MECH 576
Geometry in Mechanics

September 16, 2009

Pedal Point and Quadric Axes

1 Fußpunkt

Consider the following very easy problem. It may be effectively used, by solving it in conventional and not so conventional ways, to reinforce geometric thinking. Such thinking involves forcing oneself to visualize the situation as a multi-dimensional mosaic of various properties in order to select an appropriate combination of these so as to make up a satisfactory solution algorithm.

1.1 Introduction

To establish the distance between given point $M\{m_0 : m_1 : m_2 : m_3\}$ and plane $p\{P_0 : P_1 : P_2 : P_3\}$ requires point $P\{p_0 : p_1 : p_2 : p_3\}$ on p such that M and P are on line $\mathcal{P}$ normal to p. The stage is set with a descriptive geometric construction, Fig. 1. Notice p is specified by a segment on given points ABC. What we see there is the line segment PM of $\mathcal{P}$ drawn perpendicular to the horizontal and frontal line segments, BD and AE, respectively, that are constructed on p. The point P is located on the intersection of $\mathcal{P}$ and line segment GJ on p and a plane on M normal to AE. After drawing the segment of p the entire solution consists of drawing only seven images of top and front view line projections, *i.e.*,

$$BD_H, BD_F, \quad AE_H, AE_F, \quad MGJ_F, GJ_H, \quad MP_H$$

Coordinates of P can be measured directly in principal top and front views, -H- and -F-, respectively. The auxiliary view shows the edge of p as BC_1 and the line MP in true length. The distance between M and P can be measured here. A sphere centred on M and tangent to p on P is constructed and its image is shown in -H- and -F- as well. This sphere is the basis for one of the three analytical methods to be discussed presently.

- The first makes use of point position vectors, a parametric line equation and the plane equation.
- The next applies line geometry so as to systematically produce a sequence of essentially canonical equations in the Plücker coordinates of $\mathcal{P}$ and then in the coordinates of P.
- Finally, the third approach starts with the equation of a sphere centred on M and unknown radius r. The coefficients of the equation are inserted into a symmetric punctual quadric coefficient matrix and that matrix is converted to its corresponding dual planar equivalent. Pre-multiplying this with a row vector of the given coefficients (or coordinates) of p and post-multiplying by the column vector of the same coefficients yields a linear equation in r^2. With the value of r now in hand one merely multiplies the column vector by the planar matrix. The resulting auto-polar column vector of four elements contains the homogeneous coordinates of P.

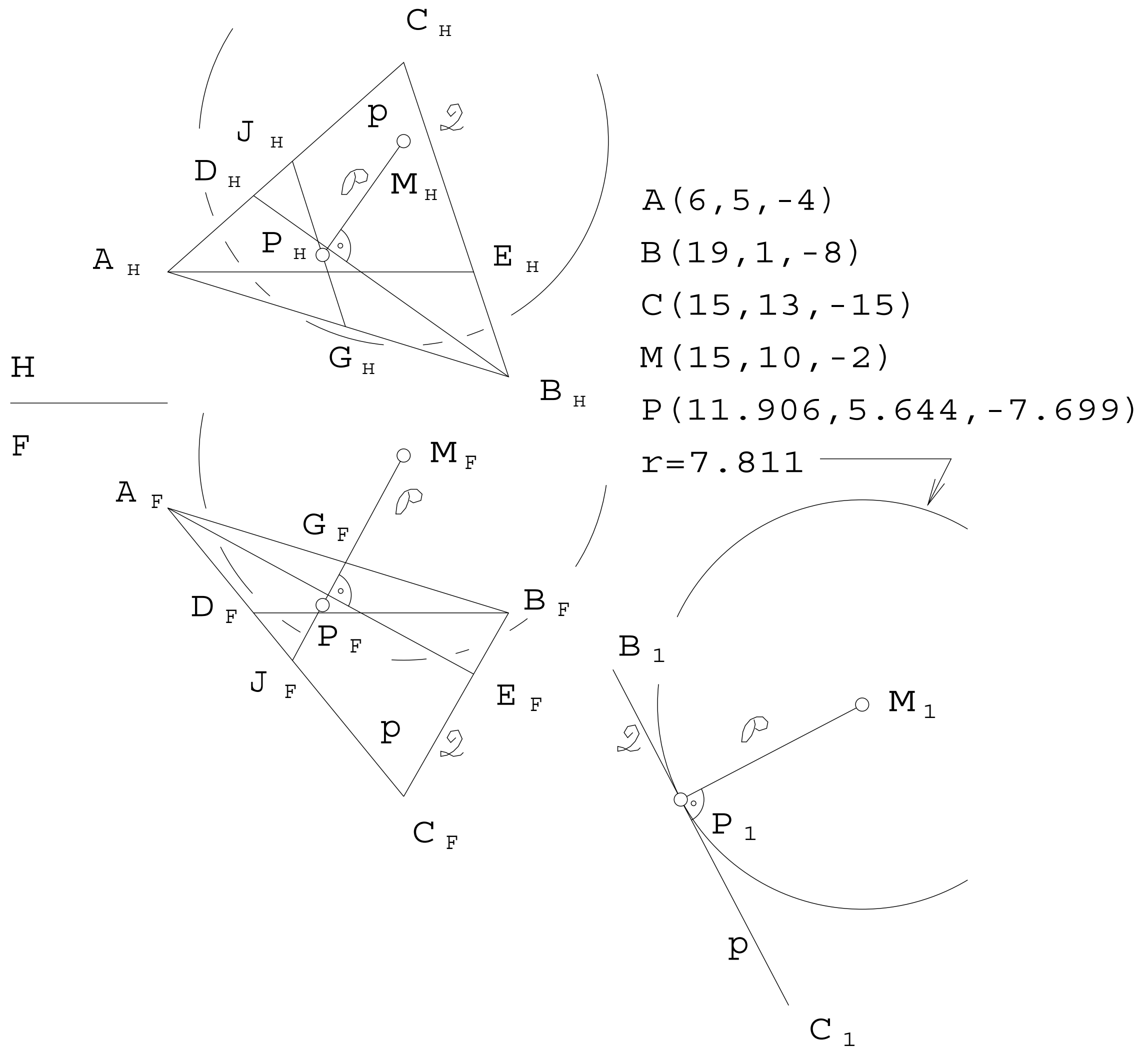

Figure 1: Normal from Point to Plane

1.2 Parametric Line Equation

The position vector of points M and P are represented by $\mathbf{m}$ and $\mathbf{p}$ while $\mathbf{n}$ is the vector normal to p. Its elements are three coefficients of the plane equation

$$P_0p_0 + P_1p_1 + P_2p_2 + P_3p_3 = 0, \;\; \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

where the point coordinates of the line we seek have been conveniently inserted. Since we seek points in Euclidean space the points are $M\{m_0 : m_1 : m_2 : m_3\}$ and $P\{p_0 : p_1 : p_2 : p_3\}$ where $m_0 = p_0 = 1$ so one may write

$$\mathbf{p} = \mathbf{m} + k\mathbf{n} \text{ where k is a scalar constant.}$$

so the plane equation becomes

$$P_0 + P_1(m_1 + n_1 k) + P_2(m_2 + n_2 k) + P_3(m_3 + n_3 k) = 0$$

and is solved for k. With k known, P may be determined. With P the length of the line segment MP can be calculated. Though the procedure is simple, formulation logic swings to and fro.

1.3 Line Normal to Plane

Examining section **1.5.3** we can write conditions that state that a line $\mathcal{P}$ on M is normal to p and intersects it on P. After appropriate substitutions have been made the seven pertinent equations are

$$-p_{23}m_0 + P_3 m_2 - P_2 m_3 = 0 \quad (1)$$
$$-p_{31}m_0 - P_3 m_1 + P_1 m_3 = 0 \quad (2)$$
$$P_1 p_{23} + P_2 p_{31} + P_3 p_{12} = 0 \quad (3)$$
$$p_0 = P_1^2 + P_2^2 + P_3^2 \quad (4)$$
$$p_1 = -P_1 P_0 + p_{12}P_2 - p_{31}P_3 \quad (5)$$
$$-p_{31}p_0 - P_3 p_1 + P_1 p_3 = 0 \quad (6)$$
$$-p_{23}p_0 + P_3 p_2 - P_2 p_3 = 0 \quad (7)$$

These equations are canonical in the sense that the first three immediately provide the three unknown moment coordinates of $\mathcal{P}$. The direction is given by $\mathbf{n}$ as before.

$$\begin{bmatrix} p_{01} \\ p_{02} \\ p_{03} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Note that Eq. 3 is the Plücker condition that states that the line direction vector is normal to the moment vector. Next, two of four homogeneous coordinates of P, p_0 and p_1, are given by the piercing point algorithm. Finally the last two, p_2 and p_3, appear as the sole unknowns in Eq. 6 and Eq. 7 that state P is on $\mathcal{P}$. As the equations are solved in the sequence given there is but one linear unknown term in each.

1.4 Sphere Tangent to Plane

Constructing a sphere centred on M and tangent to p produces the tangent point P. Eq. 8 is the scalar equation of the sphere shown in symmetrical coefficient matrix form.

$$[p_0 \;\; p_1 \;\; p_2 \;\; p_3] \begin{bmatrix} m_1^2 + m_2^2 + m_3^2 - m_0^2 r^2 & -m_0 m_1 & -m_0 m_2 & -m_0 m_3 \\ -m_0 m_1 & m_0^2 & 0 & 0 \\ -m_0 m_2 & 0 & m_0^2 & 0 \\ -m_0 m_3 & 0 & 0 & m_0^2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0 \quad (8)$$

The equation specifies P to be on the sphere. That is obvious but not helpful because sphere radius r is not known. Taking the adjoint of the matrix allows us to state that p is on the sphere, *i.e.*, a tangent plane.

$$[P_0 \;\; P_1 \;\; P_2 \;\; P_3] \begin{bmatrix} m_0^2 & m_0m_1 & m_0m_2 & m_0m_3 \\ m_0m_1 & m_1^2 - m_0^2r^2 & m_1m_2 & m_3m_1 \\ m_0m_2 & m_1m_2 & m_2 - m_0^2r^2 & m_2m_3 \\ m_0m_3 & m_3m_1 & m_2m_3 & m_3^2 - m_0^2r^2 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0 \tag{9}$$

Eq. 9 is linear in r^2. Substituting r^2 in the matrix of Eq. 9 and multiplying that by the column vector on the right yields a point column vector, Eq. 10 containing the homogeneous coordinates of P, the point *polar* to the plane p with respect to the sphere.

$$P: \begin{bmatrix} m_0^2P_0 + m_0m_1P_1 + m_0m_2P_2 + m_0m_3P_3 \\ m_0m_1P_0 + (m_1^2 - m_0^2r^2)P_1 + m_1m_2P_2 + m_3m_1P_3 \\ m_0m_2P_0 + m_1m_2P_1 + (m_2^2 - m_0^2r^2)P_2 + m_2m_3P_3 \\ m_0m_3P_0 + m_3m_1P_1 + m_2m_3P_2 + (m_3^2 - m_0^2r^2)P_3 \end{bmatrix} \tag{10}$$

1.5 Appendix

This might seem redundant. Nevertheless we take the opportunity here to introduce notation conventions and review the expansion of Grassmanian determinants to provide the coefficients of plane p on three points using the numerical values assigned to A, B, C as shown in Fig. 1. Then the algorithms

$$p = \mathcal{Q} \cap A, \;\; P_i = \sum_{j=0}^{3} Q_{ij}a_j \text{ and } P = \mathcal{P} \cap p, \;\; p_i = \sum_{j=0}^{3} p_{ij}P_j \tag{11}$$

to find the same plane p given on axial line $AB = \mathcal{Q}$ and point A, using Plücker coordinates, and by dual inference, to similarly find P the point of intersection of radial line $MP = \mathcal{P}$ and plane p are derived in symbolic form. Finally a simple demonstration that the adjoint of a homogeneous square matrix operator represents the equivalent dualistic transformation will be presented. These essential topics are recalled here to emphasize the elementary geometric background required to allow the student to develop ability to "think geometrically" and appreciate the elegant interrelation among the constructive and analytical approaches to finding the shortest distance from point to plane.

1.5.1 Conventions

- A point is specified by an upper case, unsubscripted letter and its four (three in the plane) homogeneous coordinates by single subscripted lower case letters, $P\{p_0 : p_1 : p_2 : p_3\}$. Ordinary Cartesian point coordinates in Euclidean space appear as $P(p_1/p_0, p_2/p_0, p_3/p_0)$.
- A plane is specified by a lower case unsubscripted letter and its four homogeneous coordinates (coefficients) by single subscripted upper case letters, $p\{P_0 : P_1 : P_2 : P_3\}$.
- A line in the plane is specified like a plane but with only three coefficients.

- A line in space is specified by an unsubscripted script letter and six homogeneous double subscripted letters representing line or Plücker coordinates, $\mathcal{P}\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\}$ or $\mathcal{P}\{P_{01} : P_{02} : P_{03} : P_{23} : P_{31} : P_{12}\}$. Lower case Plücker coordinates indicate a *radial* line, *i.e.*, defined on two points while upper case represents an *axial* line on two planes.

1.5.2 Plane Equation from Three Points

Expanding the following determinant, wherein the top row represents homogeneous coordinates of any variable point $X\{x_0 : x_1 : x_2 : x_3\}$ that is linearly dependent on the three given points A, B, C, produces the linear equation of plane p.

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ 1 & 6 & 5 & -4 \\ 1 & 19 & 1 & -8 \\ 1 & 15 & 13 & -15 \end{vmatrix} = 431x_0 - 76x_1 - 107x_2 - 140x_2 = 0 \tag{12}$$

1.5.3 Spanning Plane and Piercing Point

The same plane p is now expressed symbolically as Eq. 13 in terms of the same three points except rather than expanding just on the top row minors and computing four 3×3 determinants the job will be done by computing twelve 2×2 determinants thus generating the radial Plücker coordinates of $\mathcal{Q} = BC$ while the point A retains its isolated rôle. The conventional spanning plane algorithm is generated by making a radial to axial Plücker point conversion. Then the piercing point algorithm may be inferred by dual interchange of "plane" and "point". It is pointed out here that if all $P_i = 0$ then we obtain the condition that $A \in \mathcal{Q}$. Furthermore if one evaluated all q_i for the piercing point Q of line $\mathcal{Q} = BC$ with the plane $p = ABC$ these would turn out to be all zero because $\mathcal{Q} = BC$ is clearly a line on plane $p = ABC$. Note that any two of the four equations for P_i or p_i are linearly independent so one may only choose two of the four when formulating problem constraints.

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix} = 0 \tag{13}$$

The expansion gives the following scalar plane equation, still in terms of point coordinates.

$$\begin{aligned} &[(b_2c_3 - b_3c_2)a_1 - (b_1c_3 - b_3c_1)a_2 + (b_1c_2 - b_2c_1)a_3]x_0 \\ &-[(b_2c_3 - b_3c_2)a_0 - (b_0c_3 - b_3c_0)a_2 + (b_0c_2 - b_2c_0)a_3]x_1 \\ &+[(b_1c_3 - b_3c_1)a_0 - (b_0c_3 - b_3c_0)a_1 + (b_0c_1 - b_1c_0)a_3]x_2 \\ &-[(b_1c_2 - b_2c_1)a_0 - (b_0c_2 - b_2c_0)a_2 + (b_0c_1 - b_1c_0)a_2]x_3 = 0 \end{aligned} \tag{14}$$

Converting the differences of products to radial Plücker coordinates first, then to axial gives us the plane equation coefficients (coordinates).

$$\begin{aligned}
P_0 &= q_{23}a_1 + q_{31}a_2 + q_{12}a_3 = Q_{01}a_1 + Q_{02}a_2 + Q_{03}a_3 \\
P_1 &= -q_{23}a_0 + q_{03}a_2 - q_{02}a_3 = -Q_{01}a_0 + Q_{12}a_2 - Q_{31}a_3 \\
P_2 &= -q_{31}a_0 - q_{02}a_1 + q_{01}a_3 = -Q_{02}a_0 - Q_{12}a_1 + Q_{23}a_3 \\
P_3 &= -q_{12}a_0 + q_{02}a_1 - q_{01}a_2 = -Q_{03}a_0 + Q_{31}a_1 - Q_{23}a_2
\end{aligned} \tag{15}$$

To apply the algorithms as presented literally in Eq. 11 one must observe the conventions $Q_{ij} = -Q_{ji}$ and $Q_{ij} = 0$ where $i = j$.

1.5.4 The Adjoint Matrix Is the Dual of Its Transformation

The three linearly independent points A, B, C form vertices of a triangle with respectively opposite sides on lines a, b, c. These are specified by planar homogeneous coordinates, thus.

$$A\{a_0 : a_1 : a_2\},\ \ B\{b_0 : b_1 : b_2\},\ \ C\{c_0 : c_1 : c_2\},\ \ a\{A_0 : A_1 : A_2\},\ \ b\{B_0 : B_1 : B_2\},\ \ c\{C_0 : C_1 : C_2\}$$

Using a singular, dummy point $X\{x_0 : x_1 : x_2\}$ successively on a, b, c, the following Grassmannian top row determinant minor expansions produce the following homogeneous planar line coordinates.

$$a : \begin{vmatrix} x_0 & x_1 & x_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \Rightarrow \{b_1c_2 - b_2c_1 : b_2c_0 - b_0c_2 : b_0c_1 - b_1c_0\} \equiv \{A_0 : A_1 : A_2\}$$

$$b : \begin{vmatrix} x_0 & x_1 & x_2 \\ c_0 & c_1 & c_2 \\ a_0 & a_1 & a_2 \end{vmatrix} \Rightarrow \{c_1a_2 - c_2a_1 : c_2a_0 - c_0a_2 : c_0a_1 - c_1a_0\} \equiv \{B_0 : B_1 : B_2\}$$

$$c : \begin{vmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} \Rightarrow \{a_1b_2 - a_2b_1 : a_2b_0 - a_0b_2 : a_0b_1 - a_1b_0\} \equiv \{C_0 : C_1 : C_2\}$$

Now examine the nonsingular matrices containing rows of these point and line coordinates. They are duals of the same figure; a given triangle. The superscripts A and D stand for adjoint and dual, respectively. The following sequence states this equivalence.

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{bmatrix}^A \equiv \begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{bmatrix}^D \equiv \begin{bmatrix} A_0 & A_1 & A_2 \\ B_0 & B_1 & B_2 \\ C_0 & C_1 & C_2 \end{bmatrix} \equiv \begin{bmatrix} b_1c_2 - b_2c_1 & b_2c_0 - b_0c_2 & b_0c_1 - b_1c_0 \\ c_1a_2 - c_2a_1 & c_2a_0 - c_0a_2 & c_0a_1 - c_1a_0 \\ a_1b_2 - a_2b_1 & a_2b_0 - a_0b_2 & a_0b_1 - a_1b_0 \end{bmatrix}$$

Clearly, the last matrix in the sequence is the adjoint in terms of the two on the left populated by homogeneous point coordinates a_j, b_j, c_j where $j = 0, 1, 2$, is the column index. A word of caution. An entirely different triangle is produced if one plots

$$\left(\frac{A_1}{A_0}, \frac{A_2}{A_0}\right),\ \ \left(\frac{B_1}{B_0}, \frac{B_2}{B_0}\right),\ \ \left(\frac{C_1}{C_0}, \frac{C_2}{C_0}\right)\ \text{as point coordinates in the plane.}$$

2 Principal Axes of a Quadric

Finding principal axes is essentially a matrix diagonalization process presented in a geometrical context.

2.1 Introduction

A quadric surface in three-dimensional Euclidean space may be specified algebraically in the following two ways.

- The scalar point or plane equation; the result of an inner product of ten planar and punctual quadratic forms.

$$\begin{aligned} a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{03}x_0x_3 + a_{11}x_1^2 + 2a_{12}x_1x_2 \\ +2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} A_{00}X_0^2 + 2A_{01}X_0X_1 + 2A_{02}X_0X_2 + 2A_{03}X_0X_3 + A_{11}X_1^2 + 2A_{12}X_1X_2 \\ +2A_{13}X_1X_3 + A_{22}X_2^2 + 2A_{23}X_2X_3 + A_{33}X_3^2 = 0 \end{aligned} \tag{17}$$

- A symmetric matrix of planar, a_{ij} or punctual, A_{ij}, coefficients. The two equations, Eq. 16 and 17, are the result of the following two matrix multiplications.

$$[x_0 \;\; x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{18}$$

$$[X_0 \;\; X_1 \;\; X_2 \;\; X_3] \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0 \tag{19}$$

These equations state that *any* point $X\{x_0 : x_1 : x_2 : x_3\}$, that satisfies Eq. 16 or 18, is on the quadric surface whose coefficients are a_{ij}. Similarly, any *plane*, whose coordinates, *i.e.*, linear equation coefficients, are $x\{X_0 : X_1 : X_2 : X_3\}$, is tangent to the surface whose coefficients are A_{ij} as given by Eq. 17 or 19. Geometrically, a quadric may be uniquely specified on nine given points or tangent to nine given planes. Any combination of points and tangent planes, nine in all, constitutes a unique specification if each plane, if these are the fewer in number, contains one of the points and *vice-versa*. If tangent planes are separate from the points, *e.g.*, there are given five points and four planes to define a quadric but no point is on a plane there are, in general, $3^4 = 81$ possible solutions. The reason for this is that one must solve equations either for all a_{ij} or for all A_{ij}. To express a linear constraint imposed by a tangent plane one must replace all 16 A_{ij} by minor(a_{ij}). This yields a cubic in a_{ij}. If there are four such tangent planes a system of four cubic equations must be solved. This sort of "mixed specification" of quadrics, although theoretically possible, seems to be quite impractical. Another geometric specification, quite practical for hyperboloids of one sheet and hyperbolic paraboloids, is the surface swept by a fourth line that moves so as to intersect three given skew lines. What is intended here is to demonstrate that the absolute conic obtained from the point equation of the hyperboloid, computed systematically using the procedure outlined in (MECH576)H1S5Bi.mws, is identical to that obtained with the three points where the three given lines intersect the absolute plane ω and two of the three tangent lines on ω. Furthermore the three eigenvectors of the absolute conic will be revealed as principal axis lines whose three absolute points of intersection are on the conic axes that can be determined with these and the quadric's centre.

2.2 Quaternion from Unit Cartesian Axis Triads

After finding centre and principal axes of a quadric the task to diagonalize the coefficient matrix yet remains. Consider the origin centred quadric and examine Fig. 2.

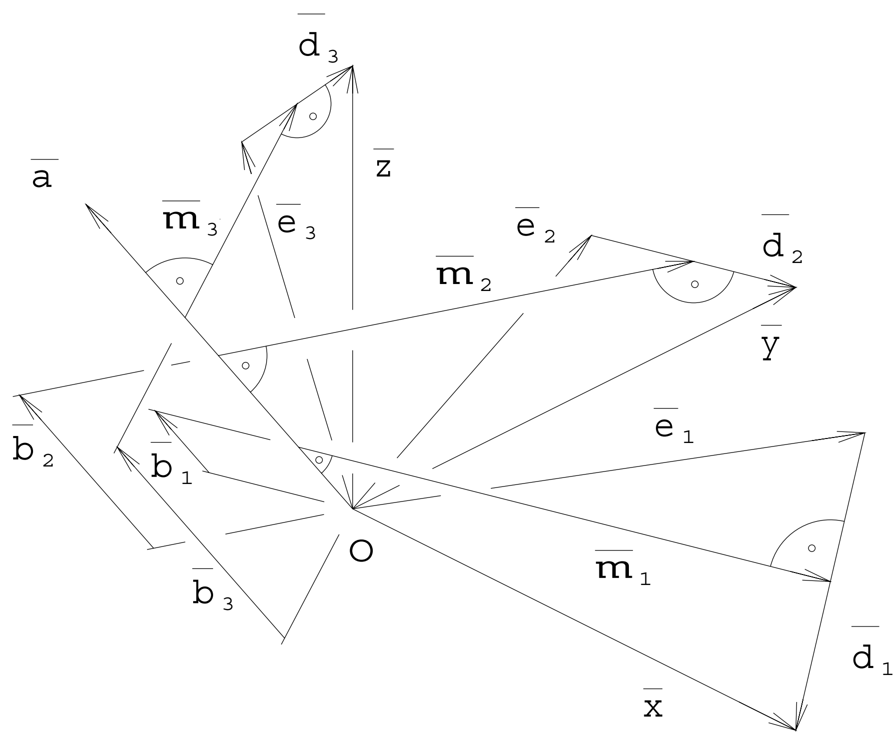

Figure 2: Rotation Axis and Cartesian Triads

The unit vectors of the Cartesian frame axes are **x**, **y** and **z**.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Unit vectors in the respective direction of the eigenvectors are obtained as $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ with respect to the frame.

(Note that these are *lines* on the absolute plane so technically it is their three points of intersection that actually represent absolute points on the three principal quadric axes and the axes themselves are defined on the quadric centre and the absolute points. Nevertheless since the absolute lines can be regarded as mutually orthogonal *planes* on the centre, using the normalized eigenvectors instead is quite acceptable. This shortcut may not be satisfactory in the case of quadrics not ruled by real lines.)

To get the quaternion elements we use the displacement vectors $\mathbf{d}_i$, the rotation axis unit vector $\mathbf{a}$ and the three vectors $\mathbf{a}\cos\alpha$, $\mathbf{a}\cos\beta$, and $\mathbf{a}\cos\gamma$, on $\mathbf{a}$ so as to locate the normal vectors $\mathbf{m}_i$ from the rotation axis to the mid points of $\mathbf{d}_i$. So we have

$$\mathbf{d}_1 = \begin{bmatrix} 1-e_{1x} \\ -e_{1y} \\ -e_{1z} \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} -e_{2x} \\ 1-e_{2y} \\ -e_{2z} \end{bmatrix}, \quad \mathbf{d}_1 = \begin{bmatrix} -e_{3x} \\ -e_{3y} \\ 1-e_{3z} \end{bmatrix}$$

and

$$\mathbf{a} = \frac{1}{\sqrt{a_x^2+a_y^2+a_z^2}} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|} = \pm\frac{\mathbf{d}_2 \times \mathbf{d}_3}{|\mathbf{d}_2 \times \mathbf{d}_3|} = \pm\frac{\mathbf{d}_3 \times \mathbf{d}_1}{|\mathbf{d}_3 \times \mathbf{d}_1|}$$

Using the first product gives

$$\mathbf{a} = \frac{\begin{bmatrix} e_{1y}e_{2z} + (1-e_{2y})e_{1x} \\ e_{1z}e_{2x} + (1-e_{1x})e_{2x} \\ (1-e_{1x})(1-e_{2y}) - e_{1y}e_{2x} \end{bmatrix}}{\sqrt{a_x^2+a_y^2+a_z^2}}$$

Axis direction cosines can be written now.

$$\begin{aligned} \cos\alpha = \mathbf{a}\cdot\mathbf{x} &= [e_{1y}e_{2z} + (1-e_{2y})e_{1x}]/\sqrt{a_x^2+a_y^2+a_z^2} \\ \cos\beta = \mathbf{a}\cdot\mathbf{y} &= [e_{1z}e_{2x} + (1-e_{1x})e_{2x}]/\sqrt{a_x^2+a_y^2+a_z^2} \\ \cos\gamma = \mathbf{a}\cdot\mathbf{z} &= [(1-e_{1x})(1-e_{2y}) - e_{1y}e_{2x}]/\sqrt{a_x^2+a_y^2+a_z^2} \\ \text{note that } \mathbf{b}_1 = (\mathbf{a}\cdot\mathbf{x})\mathbf{a},\ \ \mathbf{b}_2 &= (\mathbf{a}\cdot\mathbf{y})\mathbf{a},\ \ \mathbf{b}_3 = (\mathbf{a}\cdot\mathbf{z})\mathbf{a} \text{ in Eq. 21, below.} \end{aligned} \tag{20}$$

Vectors $\mathbf{m}_i$ are required before the rotation half-angle $\frac{\phi}{2}$ can de determined.

$$\mathbf{m}_1 = \mathbf{e}_1 + \frac{1}{2}\mathbf{d}_1 - \mathbf{b}_1, \quad \mathbf{m}_2 = \mathbf{e}_2 + \frac{1}{2}\mathbf{d}_2 - \mathbf{b}_2, \quad \mathbf{m}_3 = \mathbf{e}_3 + \frac{1}{2}\mathbf{d}_3 - \mathbf{b}_3 \tag{21}$$

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