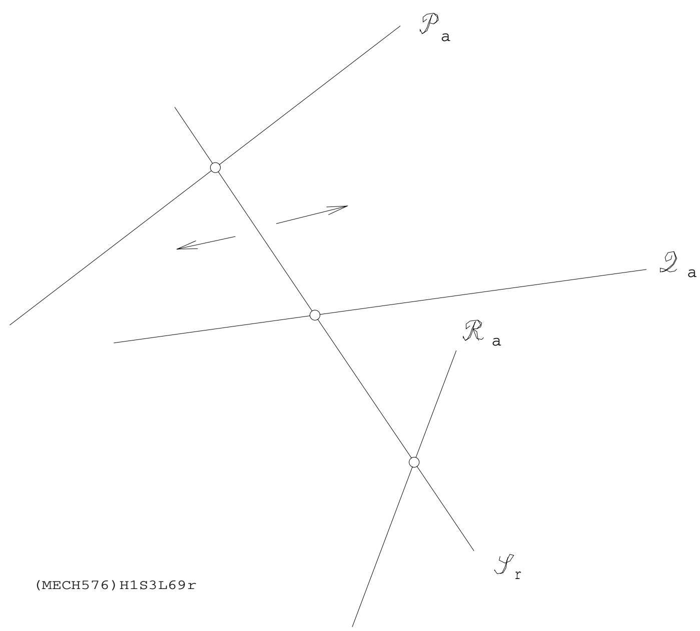

Figure 1: Ruling Line $\mathcal{S}_r$ on Given Lines $\mathcal{P}_a$, $\mathcal{Q}_a$, $\mathcal{R}_a$,

$$\begin{bmatrix} P_{01} & P_{02} & P_{03} & P_{23} & P_{31} & P_{12} \\ Q_{01} & Q_{02} & Q_{03} & Q_{23} & Q_{31} & Q_{12} \\ R_{01} & R_{02} & R_{03} & R_{23} & R_{31} & R_{12} \\ 0 & -s_3 & s_2 & -s_0 & 0 & 0 \\ s_3 & 0 & -s_1 & 0 & -s_0 & 0 \end{bmatrix} \begin{bmatrix} s_{01} \\ s_{02} \\ s_{03} \\ s_{23} \\ s_{31} \\ s_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Because symbolic expressions for s_{ij} are lengthy, they are not written out. However these are substituted into the Plücker condition

$$s_{01}s_{23} + s_{02}s_{31} + s_{03}s_{12} = 0 \tag{4}$$

The result is factorable. One factor is the desired implicit equation of the quadric, Eq. 5.

$$
\begin{aligned}
&[P_{01}(Q_{31}R_{12}-Q_{12}R_{31})+Q_{01}(R_{31}P_{12}-R_{12}P_{31})\\
&\qquad +R_{01}(P_{31}Q_{12}-P_{12}Q_{31})]\,s_1^2\\
&+[P_{02}(Q_{12}R_{23}-Q_{23}R_{12})+Q_{02}(R_{12}P_{23}-R_{23}P_{12})\\
&\qquad +R_{02}(P_{12}Q_{23}-P_{23}Q_{12})]\,s_2^2\\
&+[P_{03}(Q_{23}R_{31}-Q_{31}R_{23})+Q_{03}(R_{23}P_{31}-R_{31}P_{23})\\
&\qquad +R_{03}(P_{23}Q_{31}-P_{31}Q_{23})]\,s_3^2\\
&+[P_{23}(Q_{31}R_{02}-Q_{02}R_{31}+R_{12}Q_{03}-R_{03}Q_{12})\\
&\quad +Q_{23}(R_{31}P_{02}-R_{02}P_{31}+P_{12}R_{03}-P_{03}R_{12})\\
&\quad +R_{23}(P_{31}Q_{02}-P_{02}Q_{31}+Q_{12}P_{03}-Q_{03}P_{12})]\,s_2s_3\\
&+[P_{31}(Q_{12}R_{03}-Q_{03}R_{12}+R_{23}Q_{01}-R_{01}Q_{23})\\
&\quad +Q_{31}(R_{12}P_{03}-R_{03}P_{12}+P_{23}R_{01}-P_{01}R_{23})\\
&\quad +R_{31}(P_{12}Q_{03}-P_{03}Q_{12}+Q_{23}P_{01}-Q_{01}P_{23})]\,s_3s_1\\
&+[P_{12}(Q_{23}R_{01}-Q_{01}R_{23}+R_{31}Q_{02}-R_{02}Q_{31})\\
&\quad +Q_{12}(R_{23}P_{01}-R_{01}P_{23}+P_{31}R_{02}-P_{02}R_{31})\\
&\quad +R_{12}(P_{23}Q_{01}-P_{01}Q_{23}+Q_{31}P_{02}-Q_{02}P_{31})]\,s_1s_2\\
&+[P_{01}(Q_{31}R_{02}-Q_{02}R_{31}-R_{12}Q_{03}+R_{03}Q_{12})\\
&\quad +Q_{01}(R_{31}P_{02}-R_{02}P_{31}-P_{12}R_{03}+P_{03}R_{12})\\
&\quad +R_{01}(P_{31}Q_{02}-P_{02}Q_{31}-Q_{12}P_{03}+Q_{03}P_{12})]\,s_0s_1\\
&+[P_{02}(Q_{12}R_{03}-Q_{03}R_{12}-R_{23}Q_{01}+R_{01}Q_{23})\\
&\quad +Q_{02}(R_{12}P_{03}-R_{03}P_{12}-P_{23}R_{01}+P_{01}R_{23})\\
&\quad +R_{02}(P_{12}Q_{03}-P_{03}Q_{12}-Q_{23}P_{01}+Q_{01}P_{23})]\,s_0s_2\\
&+[P_{03}(Q_{23}R_{01}-Q_{01}R_{23}-R_{31}Q_{02}+R_{02}Q_{31})\\
&\quad +Q_{03}(R_{23}P_{01}-R_{01}P_{23}-P_{31}R_{02}+P_{02}R_{31})\\
&\quad +R_{03}(P_{23}Q_{01}-P_{01}Q_{23}-Q_{31}P_{02}+Q_{02}P_{31})]\,s_0s_3\\
&+[P_{01}(Q_{02}R_{03}-Q_{03}R_{02})+P_{02}(Q_{03}R_{01}-Q_{01}R_{03})\\
&\qquad +P_{03}(Q_{01}R_{02}-Q_{02}R_{01})]\,s_0^2=0
\end{aligned}
\tag{5}
$$

Producing the ten numerical coefficients of Eq. 5 to derive the implicit equation of a desired surface requires the specification of the three given lines $\mathcal{Q},\mathcal{R},\mathcal{S}$. This can be done quite easily if one thinks "geometrically". Imagine the following six points.

1. The first point locates the desired centre of symmetry of the desired ruled surface.
2. The second specifies another point on the axis.
3. The third is on the surface, on a principal axis of the minimal elliptical section, on the first point, perpendicular to the axis.
4. The fourth is another on a generator, on the third. Now the first line in the regulus to be ruled by moving $\mathcal{S}$ is defined by the third and fourth points.

5. The fifth is on the surface, on the *other* principal axis of the minimal elliptical section, on the first point, perpendicular to the axis.

6. The sixth is another on a generator, on the fifth. Now the second line in the regulus to be ruled by moving $\mathcal{S}$ is defined by the fifth and sixth points.

The third line is produced by perpendicular reflection, on the axis, of the first. A hyperbolic paraboloid is specified differently. One chooses two lines to be ruled and a third line which is in the opposite regulus, connecting the first two. Let us assume this line is

$$\mathcal{R}'_r\{r'_{01} : r'_{02} : r'_{03} : r'_{23} : r'_{31} : r'_{12}\}$$

Then the third specification line $\mathcal{R}$ becomes a special line at infinity whose moment is the direction of the connecting line.

$$\mathcal{R}_r\{r_{01} : r_{02} : r_{03} : r_{23} : r_{31} : r_{12}\} \equiv \{0 : 0 : 0 : r'_{01} : r'_{02} : r'_{03}\}$$

In all events, the three given lines must be specified as *axial*.

2.1 A Vector Approach

For those averse to Plücker coordinates and the like, Professor J. Angeles formulated the following method to find the implicit equation of a quadric specified by three given skew lines. An assigned problem example is described below followed by his analytical vector method and the expansion of matrix $\mathbf{A}$ with the appropriate numerical values.

2.1.1 Problem

A single-sheet hyperboloid has rulings on three skew lines $\mathcal{L}_i,\ i = 1, 2, 3$. $\mathcal{L}_1$ is the X-axis, $\mathcal{L}_2$ is normal to and intersects the Y-axis at $(0, 5, 0)$ and slopes upward at $30°$ as X increases and $\mathcal{L}_3$ contains point (1,2,3) and has direction numbers $\{-1 : 2 : -3\}$. Let these three be ruled by a variable line $\mathcal{L}$. Eq. 6 presents the distance d between two lines, *e.g.*, $\mathcal{L}$ and $\mathcal{L}_i$.

$$d = \left\| \frac{(\mathbf{e} \times \mathbf{e}_i)^T}{\|\mathbf{e} \times \mathbf{e}_i\|} (\mathbf{p} - \mathbf{p}_i) \right\| \tag{6}$$

where $\mathbf{e}$ is a unit vector in the direction of $\mathcal{L}$, $\mathbf{e}_i$ is a unit vector in the direction of $\mathcal{L}_i$, p is the position vector of a given point on $\mathcal{L}$ and p_i is the position vector of a given point on $\mathcal{L}_i$. Intersection $\mathcal{L} \cap \mathcal{L}_i$ is indicated by $d = 0$ so one may write three equations, one for $\mathcal{L}$ intersecting each of $\mathcal{L}_i$, and obtain an expression in terms of $\mathbf{p}$. However there is no need to retain the denominator or the magnitude computation in Eq. 6. Furthermore this will be a homogeneous system and the unit vectors $\mathbf{e}$ and $\mathbf{e}_i$ may be replaced by un-normed direction numbers $\mathbf{d}$ and $\mathbf{d}_i$, thus.

$$(\mathbf{d} \times \mathbf{d}_i)^T (\mathbf{p} - \mathbf{p}_i) \equiv [\mathbf{d}_i \times (\mathbf{p} - \mathbf{p}_i)]^T \mathbf{d} = 0$$

This may be abbreviated to $\mathbf{Ad} = \mathbf{0}$ and is equivalent to the combination Eq. 3 and 4. Since any ruling line on a quadric cannot have vanishing direction numbers and because we are not particularly interested in the direction of any such arbitrary line -a direction that one cannot extract anyway because it is based on a zero length assumption- the system is *not* used to solve for $\mathbf{d}$ but to get the point form quadric equation by exploiting the fact that since $\|\mathbf{d}\| \neq 0$ therefore $|\mathbf{A}| = 0$. For the problem stated above we get

$$\mathbf{p}\{x_1 : x_2 : x_3\}, \ \ \mathbf{p}_1\{p_1 : p_2 : p_3\} = \{1 : 0 : 0\}, \ \ \mathbf{p}_2\{q_1 : q_2 : q_3\} = \{0 : 5 : 0\}$$

$$\mathbf{p}_3\{r_1 : r_2 : r_3\} = \{1 : 2 : 3\}, \ \ \mathbf{e}_1\{p_{01} : p_{02} : p_{03}\} = \{1 : 0 : 0\}$$

$$\mathbf{e}_2\{q_{01} : q_{02} : q_{03}\} = \{\sqrt{3} : 1 : 0\}, \ \ \mathbf{e}_3\{r_{01} : r_{02} : r_{03}\} = \{-1 : 2 : -3\}$$

$|\mathbf{A}|$, Eq. 7, is much simpler to formulate than Eq. 3.

$$\begin{vmatrix} p_{02}(x_3-p_3)-p_{03}(x_2-p_2) & p_{03}(x_1-p_1)-p_{01}(x_3-p_3) & p_{01}(x_2-p_2)-p_{02}(x_1-p_1) \\ q_{02}(x_3-q_3)-q_{03}(x_2-q_2) & q_{03}(x_1-q_1)-q_{01}(x_3-q_3) & q_{01}(x_2-q_2)-q_{02}(x_1-q_1) \\ r_{02}(x_3-r_3)-r_{03}(x_2-r_2) & r_{03}(x_1-r_1)-r_{01}(x_3-r_3) & r_{01}(x_2-r_2)-r_{02}(x_1-r_1) \end{vmatrix} = 0 \quad (7)$$

The quadric, produced with the data given, has the implicit equation, Eq. 8.

$$(20 - 60\sqrt{3})x_3 - 3x_1x_2 - 10x_1x_3 + (15\sqrt{3})x_2x_3 + 10\sqrt{3}x_3^2 = 0 \quad (8)$$

The only detraction that may be made about the vector approach is that, aside from requiring a small algebraic "leap-of-faith", it does not successfully handle degenerate quadrics like line pencils, and cones and cylinders of revolution. Nevertheless its vector subspace is quite homogeneous thus free of norms. However it must be admitted that the combination of Eqs. 3 and 4, when tested by presenting the three principal axes, comes up empty too.

2.2 Some Special Examples

Fig. 2 shows the surface produced by ruling the lines $\mathcal{P}_a\{0 : 0 : 0 : 0 : 0 : 1\}$, $\mathcal{Q}_a\{0 : -c\cos\theta : c\sin\theta : 0 : \sin\theta : \cos\theta\}$, $\mathcal{R}_a\{0 : 0 : 1 : 0 : 0 : 0\}$. The first two represent the jig slots to lay up wooden laths to produce a laminated, twisted guitar neck blank. Imagine $\mathcal{P}$ is an edge of a vertical slot while $\mathcal{Q}$ is an edge of a slot inclined with respect to $\mathcal{P}$ by angle θ and distance c away from $\mathcal{P}$. $\mathcal{R}$ is an absolute line whose moment vector is in the direction of $\mathcal{P}$. It constrains the laths to fan towards $\mathcal{R}$ while remaining normal to $\mathcal{P}$. The surface illustrated in Fig. 2 was produced with $c = 1$, say, one metre and $\theta = \frac{\pi}{9}$.
Apart from being a (slightly) degenerate quadric with a real absolute conic -like an ordinary hyperboloid of one sheet- that decomposes into a pair of distinct lines -unlike the hyperboloid of one sheet- the hyperbolic paraboloid appears, not only as a valid test example for the algorithm implied by Eq. 5, in contexts other than description of a novel guitar neck, *viz.*,

- Imagine the shortest distance from a line to a circle in three dimensional space. This is modeled in [3] as the intersection among a sphere, a diametral plane and a hyperbolic paraboloid ruled on the given line, a line normal to the plane and on the sphere centre and a third absolute line whose moment vector is in the direction of either real line.

- On the other hand consider the family of sphero-cylindrical motions [1, 2], referred to as BBM-I, after early kinematics researchers R. Bricard and E. Borel. Here *all* points on the end effector EE move on a spheres centred on a special cubic Müller surface. A mechanism

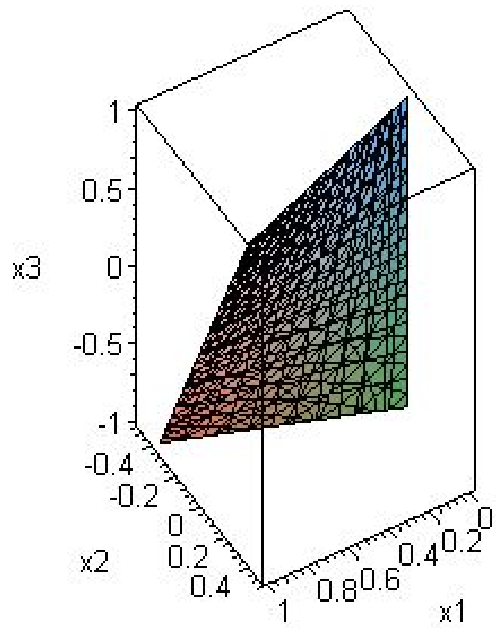

Figure 2: Hyperbolic Paraboloidal Surface of Weldon's Guitar Neck Lay Up

that illustrates an element of such motion appears in Fig. 3. Note that, in spite of an intuitive similarity, a given pose of these one degree of freedom mechanisms embed, in general, a proper hyperboloid of one sheet and *not* a hyperbolic paraboloid regulus like that of Weldon's guitar neck surface.

Fig. 4 shows three strings, AB, BR and CQ that, when the mechanism is rotated about its upward vertical axis by an positive angle $\pi/2$, rule the hyperboloid shown in Fig. 5. The equation of this specific quadric is written as Eq. 9.

$$12\sqrt{10}x_0x_3 - 10x_1x_2 - 3x_3^2 = 0 \tag{9}$$

3 The Inverse Problem: Find Ruling Lines on a Point

How a one parameter set of lines ruling a hyperboloid of one sheet -any quadric in general- may be obtained from its given implicit point equation is largely a matter of preferred parametrization. Two possibilities will be examined and presented as examples.

3.1 A General Quadric

Recall the coefficient matrix and expanded scalar form of the implicit equation.

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{10}$$

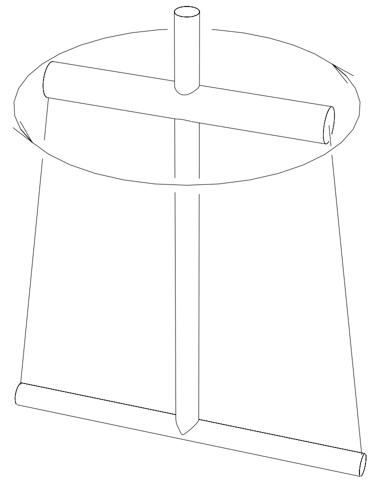

Figure 3: Borel-Bricard Motion, Type-I

$$a_{00}x_0^2+2a_{01}x_0x_1+2a_{02}x_0x_2+2a_{03}x_0x_3+a_{11}x_1^2+2a_{12}x_1x_2+2a_{13}x_1x_3+a_{22}x_2^2+2a_{23}x_2x_3+a_{33}x_3^2=0 \quad (11)$$

The quadric intersects the absolute plane $x_0 = 0$ to produce a conic

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0 \quad (12)$$

Choosing $x_0 = 1$ and any desired or convenient values for, say, x_1 and x_2, that produce a pair of real values of x_3 when substituted into Eq. 11, locates two points on the quadric. Call these points $P\{p_0 : p_1 : p_2 : p_3\}$ and $P'\{p'_0 : p'_1 : p'_2 : p'_3\}$. The polarity relation, Eq. 13

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_{00}p_0 + a_{01}p_1 + a_{02}p_2 + a_{03}p_3 \\ a_{01}p_1 + a_{11}p_1 + a_{12}p_2 + a_{13}p_3 \\ a_{02}p_0 + a_{12}p_1 + a_{22}p_2 + a_{23}p_3 \\ a_{03}p_0 + a_{13}p_1 + a_{23}p_2 + a_{33}p_3 \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (13)$$

yields a vector of homogeneous plane coordinates or coefficients. This plane intersects plane $x_0 = 0$ in the absolute line, Eq. 14.

$$P_1x_1 + P_2x_2 + P_3x_3 = 0 \quad (14)$$

Solving Eqs. 12 and 14 as a pair of homogeneous equations allows one to choose, say, $x_1 = 1$ and hence solve for a pair of values of x_3 whose corresponding values of x_2 can be obtained straightaway with back-substitution into Eq. 14. Thus we obtain two absolute points, $Q\{0 : 1 : q_2 : q_2\}$ and $Q'\{0 : 1 : q'_2 : q'_3\}$, on the polar tangent plane via Eq. 15.

$$\begin{aligned}(P_2^2a_{33} - 2P_2P_3a_{23} + P_3^2a_{33})x_3^2 - 2(P_1P_2a_{23} - P_1P_3a_{22} - P_2^2a_{13} + P_2P_3a_{12})x_3 \\ +(P_1^2a_{22} - 2P_1P_2a_{12} + P_2^2a_{11}) = 0\end{aligned} \quad (15)$$

Note that there is a distinct pair Q, Q' for *each* of P, P' making two pairs of lines, one of each pair in each quadric regulus. Every choice of p_1, p_2 on the quadric yield four ruling lines.

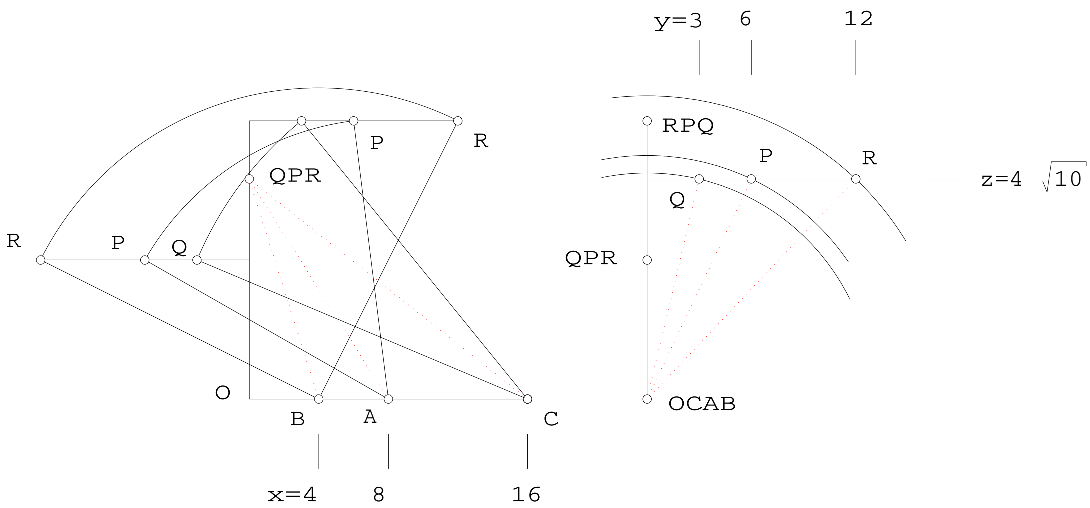

Figure 4: A Construction to Obtain a Number of Strings on a BBM-I, Kernel Mechanism

3.2 A Standard Form Quadric

Presented with a quadric centred on the origin and with its three principal axes aligned on the Cartesian ones allows a more systematic approach. In fact one may prefer to convert a quadric, in some general pose, to standard form and then reconvert ruling lines to their original, general frame. An approach that may be suggested is illustrated with a hyperboloid of one sheet whose "throat" ellipse is on the plane $x_3 = 0$, centred on the origin and major principal axis on the x_1-axis of the frame. This quadric has the scalar point equation given by Eq. 16.

$$-a^2b^2c^2x_0^2 + b^2c^2x_1^2 + a^2c^2x_2^2 + a^2b^2x_3^2 = 0 \tag{16}$$

Note that major and minor axes of the throat ellipse are a and b, respectively. The absolute conic of this standard form quadric is given by Eq. 17.

$$b^2c^2x_1^2 + a^2c^2x_2^2 + a^2b^2x_3^2 = 0 \tag{17}$$

Seeking a convenient parametrization to define points P one may consider the ruling ellipses in planes $x_3 = p_3$ by choosing a value of $\pm p_3$ with $x_0 = 1$, of course. The ruling ellipse in question is given by Eq. 18.

$$\frac{c^2x_1^2}{a^2(c^2 - p_3^2)} + \frac{c^2x_2^2}{b^2(c^2 - p_3^2)} - 1 = 0 \tag{18}$$

Consider a natural option; that of an angle θ about x_3 with $\theta = 0$ defined by rays on plane $x_2 = 0$. Therefore any point P specified by p_3 is given by the column vector in Eq. 19

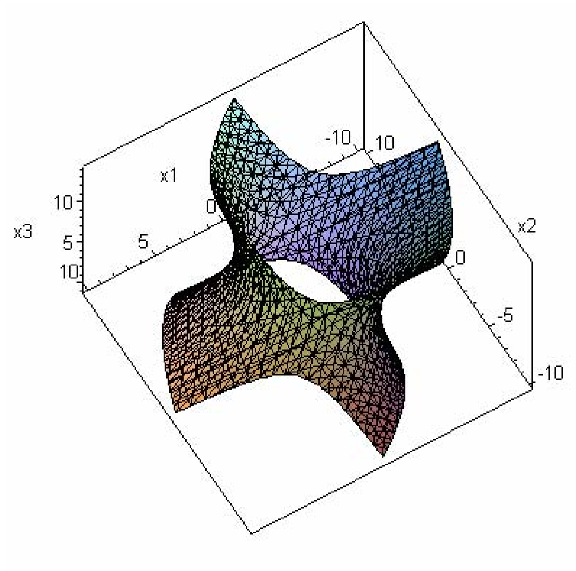

Figure 5: Hyperboloid Embedded in a Specific Pose of a BBM-I

$$\begin{bmatrix} -a^2b^2c^2 & 0 & 0 & 0 \\ 0 & b^2c^2 & 0 & 0 \\ 0 & 0 & a^2c^2 & 0 \\ 0 & 0 & 0 & -a^2b^2 \end{bmatrix} \begin{bmatrix} \pm\sqrt{A^2\sin^2\theta + B^2\cos^2\theta} \\ AB\cos\theta \\ AB\sin\theta \\ p_3 \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{19}$$

where $A^2 = a^2(c^2 - p_3^2)/c^2$, $B^2 = b^2(c^2 - p_3^2)/c^2$. As before, the tangent plane on P gives the absolute line

$$P_1x_1 + P_2x_2 + P_3x_3 = 0$$

and this intersects the conic, Eq. 17, in two points Q, Q'. The quadratic in x_3 is much simpler than its general counterpart, Eq. 15, *i.e.*, Eq. 20.

$$(P_3a^2c^2)x_3^2 + (2P_1P_3a^2c^2 - P_2^2a^2b^2)x_3 + (P_1^2a^2c^2 + P_2^2b^2c^2) = 0 \tag{20}$$

References

[1] Husty, M.L. and Zsombor-Murray (1994) "A Special Type of Stewart-Gough Platform", *Advances in Robot Kinematics and Computational Geometry*, Kluwer, ISBN 0-7923-2983-X, pp.449-458.

[2] Zsombor-Murray, P.J. (1992) "A Spherocylindrical Mechanism and Intersection among Other Quadratic(*sic.*) Surfaces", *Proc.* 5th *Int. Conf. ECG&DG*, v.2, pp.370-376.

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