

An Improved Approach to the Kinematics of CLAVEL'S DELTA Robot

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1 The DELTA Robot

"DELTA", a three dimensional translational manipulator, appears below in Fig. 1.

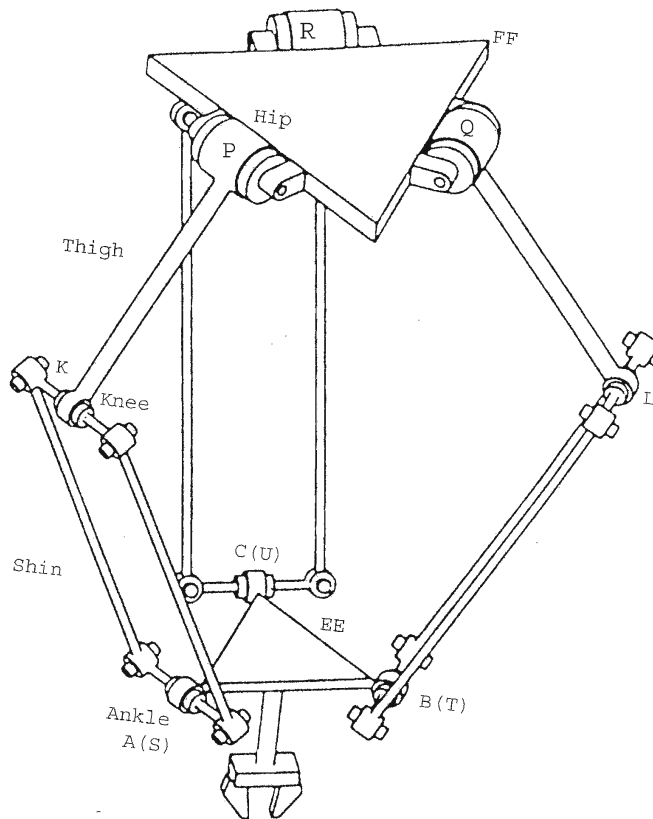


Figure 1: Symmetrical (Conventional) DELTA Robot

In what follows, inverse and direct kinematic analysis procedures –IK and DK– will be outlined. These are both based essentially on an intersection-of-line-and-sphere conceptual model and a simple translational mapping. The manipulator consists of a base or fixed frame FF connected to a moving frame or end effector EE via three legs. Starting at FF there are the motorized (actuated) “hips” P, Q, R that move the rigid “thighs” through angles θ, φ, ψ , respectively. It may be convenient to measure these as clockwise, thighs moving downwards in a right hand screw sense as one proceeds around FF in the sequence P, Q, R and taking zero as thighs horizontal like a ballerina doing the “splits”. These three angles are to be determined in the inverse problem. This places the “knees” at K, L, M . These positions are given in the case of the direct problem. The “shins”, that connect knees to “ankles” A, B, C when referring to the mapping “home” position and S, T, U when referring to the mapped final displacement position, are II-joints (parallelogram) whereon all points move on spheres. In IK the centres are S, T, U and K, L, M move on the surface. In DK the centres are K, L, M and S, T, U move on the surface.

2 Procedure: Inverse Kinematics (IK)

A method will be outlined to establish knee positions K, L, M given the ankle positions S, T, U . As opposed to direct kinematics, each leg is treated separately. Knowing the ankle position S , define sphere α .

$$\alpha : (x_1 - s_1)^2 + (x_2 - s_2)^2 + (x_3 - s_3)^2 - s = 0 \quad (1)$$

Knowing the hip position P define sphere β , centred on P and radius $KP = \sqrt{p}$, and plane γ on K and P and containing the absolute point $Z\{0 : 0 : 0 : 1\}$ so that γ is normal to the hip axis at P . Together, β and γ define a circle that intersects α .

$$\begin{aligned} \beta : (x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 - p &= 0 \\ \gamma : G_0 + G_1x_1 + G_2x_2 + G_3x_3 &= 0 \end{aligned} \quad (2)$$

Subtracting sphere equations β from α gives a second plane.

$$\alpha - \beta : 2(p_1 - s_1)x_1 + 2(p_2 - s_2)x_2 + 2(p_3 - s_3)x_3 - (p_1^2 + p_2^2 + p_3^2) + (s_1^2 + s_2^2 + s_3^2) + p - s = 0 \quad (3)$$

Proceeding with α , Eq. 1, γ , the second of Eq. 2 and Eq. 3 substituted for Eqs. 13, 14 and 15, below, one solves for x_i , the two possible positions for K . This procedure must be repeated for the other two legs and one must select judiciously between the solutions. *I.e.*, usually one should choose the more “bow-legged” rather than “knock-kneed” stance for the legs. This also applies, more or less, to direct kinematic solutions.

3 Procedure: Direct Kinematics (DK)

A method will be outlined to establish the positions of the three anchor points or ankles, initially placed ideally on EE at points $A(0, 0, 0)$, $B(b_1, 0, 0)$, $C(c_1, c_2, 0)$. These are moved onto the three spheres, σ, τ, ν , in FF, that are swept when the three actuators at the hips, P, Q, R , put the sphere centres on points $K(k_1, k_2, k_3)$, $L(l_1, l_2, l_3)$, $M(m_1, m_2, m_3)$ on the “knees” and the anchor points on the other end of the shins trace the spheres of known, respective radii, $\sqrt{s}, \sqrt{t}, \sqrt{u}$. The necessary translation is accomplished by a simple point *translation* matrix $[\mathbf{T}]$. The transformation puts $A \rightarrow S, B \rightarrow T, C \rightarrow U$, where $S(s_1, s_2, s_3), T(t_1, t_2, t_3), U(u_1, u_2, u_3)$ are points on the respective spheres.

- Three sphere equations are

$$\sigma : (s_1 - k_1)^2 + (s_2 - k_2)^2 + (s_3 - k_3)^2 - s = 0 \quad (4)$$

$$\tau : (t_1 - l_1)^2 + (t_2 - l_2)^2 + (t_3 - l_3)^2 - t = 0 \quad (5)$$

$$\nu : (u_1 - m_1)^2 + (u_2 - m_2)^2 + (u_3 - m_3)^2 - u = 0 \quad (6)$$

- Subtracting Eqs. 5 and 6 from the Eq. 4 produces the following two linear equations, Eqs. 7 and 8, that may then be solved simultaneously with, say, Eq. 4.

$$\begin{aligned} \sigma - \tau : 2(l_1t_1 - k_1s_1) + 2(l_2t_2 - k_2s_2) + 2(l_3t_3 - k_3s_3) \\ + k_1^2 - l_1^2 + k_2^2 - l_2^2 + k_3^2 - l_3^2 + t - s = 0 \end{aligned} \quad (7)$$

$$\begin{aligned} \sigma - \nu : 2(m_1u_1 - k_1s_1) + 2(m_2u_2 - k_2s_2) + 2(m_3u_3 - k_3s_3) \\ + k_1^2 - m_1^2 + k_2^2 - m_2^2 + k_3^2 - m_3^2 + u - s = 0 \end{aligned} \quad (8)$$

- Move $A \rightarrow S$, $B \rightarrow T$, $C \rightarrow U$ via the translation vector \mathbf{x} .

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{s} \end{bmatrix} \quad (10)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ b_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + b_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{t} \end{bmatrix} \quad (11)$$

$$[\mathbf{T}] \begin{bmatrix} 1 \\ c_1 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 + c_1 \\ x_2 + c_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix} \quad (12)$$

- Substitute the expressions for vectors \mathbf{s} , \mathbf{t} , \mathbf{u} , *i.e.*, the last three elements of the vectors in Eqs. 10, 11 and 12, in Eqs. 4, 7 and 8 and solve simultaneously for x_1, x_2, x_3 then compute the actual positions of S, T, U with Eqs. 10, 11 and 12. Eqs. 4, 7 and 8 become, after substitution, Eqs. 13, 14 and 15.

$$\sigma : (x_1 - k_1)^2 + (x_2 - k_2)^2 + (x_3 - k_3)^2 - s = 0 \quad (13)$$

$$\begin{aligned} \sigma - \tau : & 2(l_1 - k_1 - b_1)x_1 + 2(l_2 - k_2)x_2 + 2(l_3 - k_3)x_3 \\ & + k_1^2 + k_2^2 + k_3^2 - l_2^2 - l_3^2 - (b_1 - l_1)^2 - s + t = 0 \end{aligned} \quad (14)$$

$$\begin{aligned} \sigma - \nu : & 2(m_1 - k_1 - c_1)x_1 + 2(m_2 - k_2 - c_2)x_2 + 2(m_3 - k_3)x_3 \\ & + k_1^2 + k_2^2 + k_3^2 - (c_1 - m_1)^2 - (c_2 - m_2)^2 - m_3^2 - s + u = 0 \end{aligned} \quad (15)$$

- The advantage of this method is that it does not assume a symmetric manipulator. Upper and lower platform triangles and hip and shin lengths can be chosen arbitrarily. However the design must be configured so that all “hip”, “knee” and “ankle” R-joint axes are horizontal, *i.e.*, normal to vertical direction x_3 .
- Simultaneous solution of Eqs. 13, 14 and 15 can be advantageously done by *parametrization* of the line of intersection of planes $\sigma - \tau$, Eq. 14 and $\sigma - \nu$, Eq. 15. This avoids elimination of coordinate variables, say, x_1 and x_2 , and produces a quadratic univariate in parameter v directly. This is explained in detail after Fig. 1.

Given the sphere and platform data tabulated in Fig. 2 two translation vectors \mathbf{x} and \mathbf{x}' are found by substituting these data into Eqs. 13, 14 and 15 and solving for x_1, x_2, x_3 . The numerical versions of these equations appear as Eqs. 16.

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - 9 = 0, \quad x_1 - x_2 - x_3 - \frac{5}{3} = 0 \\ 6x_1 + 4(3 - \sqrt{3})x_2 + 2x_3 - [4(3 - \sqrt{3})^2 + 15] = 0 \end{aligned} \quad (16)$$

4 Intersection of 3 Spheres with a Parameterized Line

Compress coefficients of the two plane equations, Eq. 14 and Eq. 15, according to Eq. 17.

$$E_0 + E_1x_1 + E_2x_2 + E_3x_3 = 0, \quad F_0 + F_1x_1 + F_2x_2 + F_3x_3 = 0 \quad (17)$$

Then expand the six properly signed 2×2 minor determinants of the 2×4 matrix of the plane coordinates, *i.e.*, coefficients, E_i, F_i to establish the axial Plücker coordinates G_{ij} of their intersecting line \mathcal{G}_a .

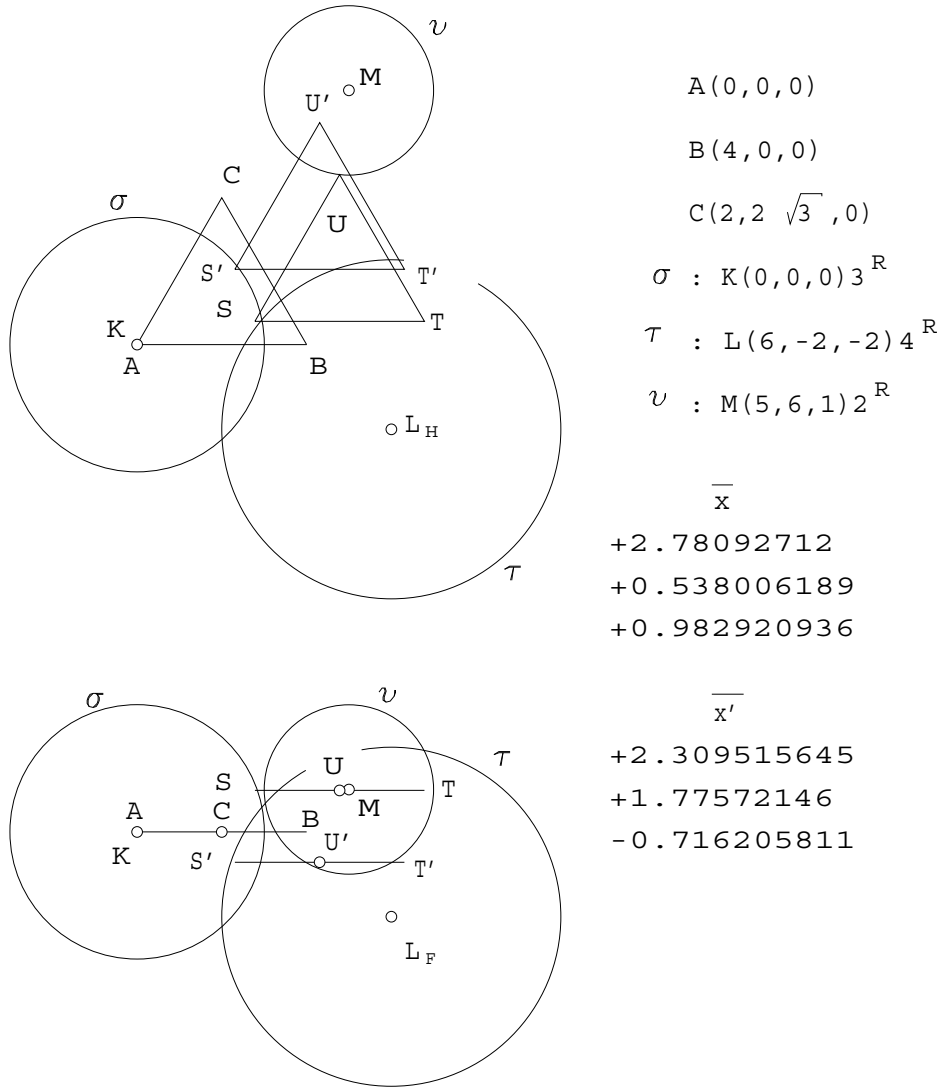


Figure 2: "Home" and Two Solution Poses of EE on a Robot with Asymmetric Thighs and Shins

$$\begin{aligned}
 G_a : \begin{vmatrix} E_0 & E_1 & E_2 & E_3 \\ F_0 & F_1 & F_2 & F_3 \end{vmatrix} &\rightarrow \{G_{01} : G_{02} : G_{03} : G_{23} : G_{31} : G_{12}\} \\
 \rightarrow \{E_0F_1 - E_1F_0 : E_0F_2 - E_2F_0 : E_0F_3 - E_3F_0 : E_2F_3 - E_3F_2 : E_3F_1 - E_1F_3 : E_1F_2 - E_2F_1\} & \quad (18)
 \end{aligned}$$

These transformations and others to follow are explained in some detail in [3]. Parametrization begins on a base point on $P\{p_0 : p_1 : p_2 : p_3\}$ on line \mathcal{G} . P may be conveniently chosen on the intersection of \mathcal{G} with its normal plane $p\{P_0 : P_1 : P_2 : P_3\}$ on the origin. Since *piercing point* of line with plane relations are usually formulated with *radial* line coordinates $\mathcal{G}(g_{ij})$ these too are expressed below. in Eq. 19.

$$\begin{aligned}
 p\{P_0 : P_1 : P_2 : P_3\} &\equiv \{0 : G_{23} : G_{31} : G_{12}\} \equiv \{0 : g_{01} : g_{02} : g_{03}\} \\
 p_0 &= \quad \quad \quad +g_{01}P_1 \quad +g_{02}P_2 \quad +g_{03}P_3 = \quad G_{23}P_1 \quad +G_{31}P_2 \quad +G_{12}P_3 \\
 p_1 &= -g_{01}P_0 \quad \quad \quad +g_{12}P_2 \quad -g_{31}P_3 = \quad \quad \quad G_{03}P_2 \quad -G_{02}P_3 \\
 p_2 &= -g_{02}P_0 \quad -g_{12}P_1 \quad \quad \quad +g_{23}P_3 = -G_{03}P_1 \quad \quad \quad +G_{01}P_3 \\
 p_3 &= -g_{03}P_0 \quad +g_{31}P_1 \quad -g_{23}P_2 = \quad G_{02}P_1 \quad -G_{01}P_2
 \end{aligned} \quad (19)$$

Now with dehomogenized coordinates of P and the normalized direction numbers of \mathcal{G}_a the line can be parameterized in v as shown in Eq. 20.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1/p_0 \\ p_2/p_0 \\ p_3/p_0 \end{bmatrix} + \begin{bmatrix} G_{23}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \\ G_{31}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \\ G_{12}v/\sqrt{G_{23}^2 + G_{31}^2 + G_{12}^2} \end{bmatrix} \quad (20)$$

Substituting for x_i in Eq. 13 from Eq. 20 leaves a quadratic in v . Returning the two values of v to Eq. 20 yields the two desired sets of piercing point coordinates. Placing the end effector (moving platform) EE artificially with A on the origin, B along the x_1 -axis and C in the plane $x_3 = 0$ and adding the vector \mathbf{x} to each, causes all to arrive on their respective spheres at S, T, U . Those who wish to read more, about the mathematical background of some of the previous calculations, are referred to Klien's little book [1] and may be inclined to browse the introductory lecture on the site mentioned in [3].

5 Parametrization of Linear Implicit Equations

Many first courses in calculus, like [2], introduce parametric equations by way of reducing given ones to implicit form and by taking derivatives of parametric functions. It is not easy to find treatment of the inverse problem, *i.e.*, how to parameterize. Although an example has been already given one may delve a little deeper by considering the simpler two dimensional parametrization of a line. Consider Fig. 5 that shows the line \mathcal{G} represented by the line g . As in the previous case the parametric origin is point $P = g \cap p$ where p is a line through O and normal to g . It is easy to see that these lines and the parametric vector equation of g can be written as shown in Eq. 21.

$$g: -12x_0 + 3x_1 + 4x_2 = 0, \quad p: 0x_0 + 4x_1 - 3x_2 = 0, \quad \mathbf{x} = \mathbf{p} + \mathbf{q}v \quad (21)$$

The vector equation expands to Eq. 22.

$$\mathbf{x} = \begin{bmatrix} x_1/x_0 \\ x_2/x_0 \end{bmatrix} = \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix} + \begin{bmatrix} 4v/5 \\ -3v/5 \end{bmatrix} \quad (22)$$

Geometrically the parametrization process is a *projection*. This can be seen by comparing the lower and upper graphs in Fig. 5. Below, the two Cartesian coordinates $x_1/x_0, x_2/x_0$ are related by an implicit equation. It does not matter which variable is chosen as "independent". The equation may be solved explicitly in both ways. Above, $x_1/x_0 = x$ and $x_2/x_0 = y$ is plotted separately against a true independent variable v that runs from P along the *real number line*. Notice the dehomogenization of point coordinates

$$\left| \begin{array}{ccc} x_0 & x_1 & x_2 \\ 0 & 4 & -3 \\ -12 & 3 & 4 \end{array} \right| \rightarrow \{25 : 36 : 48\} \rightarrow P \left(\frac{36}{25}, \frac{48}{25} \right)$$

and normalization of the direction vector \mathbf{q} .

$$\mathbf{q} = \begin{bmatrix} 4/\sqrt{4^2 + (-3)^2} \\ -3/\sqrt{4^2 + (-3)^2} \end{bmatrix}$$

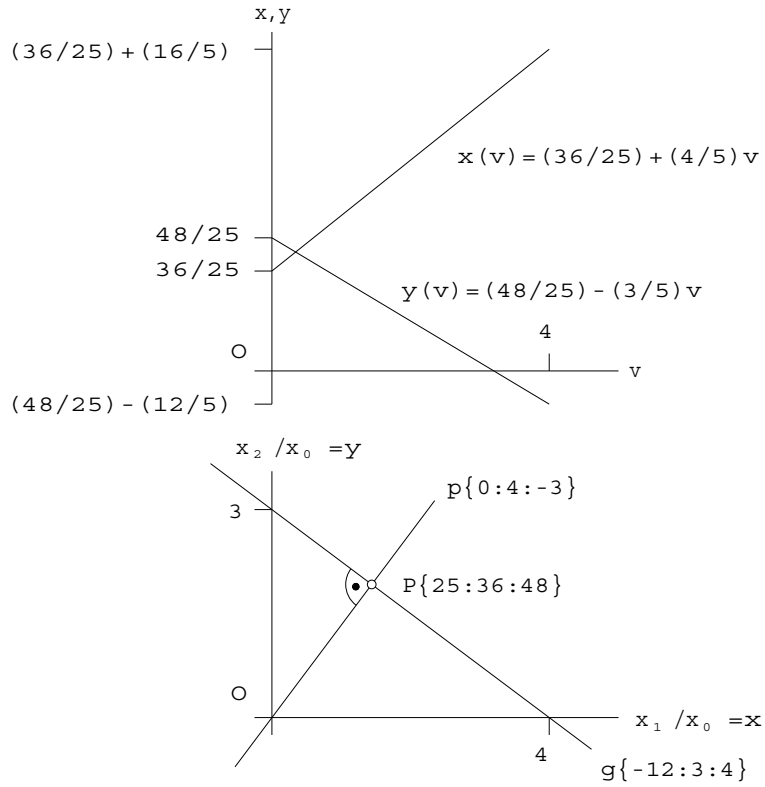


Figure 3: Parameterizing a Planar Line

References

- [1] Klein, F.C. (2004) *Elementary Mathematics from an Advanced Standpoint -Geometry-*, Dover, ISBN 0-486-43481-8.
- [2] Peterson, T.S. (1950) *Elements of Calculus*, Harper & Brothers, New York, pp.114–118.
- [3] Zsombor-Murray, P.J. (1999) "Grassmannian Reduction of Quadratic Forms", *Lecture I, MECH 576 – Geometry in Mechanics*, <<http://www.cim.mcgill.ca/~paul/GRQF69f.pdf>>