MECH 314 Dynamics of Mechanisms
Introduction to Kinematics

September 10, 2009

(MECH314)Geokin99c

1 Geometry in Kinematics

Six topics are introduced here to reveal the intimate connection between engineering kinematics and geometric thinking. These include spatial as well as planar displacement, velocity and acceleration of a rigid body. Motions of such a body are those where the distance between point pairs on the body remains constant and reflections are prohibited. This means that the spatial body cannot be turned inside-out and the planar body –imagine a 30°/60° right-triangle– cannot be flipped over. The idea here is not so much to promote graphical rather than equation-based problem solving but to lead one to appreciate the underlying geometry that emerges from the principal invariants of kinematics of rigid bodies.

1.1 Velocities of a Planar Body

Consider the translational instantaneous velocities of points on a planar rigid body and how these relate to the instant centre (point) of zero velocity that lies on the body, or some extension thereof, and the instantaneous angular velocity. Examine Fig. 1. At any instant an arbitrary point fixed to a rigid body A may experience (or have assigned to it) an arbitrary velocity represented by vector $\mathbf{v}_A$. Another point B may, at this instant, experience any velocity $\mathbf{v}_B$ so long as the component of $\mathbf{v}_B$ parallel to the line segment AB is identical to that component of $\mathbf{v}_A$.

$$\mathbf{v}_{A(=)} = \mathbf{v}_{B(=)} \tag{1}$$

The relative velocity of B with respect to A is

$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{B/A}, \quad \mathbf{r}_{B/A} = \mathbf{b} - \mathbf{a} \tag{2}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity. Lines on A and B constructed normal to $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively, intersect on point $C = C_I$, the instant centre. Note that C_I is deemed to be on the rigid body. If it happens to fall outside some arbitrary outline that is drawn, one may imagine the body to extend beyond, in fact indefinitely, like some great sheet of ice in which the finite body is frozen; its motion being taken as the motion of the solid ice sheet. Now the velocity of *any* point P can be described as a rotation about C or

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_{P/C} \tag{3}$$

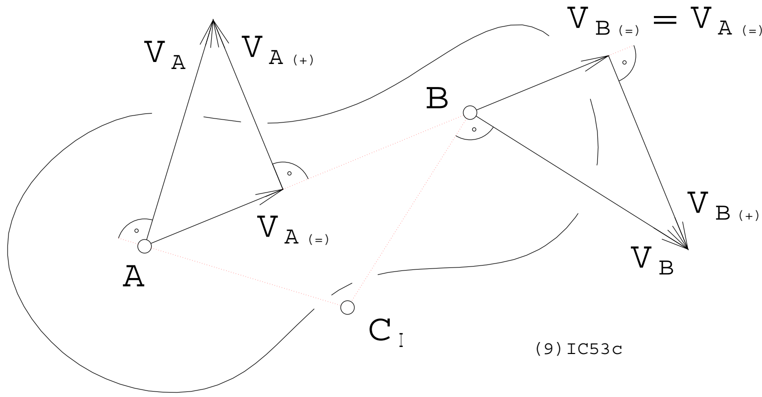

Figure 1: Instant Centre of Zero Velocity

1.2 Displacement of a Planar Body

Consider a finite displacement, *i.e.*, two distinct poses of a planar rigid body as shown in Fig. 2, and its relation to a point on the body, or its extension, that remains fixed. Again the rigid body introduced in Fig. 1 is shown with the same points A and B. This time however a second image on the right describes a displacement that has occurred. Arrows depict the translation vectors of the two points as if they had moved in straight lines. Right bisectors of the vectors intersect on $P = P_D$, called the displacement pole. Now all points on the rigid body can be seen to have moved on equiangular circular arcs centred on P. If the body undergoes a pure translation then the pole is at an indefinitely great distance normal to the pair of (now) parallel vectors. The instant centre is useful in analyzing motion between bodies in contact that roll without slipping. Such situations include cases involving wheels, gears and cams.

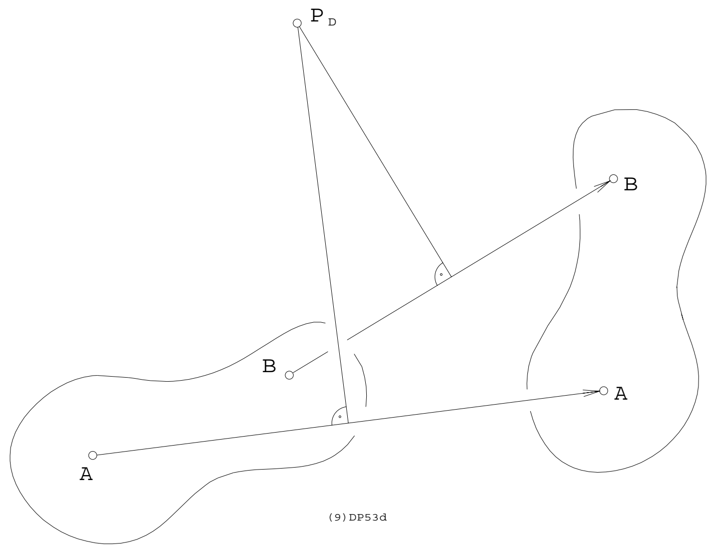

Figure 2: Displacement Pole

1.3 Acceleration and Null Acceleration Point of a Planar Body

In what follows two points, A and B, on a rigid body are considered. The acceleration of B is given as the acceleration of A plus the relative acceleration of B with respect to A, abbreviated as subscript B/A. Note $\mathbf{a}$ is an acceleration vector, identified by appropriate subscripts. What was stated above can be expressed as

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{4}$$

Relative acceleration between two points on a rigid body can be expressed as

$$\mathbf{a}_{B/A} = -\boldsymbol{\omega}^{\mathbf{2}}\mathbf{r}_{AB} + \boldsymbol{\alpha} \times \mathbf{r}_{AB} \tag{5}$$

Notice that $\boldsymbol{\omega}$ is the angular velocity vector and $\boldsymbol{\alpha}$ is the angular acceleration vector of the rigid body at this instant while $\mathbf{r}_{AB}$ is the relative position vector from A to B. If $\mathbf{a}_B = \mathbf{0}$ then B is a point of zero acceleration. Expanding Eq. 5 into Cartesian components produces

$$\begin{bmatrix} a_{Bx} \\ a_{By} \\ a_{Bz} \end{bmatrix} \begin{bmatrix} a_{Ax} \\ a_{Ay} \\ a_{Az} \end{bmatrix} + \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \times \begin{bmatrix} r_{ABx} \\ r_{ABy} \\ r_{ABz} \end{bmatrix} - (\omega_x^2 + \omega_y^2 + \omega_z^2) \begin{bmatrix} r_{ABx} \\ r_{ABy} \\ r_{ABz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

which is equivalent to three simultaneous equations Eq. 7.

$$
\begin{aligned}
a_{Ax} + [\alpha_y(b_z - a_z) - \alpha_z(b_y - a_y)] - (\omega_x^2 + \omega_y^2 + \omega_z^2)(b_x - a_x) &= 0 \\
a_{Ay} + [\alpha_z(b_x - a_x) - \alpha_x(b_z - a_z)] - (\omega_x^2 + \omega_y^2 + \omega_z^2)(b_y - a_y) &= 0 \\
a_{Az} + [\alpha_x(b_y - a_y) - \alpha_y(b_x - a_x)] - (\omega_x^2 + \omega_y^2 + \omega_z^2)(b_z - a_z) &= 0
\end{aligned} \tag{7}
$$

All problems of this type are solvable for any three unknowns, given all other parameters. *E.g.*, where is B, *i.e.*, what is $\mathbf{r}_B = [b_x \;\; b_y \;\; b_z]^\top$? Multiplying out and collecting constants yields, in this case, the system

$$
\begin{aligned}
k_{10} + k_{11} b_x + k_{12} b_y + k_{13} b_z &= 0 \\
k_{20} + k_{21} b_y + k_{22} b_y + k_{23} b_z &= 0 \\
k_{30} + k_{31} b_x + k_{32} b_y + k_{33} b_z &= 0
\end{aligned} \tag{8}
$$

1.4 Velocities of a Spatial Body

"Twist" is the spatial equivalent of angular velocity about the instant centre in the plane. It is composed of an instantaneous axis XY and lead and angular velocities of a rigid body like that shown in Fig. 3.

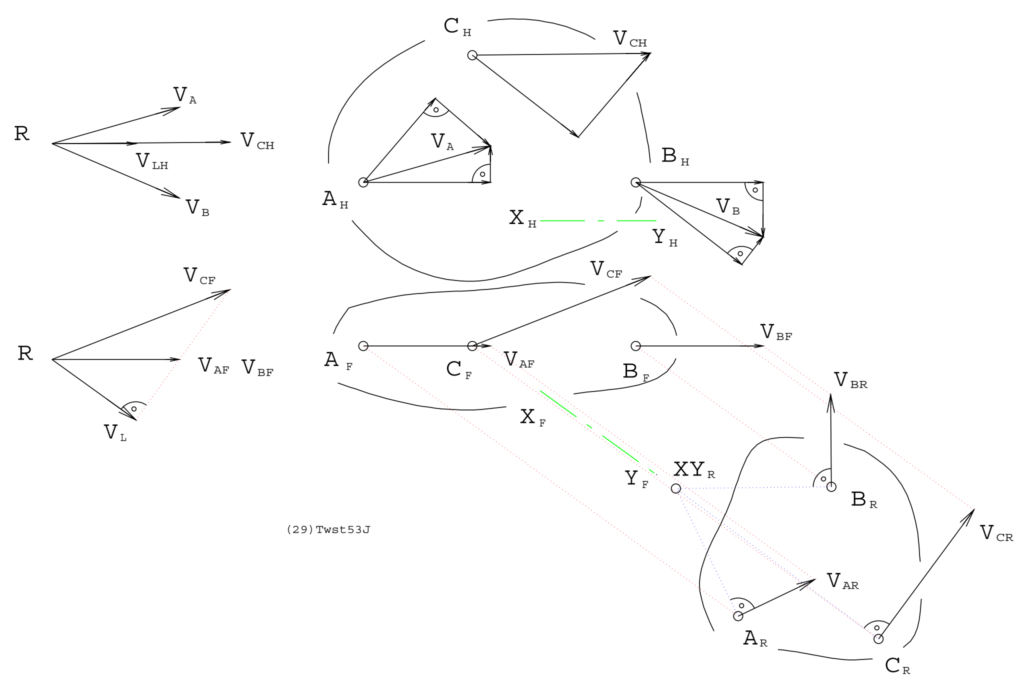

Figure 3: Instantaneous Twist

Three points A, B, C are shown. A is assigned an arbitrary instantaneous velocity V_A. Then B may have any velocity V_B as long as its component parallel to line AB is the same as that of V_A. Point C is constrained to sustain velocity V_C such that the component of V_C parallel to line BC is the same as that of V_B. Similarly, components of V_A and V_C along AC are identical. The only component of V_C that may be freely imposed is that normal to plane ABC. Notice that V_A and V_B were chosen to be coplanar. This is of course not necessary but it simplifies construction of the lead velocity vector V_L. To do this, top and front views of all three velocity vectors are plotted to scale, radiating from common point R. Note that the tips of the three velocity vectors V_{AF}, V_{BF} and V_{CF} fall in a planar line view. That was the reason for the coplanar velocity simplification; to save an auxiliary construction. Now the lead velocity V_L appears normal to this planar edge. If the points A, B, C are projected to a view where V_L would appear as a point then velocity components V_{AR}, V_{BR} and V_{CR} are normal to the twist axis whose end view XY_R is on normals to V_{AR}, V_{BR} and V_{CR} drawn respectively on A_R, B_R, C_R. Angular velocity of twist ω can be seen to be righthanded and its magnitude is determined by dividing, say, V_{AR} by the length of the normal $XY_R \rightarrow A_R$.

1.5 Displacement of a Spatial Body

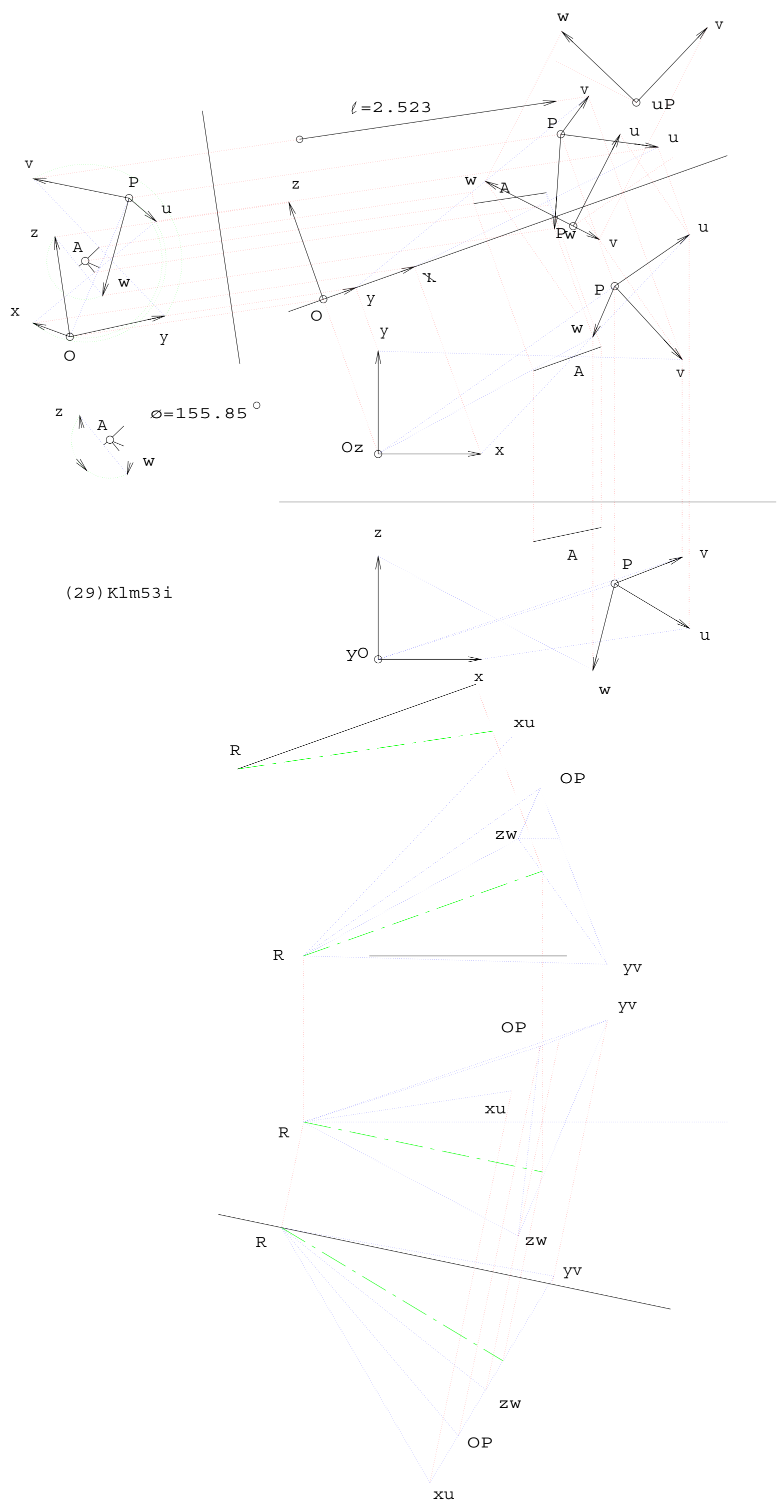

Figure 4: Finite Screw Displacement

Screw axis, rotational angle and translational lead, given two views of a unit Cartesian axis triad in an original or "home" pose and one finitely displaced, are constructed as shown in Fig. 4. These three displacement parameters correspond to the two, *viz.*, displacement pole and rotation angle, encountered in two dimensional motion and described in Fig. 2. They are constructed in the following way.

1. Notice the original pose defined on origin O and the tips of the unit axes labelled x, y, z respectively. In the corresponding displaced pose these become P, u, v, w.
2. Now consider the four displacement vectors $O \rightarrow P$, $x \rightarrow u$, $y \rightarrow v$, $z \rightarrow w$. If these are plotted, below, in two similar top and front views and radiating from a common point R we see these vector tips labelled as OP, xu, yv and zw.
3. Then examine the triangular plane segment $yv \;\; zw \;\; OP$. An edge or line view was constructed. It is noted with satisfaction that xu also falls in the plane. A normal to the plane extended from R is shown as a dash-dotted centre line. All four point displacement vectors project a common component length onto this direction. The length of this normal is the lead of the screw.
4. If an end or point view of this normal direction is constructed, only those components of the displacement vectors that are normal to the screw axis project on it. They can be accounted for by rotation about the axis.
5. The point view construction is accomplished with the line segment labelled A that is on the axis. But we don't yet know where it is located. But any line parallel to the axis will do to obtain a projection in which the axis appears as a point.
6. Once this has been done, right bisectors of the displacement vector projections are taken. These intersect on the point view of A which can be returned by projection to the original reference frame of top and front view.
7. Finally each displacement vector in the axial point view projection subtends a chord of a circular arc of rotation about the screw axis. Note that this has been measured as $\phi = 155.85°$ in this example while the lead is $l = 2.523$.

2 Twist and Screw Axis and Lead (Pitch)

Look at Figs. 3 and 4. The first task is to find the lead vector $\mathbf{v}_L$ with three point velocities $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C$ or lead vector $\overrightarrow{OP}$ with displacement vectors $\overrightarrow{xu}, \overrightarrow{yv}, \overrightarrow{zw}$. Call the lead vector $\mathbf{l}$ and the three, that project onto it with identical magnitude and sense, $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$$\mathbf{l} = \left[\frac{(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}| \;\; |\mathbf{c}-\mathbf{a}|} \cdot \left\{ \begin{array}{c} \mathbf{a} \\ \text{or} \\ \mathbf{b} \\ \text{or} \\ \mathbf{c} \end{array} \right\} \right] \frac{(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}| \;\; |\mathbf{c}-\mathbf{a}|} \tag{9}$$

Now subtract $\mathbf{l}$ from each of the three to get the rotational components $\mathbf{a}_r, \mathbf{b}_r, \mathbf{c}_r$, *i.e.*, those normal to the common lead.

$$\mathbf{a}_r = \mathbf{a} - \mathbf{l}, \quad \mathbf{b}_r = \mathbf{b} - \mathbf{l}, \quad \mathbf{c}_r = \mathbf{c} - \mathbf{l} \tag{10}$$

Using points A and B, in the case of twist, or x and y at the tips of the undisplaced Cartesian axis triad –call these both points A and B– place a plane a on A with normal $\mathbf{a}_r$ and plane b on B with normal $\mathbf{b}_r$. A plane c on C with normal $\mathbf{c}_r$ might be used for intersection checking. Homogeneous coordinates (coefficients) of planes a and b are presented in Eq. 11.

$$\begin{aligned} a\{A_0 : A_1 : A_2 : A_3\} &= \{A_0 : a_{rx} : a_{ry} : a_{rz}\} \\ b\{B_0 : B_1 : B_2 : B_3\} &= \{B_0 : b_{rx} : b_{ry} : b_{rz}\} \\ A_0 = -(a_{rx}a_x + a_{ry}a_y + a_{rz}a_z), \quad B_0 &= -(b_{rx}b_x + b_{ry}b_y + b_{rz}b_z) \end{aligned} \tag{11}$$

The screw or twist axis $\mathcal{X}_a$ is given by its axial Plücker or *line* coordinates by the intersection of planes $a \cap b$; $a \cap c$ and $b \cap c$ should give the same result. The appropriate expansion of six 2×2 array determinants gives the appropriate differences of products, Eq. 12.

$$\begin{aligned} \begin{vmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \end{vmatrix} &\rightarrow \mathcal{X}_a\{X_{01} : X_{02} : X_{03} : X_{23} : X_{31} : X_{12}\} \\ &= \{(A_0B_1 - A_1B_0) : (A_0B_2 - A_2B_0) : (A_0B_3 - A_3B_0) \\ &\quad : (A_2B_3 - A_3B_2) : (A_3B_1 - A_1B_3) : (A_1B_2 - A_2B_1)\} \end{aligned} \tag{12}$$

Notice the last three elements are the direction numbers of the axis line $\mathcal{X}_a$ and the first three are the moment these numbers exert about the origin.

(Burmester3PP)GeoKin99c.tex