MECH 261/262
Measurement Lab (& Statistics)

March 5, 2007

End Loaded Cantilever

1 "Diving Board"

Imagine a diving board imbedded or fixed solidly at one end, O at $x = 0$, in a rigid wall. The "diver" stands on the other, far end. His weight exerts a vertical force F and the board bends downward such that the far end is bent down, with respect to O, a distance $y = \delta$. The board has a total length $x = L$ and a strain gage is glued to the top, tensile side of the board. The centre of the gage is located at $x = C$. The board is of width b and thickness h. This set up will now be used to demonstrate how a similar arrangement, using a tiny steel beam, maybe used as a displacement, δ, transducer as well as a force, F, transducer. Notice, because deflections are small, the length of the bent beam is only negligibly greater that L.

2 Deflection

The bending moment at any distance x from O can be expressed as

$$M(x) = F(L - x) \tag{1}$$

The fundamental differential equation relating deflection $y(x)$ can be written as

$$\frac{d^4y}{dx^4} = \frac{w(x)}{EI} \tag{2}$$

where E is the Young's modulous of steel $E \approx 30 \times 10^6{}_{\text{psi}} \approx 2.07 \times 10^{11}{}_{\text{Pa}}$ and I is the second moment of area of the cross-section $b \times h$ with respect to the plane $y = 0$. this is calculated as follows.

$$I = 2b \int_0^{\frac{h}{2}} y^2 dy = \frac{bh^3}{12} \tag{3}$$

One cannot integrate Eq. 2 so the first two integrations are skipped and we write instead Eq. 4.

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{4}$$

This can be integrated to produce the slope equation Eq. 5.

$$\frac{dy}{dx} = \int d\left(\frac{dy}{dx}\right) = \frac{F}{EI}\int (L - x)dx = \frac{F}{EI}\left(Lx - \frac{x^2}{2}\right) + k_1 \tag{5}$$

$$\text{Since } \left.\frac{dy}{dx}\right]_{x=0} = 0 \text{ therefore } k_1 = 0$$

Then integrating again we get Eq. 6.

$$y = \frac{F}{EI}\int\left(Lx - \frac{x^2}{2}\right)dx + k_0 = \frac{F}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + k_0 \tag{6}$$

$$\text{Since} \quad y\big]_{x=0} = 0 \;\; \text{therefore} \;\; k_0 = 0$$

Obviously the displacement *-vs-* force transducer behaviour is summed up in Eq. 7.

$$\delta = y(L) = \frac{FL^3}{3EI} \tag{7}$$

3 Stress, Strain, Force and Displacement

The bending stress in a rectangular beam at $x = C$ is given by Eq. 8

$$\sigma_C = \frac{M(C)h}{2I} \tag{8}$$

Together with Hooke's Law relating stress and strain, which our gage measures via the out of balance voltage of a single arm tensile bridge, we get the strain at $x = C$.

$$\epsilon = \frac{\sigma}{E}, \;\; \epsilon_C = \frac{F(L-C)h}{2EI} \tag{9}$$

The final result desired for a force transducer is obtained by rearranging Eq. 9.

$$F = \frac{2EI\epsilon_C}{h(L-C)} \tag{10}$$

The final result for a displacement transducer is obtained by incorporating Eq. 7.

$$\delta = \frac{2L^3\epsilon_C}{3h(L-C)} \tag{11}$$

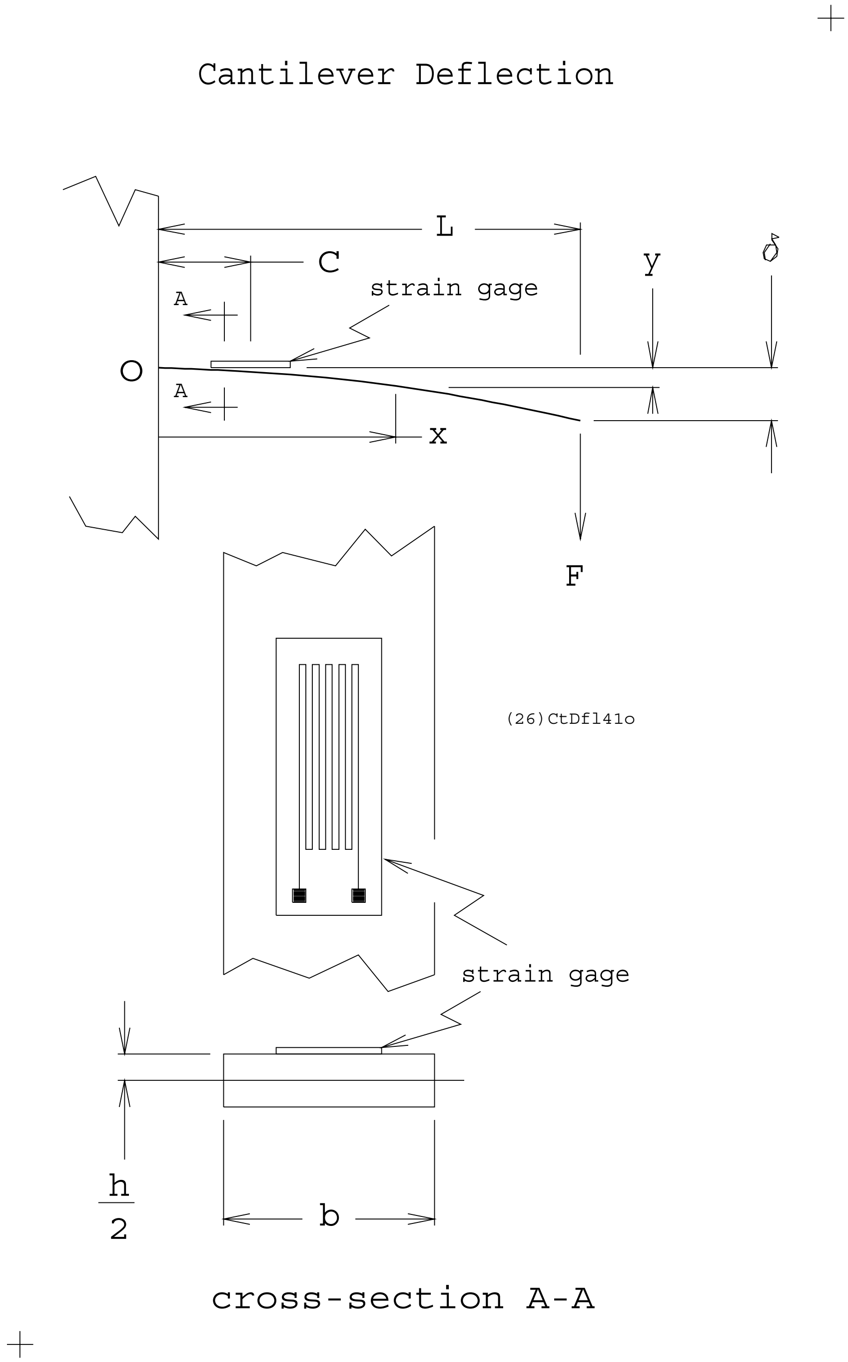

Figure 1: Cantilever Force/Linear Displacement Transducer