3R Wrist Positioning – Its Geometric Background

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Outline

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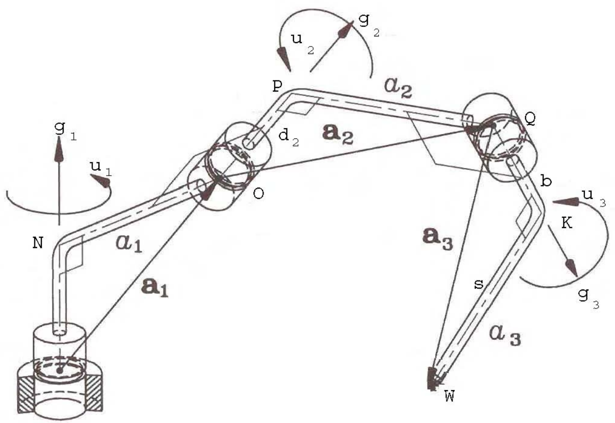
- Solve for u_3. Use two equation system to find u_1 and three equation system to find u_2.

Conventional Layout



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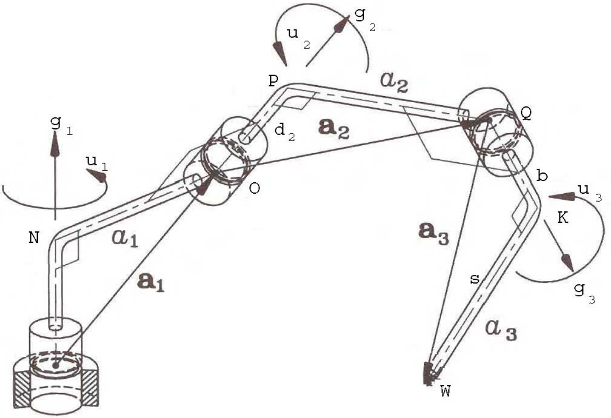

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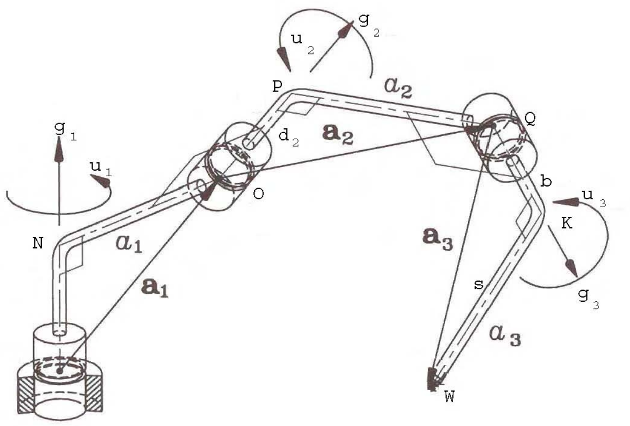

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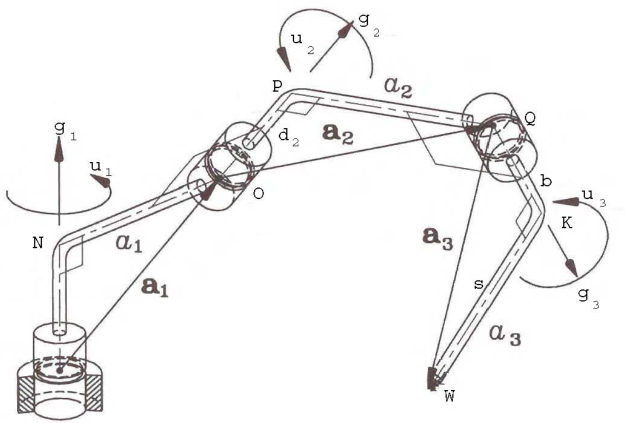

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- Consistent with geometric approach. Skew axis angles $\alpha_{1,2}$ not shown.

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- $b_1 = 0,\ b_2 = d_2\ b_3 = b$

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- Aside from back-substitution to find angles $u_{1,2}$ the problem is solved **but why is this sort of octic factorable?**

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- Computational process, reducing a sphero-quartic surface and some questions.

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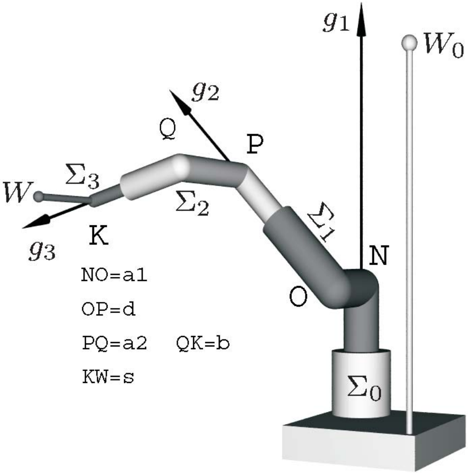

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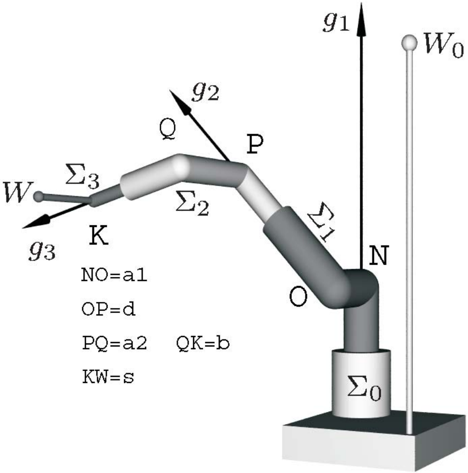

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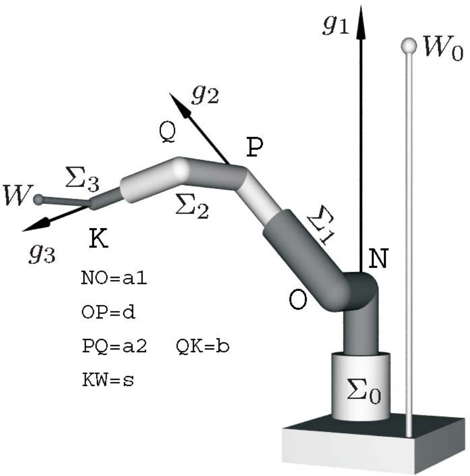

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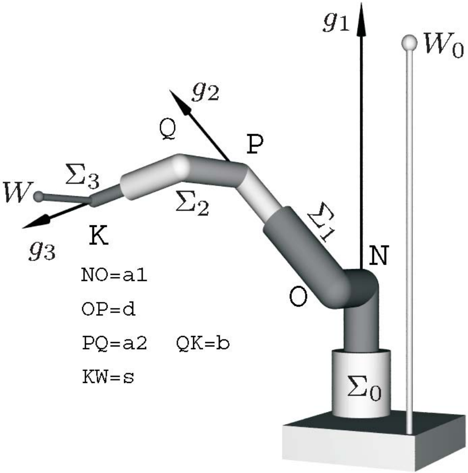

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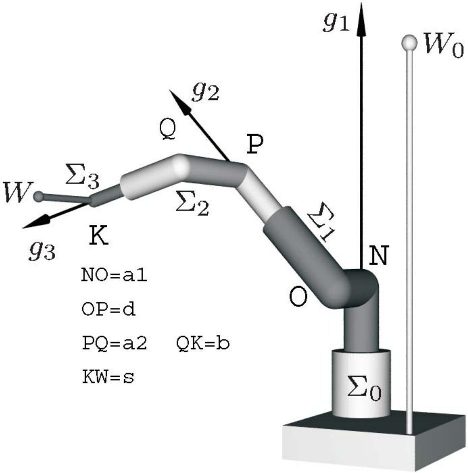

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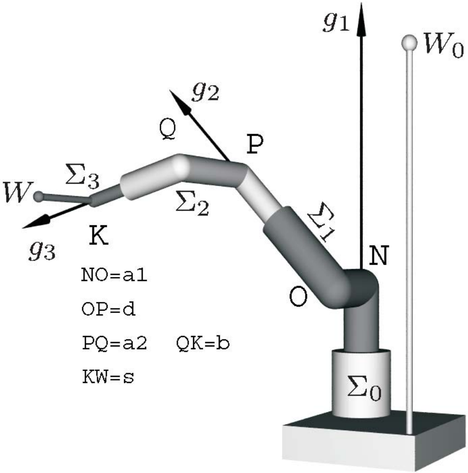

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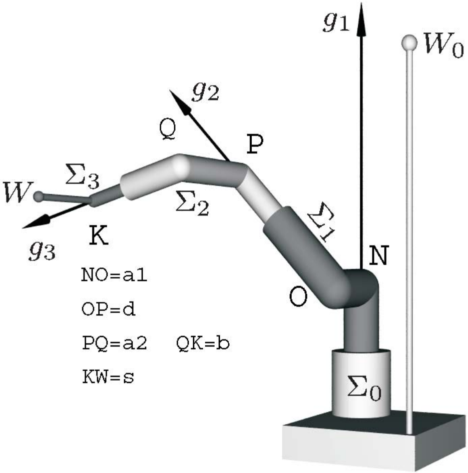

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- Still missing, angles α_1 between g_1 and g_2 and α_2 between g_2 and g_3 are easy to visualize here.

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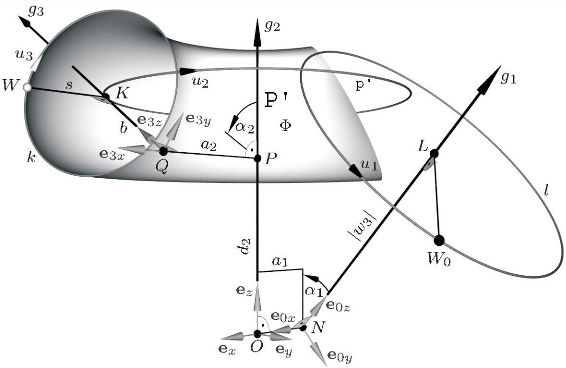

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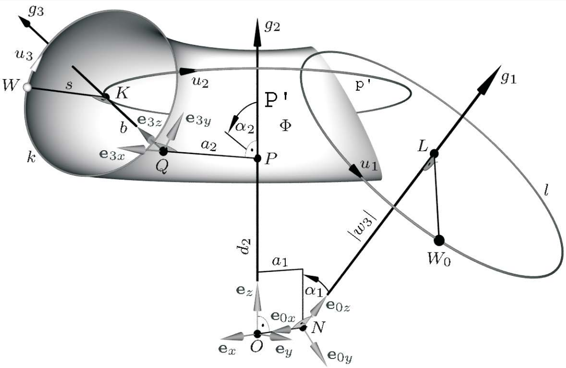

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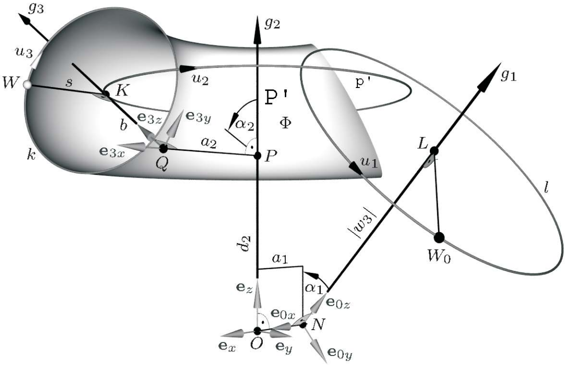

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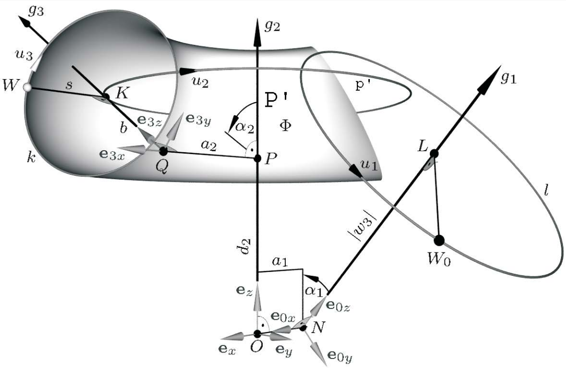

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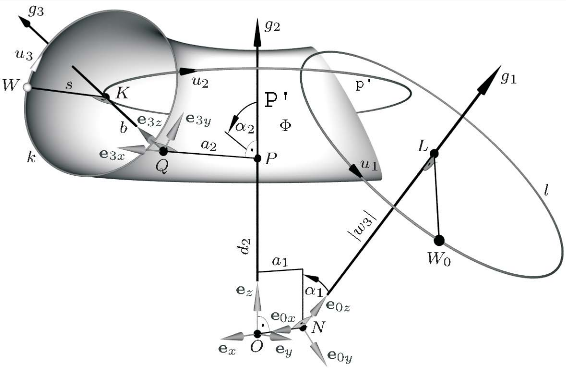

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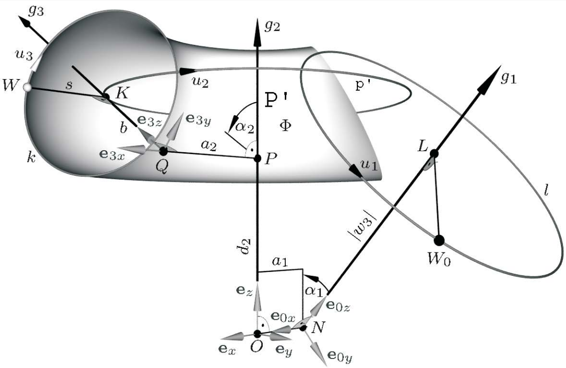

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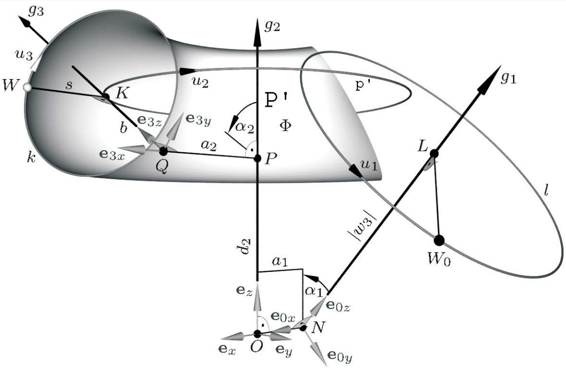

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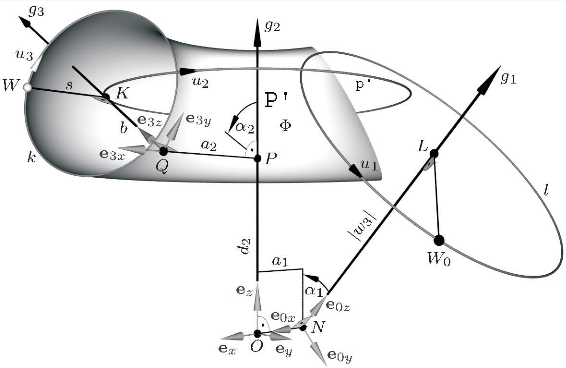

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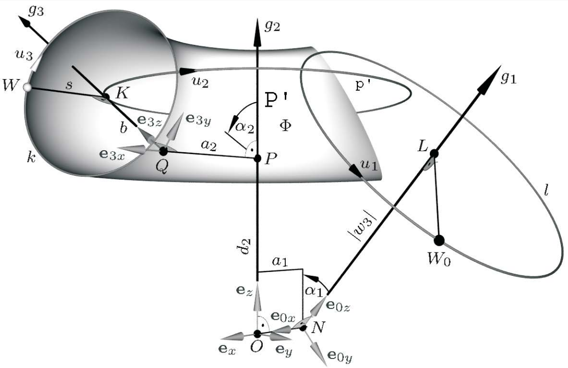

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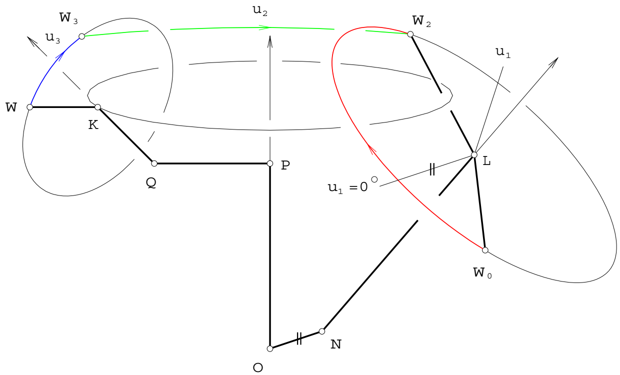
3 Joints Angles and 3 Circular Arcs to a Solution



- Excursion on surface Φ meridian due to rotation u_3 followed by rotation u_2 along latitude circle places W at one of four desired locations on circle l at W_2.

$I(u_1)$ and $\Phi(u2, u3)$ Yields $W(x, y, z)$

- $I(u_1): \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a_1 \\ w_3 sin\alpha_1 \\ w_3 \cos\alpha_1 \end{bmatrix} + \cos u_1 \begin{bmatrix} w_1 \\ w_2 \cos\alpha_1 \\ -w_2 \sin\alpha_1 \end{bmatrix} + \sin u_1 \begin{bmatrix} w_2 \\ -w_1 \cos\alpha_1 \\ w_1 \sin\alpha_1 \end{bmatrix}, \quad \Phi(u_2, u_3): \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$l(u_1)$ and $\Phi(u2, u3)$ Yields $W(x, y, z)$

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- $$= \begin{bmatrix} \cos u_2(a_2 + s\cos u_3) - \sin u_2(-b\sin a_2 + s\cos\alpha_2 \sin u_3) \\ \sin u_2(a_2 + s\cos u_3) + \cos u_2(-b\sin\alpha_2 + s\cos\alpha_2 \sin u_3) \\ d_2 + b\cos\alpha_2 + s\sin\alpha_2 \sin u_3 \end{bmatrix}$$

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- $[(x^2 + y^2 + z^2 - 2d_2 z - a_2^2 + d_2^2 - b^2 - s^2)\sin\alpha_2]^2$
  $+[(z - d_2 - b\cos\alpha_2)^2 - s^2 \sin^2\alpha_2]4a_2^2 = 0 \leftarrow \Phi(x, y, z)$

Eliminating u_2, u_3 Produces Quartic Univariate in τ

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- $\tau \rightarrow \cos u_1, \ \sin u_1$, up to four real solutions.

Computing u_3, u_2

- From parametrization of $\Phi(u_2, u_3) \to z$ recall $\sin u_3 = \frac{z - d_2 - b\cos\alpha_2}{s\sin\alpha_2}$.

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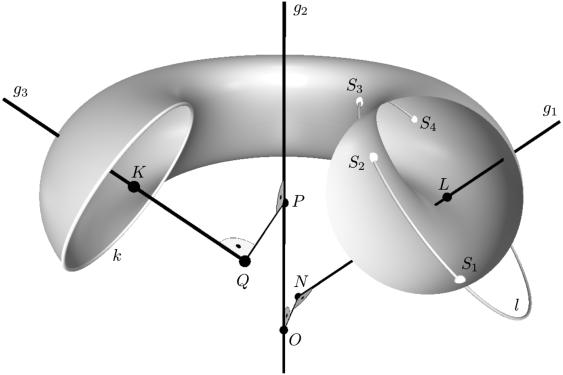
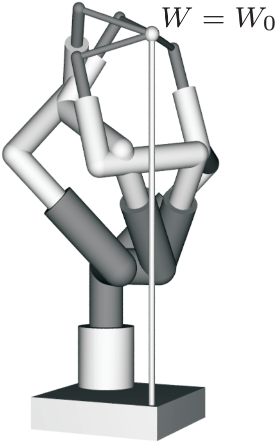
- From parametrization of $\Phi(u_2, u_3) \to x, y$ recall
- $\cos u_2 = \frac{px+qy}{x^2+y^2}$, $\sin u_2 = \frac{py-qx}{x^2+y^2}$.

Summing Up: 4 Solutions and 4 Postures





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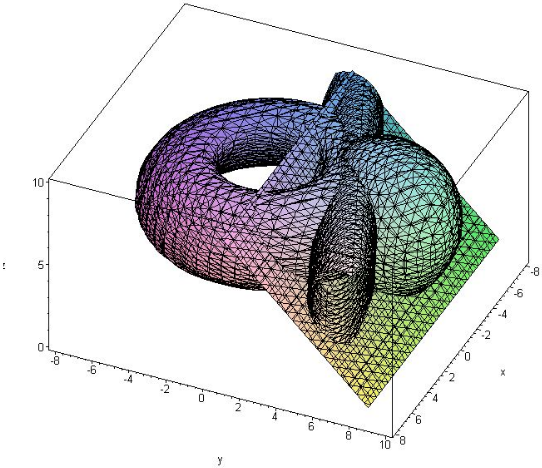
- Thus producing a quadric surface that yields the four values of S_i when combined with the sphere and plane.

All Four Surfaces Together



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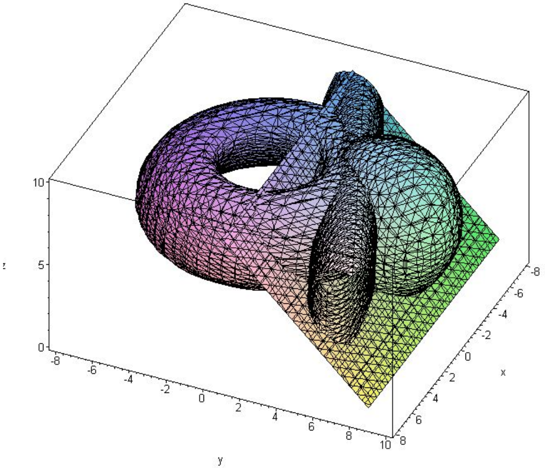

- Eliminating the absolute conic $C_\infty = x^2 + y^2 + z^2$ from the cyclid with the linear form from the sphere produces a cylinder with an elliptic right section.
- Intersection of the sphere, cylinder and plane yields the four required points $S_i, \quad i = 1, 2, 3, 4.$

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- Separating joint variables in this way renders the coefficients in the equation set quite compact
- And Identifies the cyclid as the invariant or architectural geometric element; the circle is the variable element.

Questions to Be Investigated

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- How may the cyclid-circle model be exploited to gain insight in velocity analysis? ... acceleration ... ?

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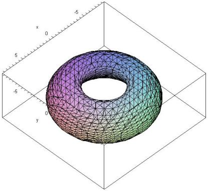

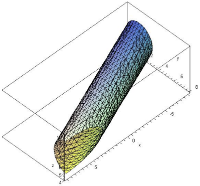

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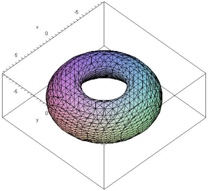

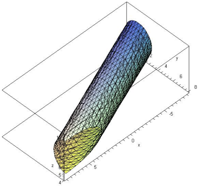

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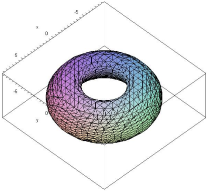

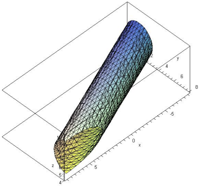

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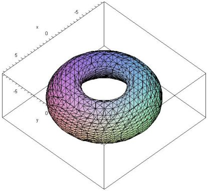

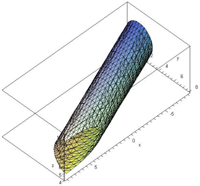

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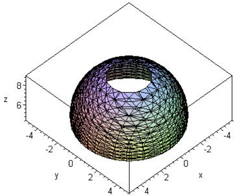

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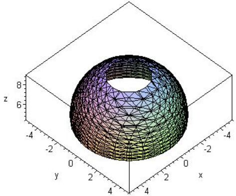

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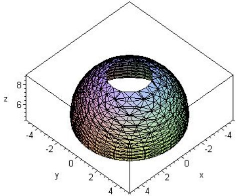

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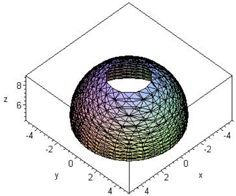

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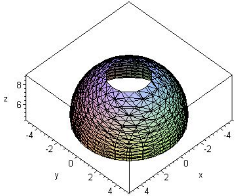

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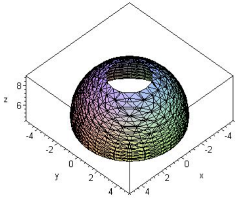

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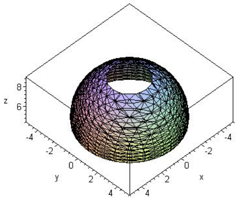

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