SYNTHESIS AND ANALYSIS OF A CONSTRAINED SPHERICAL PARALLEL MANIPULATOR

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Abstract This paper presents a methodology for constraining a spherical parallel manipulator so that it guides a body through five task positions with one degree-of-freedom. A dimensional synthesis procedure is used to constrain links of a 3-RRR spherical parallel manipulator using two spherical RR chains to obtain a spherical 10-bar linkage. The dimensions of the spherical parallel manipulator are arbitrary but we consider a design that is symmetrical, except for the platform which is an isosceles spherical triangle. Inverse kinematics analysis of the spherical manipulator provides a set of relative task positions that are used to formulate the synthesis equations for spherical RR chains. The primary challenge is the analysis of the resulting four loop spherical linkage in order to animate is movement.

Keywords: kinematic synthesis, spherical parallel manipulator, spherical linkage synthesis, 10-bar spherical linkage

1. Introduction

In this paper, we constrain a spherical parallel manipulator to obtain a 10-bar spherical linkage. This paper extends recent dimensional synthesis results for planar eight-bar linkages (Soh and McCarthy, 2007) to the design of a spherical linkage.

We begin with an arbitrarily specified 3-RRR spherical parallel manipulator and add two RR spherical chains to constrain its movement. Inverse kinematic analysis of this manipulator for five task positions of the end-effector yield five configurations for each link in the articulated system. This provides the data necessary to formulate dimensional synthesis equations for a spherical RR chain connecting any two links in the articulated system (McCarthy, 2000).

The result is a constrained spherical parallel manipulator that moves through five task positions with one degree-of-freedom.

2. Literature review

This paper focusses constrains a 3-RRR spherical parallel manipulator, Figure 1, to obtain a desired movement. The forward and inverse kinematics of this manipulator was presented by Alizade et al., 1994, and Bulca et al., 1999 analyzed its workspace.

Kong and Gosselin, 2004 classify the variety of structures available for spherical parallel manipulators, see also Gallardo et al., 2007. The focus of these papers are on ways that links and joints can be connected to obtain a "type" or topology for the system, and are examples of *type synthesis*. Our goal is to determine the dimensions of a given manipulator system, which is termed *dimensional synthesis*.

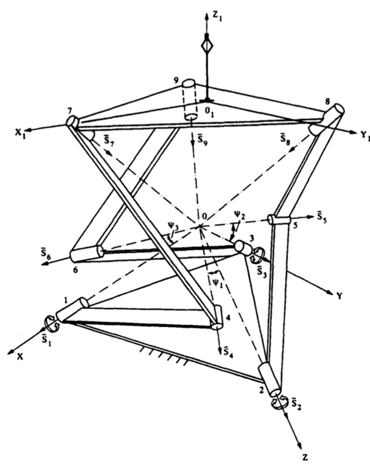

Figure 1. The 3-RRR Spherical parallel manipulator formed by an end-effector supported by three spherical 3R chains.

Once the spherical manipulator is constrained to a 10-bar linkage, it must be analyzed to determine its movement as a function of the input crank angle. We rely on the results of Wampler, 2004 to analyze this four loop spherical linkage.

3. Spherical Parallel Manipulators

A spherical parallel manipulator is a system of rigid bodies assembled so the end-effector moves about a fixed point in space. The 3-RRR spherical parallel manipulator is a three legged platform manipulator that is constructed so the joint axes of all of the revolute joints pass through this same fixed point. The result is that all of the links of the system undergo pure spatial rotation about this fixed point.

The kinematics equations of a spherical 3R robot equate the 3×3 rotation transformation $[D]$ between the end-effector and the base frame to the sequence of local coordinate transformations around the joint axes and along the links of the chain,

$$[D] = [G][Z(\theta_1)][X(\alpha_{12})][Z(\theta_2)][X(\alpha_{23})][Z(\theta_3)][H]. \tag{1}$$

The parameters θ_i define the movement at each joint and $\alpha_{i,j}$ define the angular length of the links. The transformation $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the task frame relative to the end-effector frame. The matrix $[D]$ defines the coordinate transformation from the world frame F to the task frame M.

4. The Spherical RR Constraint

A fundamental step of our spherical ten-bar synthesis methodology consists of sizing two spherical RR chains that constrains the three RRR spherical robot to one degree-of-freedom. We expand the RR synthesis equations, McCarthy, 2000, to apply to this situation. Also see Suh and Radcliffe, 1978, and Alizadeh and Kilit, 2005.

Let $[B_{l,j}]$ be five positions of the lth moving link, and $[B_{k,j}]$ be the five positions of the kth moving link measured in a world frame F, $j = 1, \ldots, 5$. Let $\mathbf{g}$ be the coordinates of the R-joint attached to the lth link measured in the link frame B_l. Similarly, let $\mathbf{w}$ be the coordinates of the other R-joint measured in the link frame B_k. The five positions of these points as the two moving bodies move between the task configurations are given by

$$\mathbf{G}^j = [B_{l,j}]\mathbf{g} \quad \text{and} \quad \mathbf{W}^j = [B_{k,j}]\mathbf{w} \tag{2}$$

Now, introduce the relative displacements

$$[R_{1j}] = [B_{l,j}][B_{l,1}]^{-1} \quad \text{and} \quad [S_{1j}] = [B_{k,j}][B_{k,1}]^{-1}, \tag{3}$$

so these equations become

$$\mathbf{G}^j = [R_{1j}]\mathbf{G}^1 \quad \text{and} \quad \mathbf{W}^j = [S_{1j}]\mathbf{W}^1 \tag{4}$$

where $[R_{11}] = [S_{11}] = [I]$ are the identity transformations.

The point $\mathbf{G}^j$ and $\mathbf{W}^j$ define the ends of a rigid angular link of length ρ, therefore we have the constraint equations

$$[R_{1j}]\mathbf{G}^1 \cdot [S_{1j}]\mathbf{W}^1 = \|\mathbf{G}^j\|\|\mathbf{W}^j\| \cos\rho \tag{5}$$

These five equations can be solved to determine the design parameters of the spherical RR constraint, $\mathbf{G}^1 = (u, v, w)$, $\mathbf{W}^1 = (x, y, z)$ and ρ. We will refer to these equations as the *synthesis equations* for the spherical RR link.

To solve the synthesis equations, it is convenient to introduce the displacements

$$[D_{1j}] = [R_{1j}]^T [S_{1j}], \tag{6}$$

so these equations become

$$\mathbf{G}^1 \cdot [D_{1j}]\mathbf{W}^1 = \|\mathbf{G}^j\|\|\mathbf{W}^j\| \cos\rho. \tag{7}$$

Subtract the first of these equations from the remaining to cancel the scalar terms $\|\mathbf{G}^j\|\|\mathbf{W}^j\| \cos\rho$, and the square terms in the variables u, v, w and x, y, z. The resulting four bilinear equations can be solved algebraically, or numerically using something equivalent to *Mathematica's* `Nsolve` function by setting $w = z = 1$ to obtain the desired pivots.

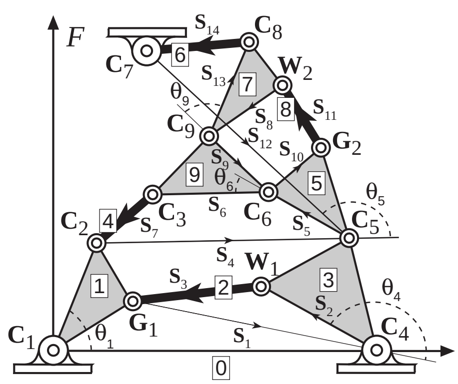

Figure 2. This shows our conventions for the synthesis and analysis of the spherical parallel platform.

5. Synthesis of a Spherical Ten-Bar Linkage

To illustrate the proposed synthesis process, we attempt to constrain a spherical parallel manipulator to one degree of freedom by adding two spherical RR chains. Let the task positions in longitude, latitude and roll co-ordinates be as shown in Table 1. Select the angular links with dimensions such that $\alpha_{12} = \alpha_{23} = \alpha_{45} = \alpha_{56} = \alpha_{78} = \alpha_{89} = 60°$. Also, we assume that the transformation $[G]$ that defines the base of the chain relative to the fixed frame, and the transformation $[H]$ that locates the task frame relative to the end-effector frame for each of the 3R spherical chains $\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3$, $\mathbf{C}_4\mathbf{C}_5\mathbf{C}_6$, and $\mathbf{C}_7\mathbf{C}_8\mathbf{C}_9$ are as defined in Table 2.

Table 1. Five task positions for the end-effector of the spherical platform in terms of longitude θ, latitude ϕ, and roll ψ.

Task	*Position* (θ, ϕ, ψ)
1	$(90°, -60°, 90°)$
2	$(77°, -35°, 82°)$
3	$(68°, -9°, 78°)$
4	$(65°, -1°, 75°)$
5	$(64°, 7°, 67°)$

Table 2. The transformation $[G]$ and $[W]$ used to locate the base of the chain relative to the fixed frame and the task frame relative to the end-effector frame.

Spherical 3R Chain	$[G] = [Y(\theta)][X(-\phi)][Z(\psi)]$	$[H] = [Y(\theta)][X(-\phi)][Z(\psi)]$
$\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3$	$(90°, -60°, 0°)$	$(-60°, 0°, 0°)$
$\mathbf{C}_4\mathbf{C}_5\mathbf{C}_6$	$(90°, 60°, 0°)$	$(0°, -60°, 0°)$
$\mathbf{C}_7\mathbf{C}_8\mathbf{C}_9$	$(90°, -180°, 0°)$	$(60°, 0°, 0°)$

Once the various spherical platform dimensions are identified, the positions of its links for the various 3R chains, B_1, B_4, B_3, B_5, B_6, and B_7 can be determined by solving the inverse kinematics of the various spherical 3R chains. Therefore we can identify five positions $T_i^{B_1}$, and $T_i^{B_3}, i = 1, \ldots, 5$ for the design of a spherical RR chain denoted $\mathbf{G}_1\mathbf{W}_1$ in Figure 2. We compute the displacements $[D_{1j}] = ([B_{1,j}][B_{1,1}]^{-1})^T[B_{4,j}][B_{4,1}]^{-1}, i = 1, \ldots, 5$, to synthesize the spherical RR chain $\mathbf{G}_1\mathbf{W}_1$ using Eq. 7. Similarly, we identify five positions $T_i^{B_5}$, and $T_i^{B_7}, i = 1, \ldots, 5$ for the design of a spherical RR chain $\mathbf{G}_2\mathbf{W}_2$.

The five task positions listed in Table 1 yield three design candidates to constrain the spherical parallel manipulator, Table 3. Figure 3 shows the chosen design, and Figure 4 shows the constrained spherical parallel manipulator passing through each of the specified task positions.

6. Analysis of the Spherical Ten-Bar Linkage

The analysis of the spherical ten-bar linkage as shown in Figure 2 is equivalent to the displacement analysis of a single loop spherical triangle $\mathbf{G}_1\mathbf{C}_4\mathbf{W}_1$, and a three loop spherical type 3b structure $\mathbf{C}_2\mathbf{C}_3\mathbf{C}_9\mathbf{C}_6\mathbf{C}_5\mathbf{G}_2\mathbf{W}_2\mathbf{C}_8\mathbf{C}_7$. See Figure 5 for the various spherical structures having three or fewer loops.

To see how we can decompose this ten-bar spherical linkage into a spherical triangle and type 3b structure , we label the links on the spher-

Table 3. The spherical RR chain solutions obtained. The highlighted pivots were selected for the design.

S/N	$\mathbf{G}_1$	$\mathbf{W}_1$
1	$(-0.9456, -0.0093, 0.3250)$	$(-0.9138, -0.0213, 0.4057)$
2	$(-0.8793, -0.1323, 0.4575)$	$(-0.8989, -0.1716, 0.4032)$
3	$(\mathbf{-0.3041}, \mathbf{-0.6490}, \mathbf{0.6974})$	$(\mathbf{-0.7338}, \mathbf{-0.5137}, \mathbf{0.4445})$
4	$(0.5000, -0.8660, 0.0000)$	$(0.5000, 0.8660, 0.0000)$
5	$(0.35 + 1.27i, 3.44 - 0.62i, 0.56 + 3.05i)$	$(1.09 + 1.24i, 2.75 - 2.15i, 1.96 + 2.33i)$
6	$(0.35 - 1.27i, 3.44 + 0.62i, 0.56 - 3.05i)$	$(1.09 - 1.24i, 2.75 + 2.15i, 1.96 - 2.33i)$

S/N	$\mathbf{G}_2$	$\mathbf{W}_2$
1	$(0.54 - 29.18i, 20.60 - 2.00i, 20.96 + 2.72i)$	$(9.63 - 26.69i, 19.50 + 4.76i, 18.49 + 8.88i)$
2	$(0.54 + 29.18i, 20.60 + 2.00i, 20.96 - 2.72i)$	$(9.63 + 26.69i, 19.50 - 4.76i, 18.49 - 8.88i)$
3	$(0.11 - 0.01i, -0.71 - 0.01i, 0.70 - 0.00i)$	$(0.11 - 0.01i, -0.71 - 0.00i, 0.70 - 0.00i)$
4	$(0.11 + 0.01i, -0.71 + 0.01i, 0.70 + 0.00i)$	$(0.11 + 0.01i, -0.71 + 0.00i, 0.70 + 0.00i)$
5	$(\mathbf{0.1964}, \mathbf{-0.6483}, \mathbf{0.7356})$	$(\mathbf{0.2562}, \mathbf{-0.6482}, \mathbf{0.7170})$
6	$(-0.2500, -0.4330, 0.8660)$	$(0.5000, 0.8660, 0.0000)$

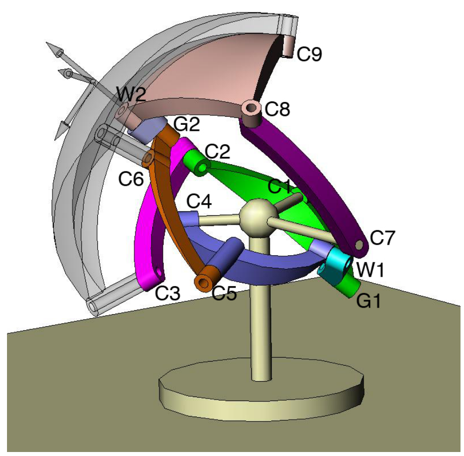

Figure 3. The resulting constrained 3-RRR spherical parallel manipulator, which is a spherical 10-bar linkage.

ical linkage as $0, 1, \ldots, 9$ as shown in Figure 2. Given an input angle θ_1, we can merge links 0 and 1 into a composite rigid link. We use (01) to represent this composite link. Now, $(01), 2, 3$ forms a spherical triangle. We solve it to obtain two solutions. For each solution, we merge them into one big composite link (0123). Now (0123) and $4, 5, 6, 7, 8, 9$ forms

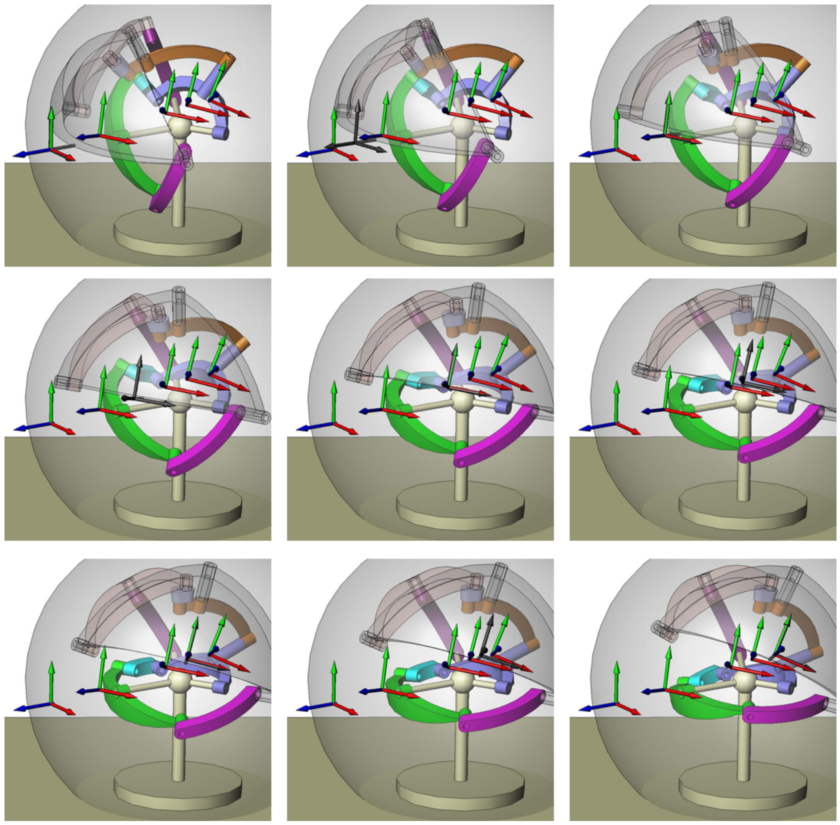

Figure 4. A sequence of images showing the movement of the spherical 10-bar linkage reaching each of the five task positions.

a spherical type 3b structure. We solve it to obtain 24 solutions, hence a total of 48 solutions for this ten-bar spherical topology.

6.1 Analyzing the Spherical Triangle

The loop equation for the spherical triangle $\mathbf{G}_1\mathbf{C}_4\mathbf{W}_1$ may be written as

$$Z_{G_1} S_1 Z_4 S_2 Z_{W_1} S_3 = I \tag{8}$$

where Z_i is a joint rotation and S_i is a side rotation and I is the identity rotation. Z_i is a rotation about the z-axis, and may be written in terms of a rotation angle θ_i as

$$Z_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

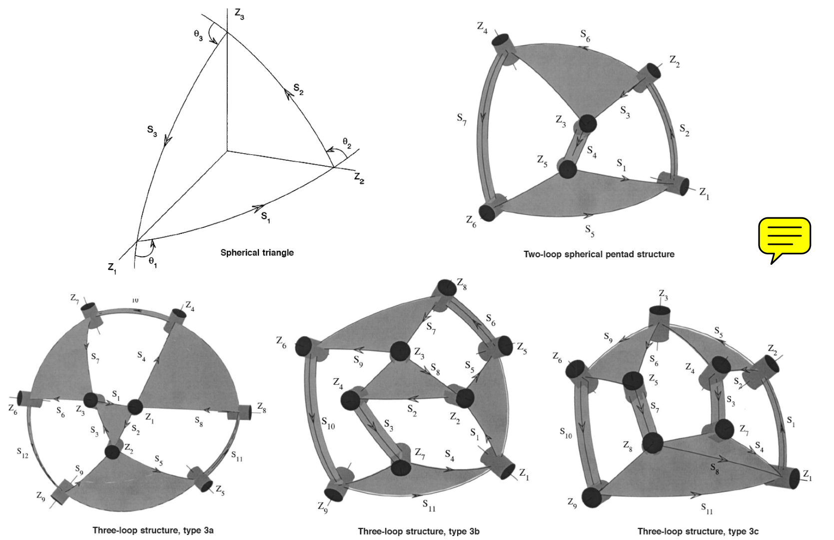

Figure 5. The various basic spherical structures having three or fewer loops. in the first row, we have the single-loop spherical triangle, and the two-loop pentad structure. The second row consists of the three-loop type 3a, type 3b, and type 3c structures.

Let $\mathbf{z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $\cos\theta_i = (1-t_i^2)/(1+t_i^2)$ and $\sin\theta_i = -2t_i/(1+t_i^2)$. If we pre and post multiply $\mathbf{z}^T$ and $S_3^T\mathbf{z}$ respectively, we can simplify Eq. 8 into

$$\mathbf{z}^T S_1 \hat{Z}_4 S_2 \mathbf{z} = \mathbf{z}^T S_3^T \mathbf{z}(1+t_4^2). \tag{10}$$

where $S_1 = R_y(2.37)$, $S_2 = R_y(2.52)$, and $S_3 = R_y(0.52)$ for $\theta_1 = 0.34rad$, and

$$\hat{Z}_i = \begin{bmatrix} 1-t_i^2 & -2t_i & 0 \\ 2t_i & 1-t_i^2 & 0 \\ 0 & 0 & 1+t_i^2 \end{bmatrix}. \tag{11}$$

We solve this quadratic equation for t_4.

6.2 Analyzing the Spherical 3b Structure

The loop equations for the three-loop spherical structure 3b $\mathbf{C}_2\mathbf{C}_3\mathbf{C}_9$ $\mathbf{C}_6\mathbf{C}_5\mathbf{G}_2\mathbf{W}_2\mathbf{C}_8\mathbf{C}_7$ can be written as

$$\begin{aligned}
&Z_2 S_4 Z_5 S_5 Z_6 S_6 Z_3 S_7 = I, \\
&Z_{W_2} S_8' Z_9 S_9' Z_6 S_{10} Z_{G_2} S_{11} = I, \\
&Z_7 S_{12}' Z_5 S_5' Z_6 S_9 Z_9 S_{13} Z_8 S_{14} = I,
\end{aligned}$$

Using the simplification process as above, one obtains the following three equations in t_5, t_6, and t_9:

$$\begin{aligned}
&f_1 : \mathbf{z}^T[S_4\hat{Z}_5 S_5\hat{Z}_6 S_6 - S_7^T(1+t_5^2)(1+t_6^2)]\mathbf{z} = 0,\\
&f_2 : \mathbf{z}^T[S_8'\hat{Z}_9 S_9'\hat{Z}_6 S_{10} - S_{11}^T(1+t_6^2)(1+t_9^2)]\mathbf{z} = 0,\\
&f_3 : \mathbf{z}^T[S_{12}'\hat{Z}_5 S_5'\hat{Z}_6 S_9\hat{Z}_9 S_{13} - S_{14}^T(1+t_5^2)(1+t_6^2)(1+t_9^2)]\mathbf{z} = 0
\end{aligned}$$

where $S_4 = R_y(2.52)$, $S_5 = R_y(1.05)$, $S_5' = R_y(1.05)R_z(-2.21)$, $S_6 = R_y(1.32)$, $S_7 = R_y(1.05)$, $S_8' = R_y(0.77)R_z(-0.99)$, $S_9 = R_y(1.32)$, $S_9' = R_y(1.32)R_z(1.20)$, $S_{10} = R_y(0.26)$, $S_{11} = R_y(0.06)$, $S_{12}' = R_y(1.62)R_z(-2.78)$, $S_{13} = R_y(1.05)$ and $S_{14} = R_y(1.05)$ for $\theta_1 = 0.34rad$.

6.3 Elimination Procedure

To solve this system, first augment f_1 with $\{1, t_9, t_9^2, t_9^3\} \otimes \{1, t_5\} \otimes \{1, t_6\}$, f_2 with $\{1, t_5, t_5^2, t_5^3\} \otimes \{1, t_6\} \otimes \{1, t_9\}$, and f_3 with $\{1, t_5\} \otimes \{1, t_6\} \otimes \{1, t_9\}$ to get 40 equations in the 64 monomials $\mathbf{m} = \{1, t_5, t_5^2, t_5^3\} \otimes \{1, t_6, t_6^2, t_6^3\} \otimes \{1, t_9, t_9^2, t_9^3\}$. The excess of monomials over the equations is $64 - 40 = 24$, so we must append 24 identities to get a system of 64 polynomials in 64 monomials. The 24 identities are formulated such that they satisfy $t_9\mathbf{m}_1 - \mathbf{m}_2 = 0$, where $\mathbf{m}_1$ is the list of monomials in the set $[(\{1, t_5, t_5^2, t_5^3\} \otimes \{1, t_2\}) \bigcup \{t_6^2, t_5t_6^2, t_6^3, t_5t_6^3\}] \otimes \{1, t_9\}$, and $\mathbf{m}_2 = t_9 \otimes \mathbf{m}_1$.

Writing the equations in block matrix form and letting $t_9 = x$ we get

$$[\hat{K}(x)]\mathbf{m} = \begin{bmatrix} K\mathbf{m} \\ x\mathbf{m}_1 - \mathbf{m}_2 \end{bmatrix} = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ I_1x + C_1 & C_2 & 0 & 0 \\ 0 & I_2x & -I_2 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{Bmatrix} = 0, \tag{12}$$

where $\mathbf{y}_1 = \mathbf{m}_1 \backslash \mathbf{y}_2, \mathbf{y}_2 = \mathbf{y}_3/x, \mathbf{y}_3 = \mathbf{m}_2 \backslash \mathbf{m}_1, \mathbf{y}_4 = \mathbf{m} \backslash (\mathbf{m}_1 \bigcup \mathbf{m}_2)$ are the four partition sets of the monomials $\mathbf{m}$. I_1,I_2 are identity matrices and C_2 are sparse, having a single entry of -1 in each row of the pair. Now we reduce K_4 to upper triangular form, and premultiply by a matrix that annihilates $\mathbf{y}_3, \mathbf{y}_4$ to obtained,

$$\begin{bmatrix} 0 & 0 & I_1 & 0 \\ 0 & I_2 & 0 & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} & \tilde{K}_{13} & U \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} & 0 \\ I_1x + C_1 & C_2 & 0 & 0 \\ 0 & I_2x & -I_2 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{Bmatrix} = 0. \tag{13}$$

Multiplying this out and dropping the trailing trivial columns we manage to reduce the system of 64 polynomial equations to a 24×24 generalize eigenvalue problem,

$$\begin{bmatrix} I_1x + C_1 & C_2 \\ \tilde{K}_{21} & \tilde{K}_{22} + \tilde{K}_{23}x \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} = 0. \tag{14}$$

Each value of t_9 has an associated eigenvector $(\mathbf{y}_1, \mathbf{y}_2)^T$. The eigenvectors are up to scale corresponding to the monomial 1 in $\mathbf{m}_1$. Divide out the scale factor to retrieve the values of the remaining joint angles t_5, t_6.

7. Conclusions

This paper presents a strategy to constrain a 3-RRR spherical parallel manipulator with two spherical RR chains to obtain a spherical 10-bar linkage. The resulting device performs a one degree-of-freedom movement through five arbitrary task positions. The analysis of this resulting four loop spherical linkage is presented using Wampler's method. An example design and analysis is presented.

8. Acknowledgements

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