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A Cylinder of Revolution on Five Points

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Abstract. Although a general quadric surface is uniquely defined on nine linearly independent given points, special and possibly degenerate quadrics can be generated on fewer if certain constraints, implied or explicit, apply. *E.g.*, coefficients of the implicit equation of a unique sphere may be generated with four given points and five constraint equations. The sphere is special but not degenerate. This article addresses a specific degenerate case, an arbitrary disposition of five given points so as to unambiguously define up to six cylinders of revolution upon them. An approach based on geometric constraints concerning the distance between any two points on the surface yields a sestic univariate in one of the cylinder axis direction numbers. Three linear variables are eliminated from the five initially formulated second order constraints. A cubic and quadratic intermediate pair of equations is produced. These contain the three homogeneous axial direction numbers. Projection of the given points onto any normal plane reveals that the five projected images lie on a circle.

Key Words: Algebraic Geometry, cylinder of revolution, degenerate quadric

MSC: 51N20, 51N05

1. Introduction

Recently representatives of *FARO Technologies, Inc.* in Florida exhibited their six revolute jointed serial robot arm equipped with end effectors like a simple miniature touch-sphere or a raster scanning laser range-finder that can capture thousands of points on a spatial surface and resolve their location to within a sphere of $25\,\mu$ radius as the user manually moves the end effector to scan the part. One example was to measure a round cylinder by touching many points on its surface. The device is capable of incorporating many more points than the minimum number required and statistically establishing a consensus that improves the accuracy of the solid model obtained. "How many represent a minimum number?" was asked. "Six" was the reply. This established a reason for this article. What are the principles underlying the minimal point definition of simple surfaces? Using analytical and descriptive geometry a robust and efficient algorithm to unambiguously identify cylinders of revolution on

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five points, by finding cylinder axis directions, is outlined and tested. The results are not new since there exists a considerable body of modern pertinent literature, *e.g.*, by Devilliers et al. [1] and Lichtblau [2]. It is interesting to note that these papers in English make no reference to earlier research in German. Nevertheless the opportunity to expose what is believed to be a somewhat novel and elegant approach, based on simple engineering vector algebra, is taken.

2. Key equation and solution

Manfred Husty of the University of Innsbruck, together with Hans-Peter Schröcker, devised and solved a system of four simultaneous equations based on earlier work by Hermann Schaal who studied the distance relation between points on a cylinder of revolution [3] and the one parameter system of such cylinders specified to lie on four points [4] to produce equations like

$$(\mathbf{x} \times \mathbf{a})^2 - 2\mathbf{a}^2(\mathbf{x} \cdot \mathbf{f}) = 0 \tag{1}$$

where $\mathbf{x}$ indicates the position vector from an origin, O, chosen to be on one of the given points, to each of the other four. $\mathbf{a}$ is any unknown vector in the direction of the cylinder's axis and all its generators. $\mathbf{f}$ is the vector from O to the axis and normal to $\mathbf{a}$. Derivation of Eq. (1) is detailed in the appendix, Section 4. The two additional conditions, necessary to obtain the six unknowns, *i.e.*, elements of $\mathbf{a}$ and $\mathbf{f}$, are

$$\mathbf{a} \cdot \mathbf{f} = 0, \quad \mathbf{a}^2 = 1\,. \tag{2}$$

The five given points are O, P, Q, R, S. There is no loss in generality if O is taken as the Cartesian origin, P is on the x-axis and Q is on the plane $z = 0$. This gives

$$[\mathbf{o}\,\mathbf{p}\,\mathbf{q}\,\mathbf{r}\,\mathbf{s}] = \begin{bmatrix} 0 & p_1 & q_1 & r_1 & s1 \\ 0 & 0 & q_2 & r_2 & s_2 \\ 0 & 0 & 0 & r_3 & s_3 \end{bmatrix}. \tag{3}$$

Eq. (1) and the first of (2) produce five equations in the six unknowns

$$\mathbf{a} = [a_1\; a_2\; a_3]^T, \quad \mathbf{f} = [f_1\; f_2\; f_3]^T : \tag{4}$$

$$\begin{aligned} a_1 f_1 + a_2 f_2 + a_3 f_3 &= 0, \\ p_1^2 a_3^2 + p_1 a_2^2 - 2p_1(a_1^2 + a_2^2 + a_3^2) f_1 &= 0, \\ q_2^2 a_3^2 + q_1^2 a_3^2 + (q_1 a_2 - q_2 a_1)^2 - 2(a_1^2 + a_2^2 + a_3^2)(q_1 f_1 + q_2 f_2) &= 0, \\ (r_2 a_3 - r_3 a_2)^2 + (r_3 a_1 - r_1 a_3)^2 + (r_1 a_2 - r_2 a_1)^2 & \\ - 2(a_1^2 + a_2^2 + a_3^2)(r_1 f_1 + r_2 f_2 + r_3 f_3) &= 0, \\ (s_2 a_3 - s_3 a_2)^2 + (s_3 a_1 - s_1 a_3)^2 + (s_1 a_2 - s_2 a_1)^2 & \\ -2\,(a_1^2 + a_2^2 + a_3^2)(s_1 f_1 + s_2 f_2 + s_3 f_3) &= 0. \end{aligned} \tag{5}$$

Elimination of f_1, f_2, f_3 from the system (5) leaves two equations, (6), in a_1, a_2, a_3:

$$\begin{aligned} b_1 a_1 a_2 a_3 + b_2 a_1 a_2^2 + b_3 a_1 a_3^2 + b_4 a_3^3 + b_5 a_2 a_3^2 & \\ + b_6 a_1^2 a_2 + b_7 a_2^3 + b_8 a_1^2 a_3 + b_9 a_2^2 a_3 &= 0, \\ c_1 a_1^2 + c_2 a_2^2 + c_3 a_3^2 + c_4 a_2 a_3 + c_5 a_1 a_3 + c_6 a_1 a_2 &= 0. \end{aligned} \tag{6}$$

Rather than taking **a** to be a unit vector, as implied by the second of Eq. (2), we set $a_3 = 1$ instead. Note that the constraints imposed by (5) are independent of the magnitude and sense of **a**. If a_3 turns out to be $a_3 = 0$, due to the disposition of the five given points, one may set either $a_2 = 1$ or $a_1 = 1$. Since a_1 does not appear as a_1^3 we choose to eliminate a_1 in Eq. (6).

$$\begin{aligned} (b_6a_2 + b_8)a_1^2 + (b_2a_2^2 + b_1a_2 + b_3)a_1 & \\ +(b_7a_2^3 + b_9a_2^2 + b_5a_2 + b_4) &= 0\,, \\ c_1a_1^2 + (c_6a_2 + c_5)a_1 + (c_2a_2^2 + c_4a_2 + c_3) &= 0\,. \end{aligned} \tag{7}$$

Upon eliminating a_1 a sestic univariate is obtained.

$$D_1a_2^6 + D_2a_2^5 + D_3a_2^4 + D_4a_2^3 + D_5a_2^2 + D_6a_2 + D_7 = 0\,. \tag{8}$$

Seven coefficients D_i, $i = 1, \ldots, 7$, in terms of the already compressed coefficients, engender the given point coordinates of O, P, Q, R, S, *i.e.*, b_j, $j = 1, \ldots, 9$, and c_k, $k = 1, \ldots, 6$.

$$\begin{aligned} D_1 &= d_1d_2 - d_3^2, \\ D_2 &= d_4d_2 + d_1d_5 - 2d_8d_3, \\ D_3 &= d_9d_2 + d_4d_5 + d_1d_6 - 2d_7d_3 - d_8^2, \\ D_4 &= d_9d_5 + d_4d_6 + d_1d_{10} - 2d_{11}d_3 - 2d_7d_8, \\ D_5 &= d_9d_6 + d_4d_{10} + d_1d_{12} - 2d_{11}d_8 - d_7^2, \\ D_6 &= d_9d_{10} + d_4d_{12} - 2d_{11}d_7, \\ D_7 &= d_9d_{12} - d_{11}^2, \end{aligned} \tag{9}$$

where

$$\begin{aligned} d_1 &= c_1b_2 - b_6c_6, & d_2 &= c_6b_7 - b_2c_2 \\ d_3 &= c_1b_7 - b_6c_2, & d_4 &= c_1b_1 - b_8c_6 - b_6c_5 \\ d_5 &= c_5b_7 + c_6b_9 - b_1c_1 - b_2c_4, & d_6 &= c_5b_9 + c_6b_5 - b_3c_2 - b_1c_4 - b_2c_3 \\ d_7 &= c_1b_5 - b_8c_4 - b_6c_3, & d_8 &= c_1b_9 - b_8c_2 - b_6c_4, \\ d_9 &= c_1b_3 - b_8c_5, & d_{10} &= c_5b_5 + c_6b_4 - b_3c_4 - b_1c_3 \\ d_{11} &= c_1b_4 - b_8c_3, & d_{12} &= c_5b_4 - b_3c_3 \end{aligned} \tag{10}$$

These 15 additional coefficients are written

$$\begin{aligned} b_1 &= 2q_2r_2(q_1 - r_1), & b_2 &= q_2r_3(p_1 - 2q_1), \\ b_4 &= q_2[r_1(r_1 - p_1) + r_2(r_2 - q_2)] + q_1r_2(p_1 - q_1), & b_3 &= q_2r_3(p_1 - 2r_1), \\ b_5 &= r_3[q_2(q_2 - 2r_2) + q_1(q_1 - p_1)], & b_6 &= q_2^2r_3, \\ b_7 &= q_1r_3(q_1 - p_1), & b_8 &= q_2[r_2(r_2 - q_2) + r_3^2], \\ b_9 &= q_2[r_1(r_1 - p_1) + r_3^2] + q_1r_2(p_1 - q_1), \end{aligned}$$

$$\begin{aligned} c_1 &= q_2[r_2s_3(r_2 - q_2) + r_3s_2(q_2 - s_2) + r_3s_3(r_3 - s_3)], \\ c_2 &= q_2r_3s_3(r_3 - s_3) + q_1^2(r_3s_2 - r_2s_3) \\ &\quad + p_1[q_2(r_3s_1 - r_1s_3) + q_1(r_2s_3 - r_3s_2)] + q_2(r_1^2s_3 - r_3s_1^2), \\ c_3 &= q_2[p_1(s_1r_3 - s_3r_1) - r_3(s_1^2 + s_2^2) + s_3(r_1^2 + r_2^2)] \\ &\quad + (r_2s_3 - r_3s_2)[q_1(p_1 - q_1) - q_2^2], \\ c_4 &= 2q_2r_3s_3(s_2 - r_2), \\ c_5 &= 2q_2r_3s_3(s_1 - r_1), \\ c_6 &= 2q_2[r_3s_2(s_1 - q_1) + r_2s_3(q_1 - r_1)]. \end{aligned} \tag{11}$$

Once values of a_2 have been determined with Eq. (8) a linear equation in a_1 can be produced by eliminating a_2^2 between Eqs. (7), thus

$$\begin{aligned}&[c_1(b_2a_2^2+b_1a_2+b_3)-(b_6a_2+b_8)(c_6a_2+c_5)]a_1\\+&[c_1(b_7a_2^3+b_9a_2^2+b_5a_2+b_4)-(b_6a_2+b_8)(c_2a_2^2+c_4a_2+c_3)]=0\,.\end{aligned} \tag{12}$$

2.1. A numerical example

The five given points

$$O(0,0,0),\quad P(3,0,0),\quad Q(2,2,0),\quad R(0,2,4),\quad S(2,0,3)$$

were coded into Eqs. (11), (10) and (9), in that order, and solved for a_2 with Eq. (8). The two real roots were used in (12) to get the corresponding values of a_1. Of course $a_3 = 1$ in all cases.

a_3	a_2	a_1
1	0.3876994216	0.01221017201
1	$2.079377226+1.357288502i$	$\cdots$
1	$-1.32199443+0.3554442093i$	$\cdots$
1	-0.01630698077	-0.3397443430
1	$-1.32199443-0.3554442093i$	$\cdots$
1	$2.079377226-1.357288502i$	$\cdots$

There are a number of ways to find the two cylinders represented by the first and fourth axial direction numbers, the only real ones, tabulated above. *E.g.*, one may find four additional points on each cylinder by parameterizing lines on four of the given points, say, P, Q, R, S, using a vector on the point $A_\omega\{0 : a_1 : a_2 : a_3\}$ multiplied by the parameter t and finding the point intersections P_a, Q_a, R_a, S_a on the plane $a\{A_0 : A_1 : A_2 : A_3\} = \{0 : a_1 : a_2 : a_3\}$ on O

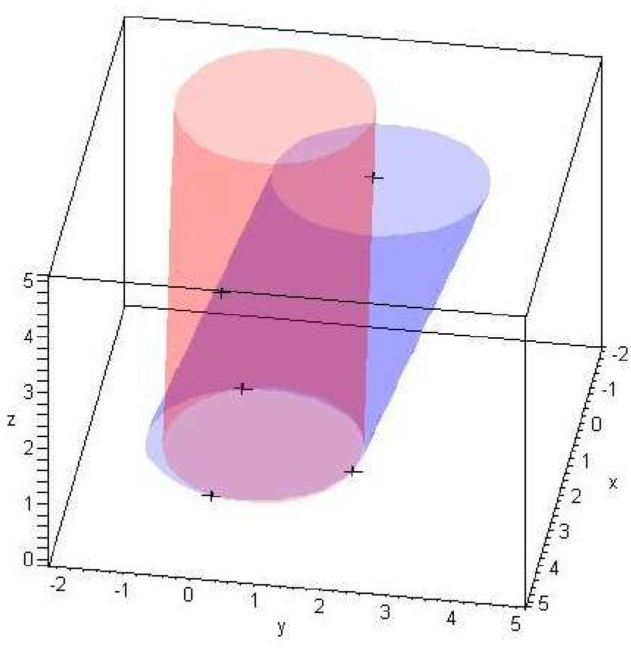

Figure 1: Two real solutions on given example with five specific points

and normal to these lines. Pertinent equations to define, say, P_a and find t are as follows.

$$P_a:\ \mathbf{p}_a = \mathbf{p} + \mathbf{a}_\omega t, \qquad \begin{bmatrix} p_{A_1} \\ p_{A_2} \\ p_{A_3} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} a_1 t \\ a_2 t \\ a_3 t \end{bmatrix},$$

$$(p_1 + a_1 t)a_1 + (p_2 + a_2 t)a_2 + (p_3 + a_3 t)a_3 = 0\,.$$

Then these nine points may be used to find the ten quadric coefficients on the cylinder by Grassmannian expansion of the 10×10 determinant of the singular matrix of the ten quadratic variable point forms in the implicit equation of the quadric. These two implicit equations are plotted to show the two real cylinders in Fig. 1.

But are these two quadrics really and truly cylinders of revolution? What one sees in Fig. 2 is a descriptive geometric construction that is equivalent to projecting all five points, including O, onto plane a, finding a circular section centre on the intersection of right bisectors of chords, say, $O_A P_A$ and $Q_A R_A$, and establishing that the distances from such a centre to all five point projections are identical.

This constructive verification, that involves only an end view of any vector $\mathbf{a}[a_1\ \ a_2\ \ a_3]^T$, seems to be more convenient than algebraically finding four planes and a line to establish the condition of equidistance.

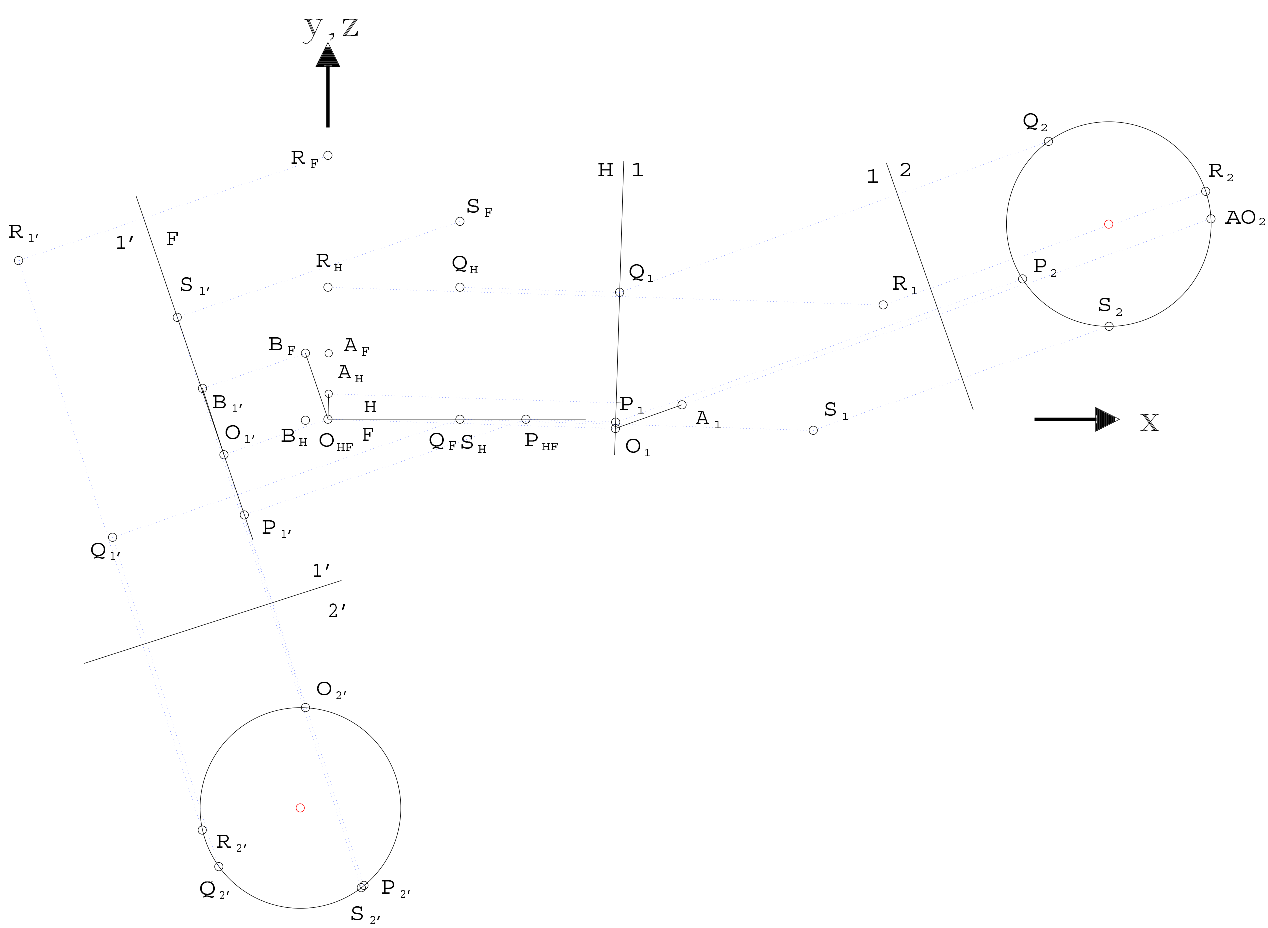

Figure 2: Solutions verified by projection in computed axial directions

2.2. Six solutions

It is not hard to imagine a cube of unit edge length where $O(0,0,0)$, $P(1,0,0)$, $Q(1,1,0)$, $R(0,0,1)$, $S(1,1,1)$ are vertices. Three cylinders of revolution circumscribe the three pairs of parallel face squares.

A fourth may be constructed perpendicular to a plane normal to parallel face diagonals OQ and RS. Another cylinder whose generators are parallel to the cube's space diagonal that is coplanar with the one on P also contains all five points which appear on the vertices of a regular hexagon when the cylinder is viewed in a projection normal to cylinder axis. However there does not appear to be a sixth cylinder. There is a double solution. However A. Gfrerrer at TU-Graz has shown that six distinct real solutions may indeed exist, *e.g.*, on the six points $O(0,0,0)$, $P(5,0,0)$, $Q(0,5,0)$, $R(0,0,5)$, $S(5,4,4)$.

3. Conclusion

Although this particular problem may appear to have been thoroughly and sufficiently worked over, nevertheless there remain related issues as yet unsolved. These involve the definition of other special quadrics that may be specified by less than nine points, *e.g.*, surfaces of revolution, and the incorporation of over-determined point sets such as may be encountered in inspection and metrology based on digitized optical data. For instance, the *FARO* robot introduced in Section 1 can measure cylindrical shafts and holes using many more than five surface points however no references that deal with the nature and efficiency of the procedure, used to process redundant data, have been discovered.

4. Appendix

Fig. 3 shows an axial view of the cylinder. O is one of the five given points. There is no loss in generality to take it as the Cartesian origin. A is any point on the generator on O while X represents any one of the other four given points on the cylinder. $\mathbf{x}$ spans OX, $\mathbf{a}$ spans OA and is parallel to the axis while $\mathbf{f}$ spans from O to the axis and is normal to it. A constructive solution follows. Note that $\mathbf{x}\times\mathbf{a}/|\mathbf{a}|$ and $(\mathbf{x}\cdot\mathbf{f})\mathbf{f}/\mathbf{f^2}$ represent a vector whose *length* is that of the projection of $\mathbf{x}$ normal to axial direction parallel to $\mathbf{a}$ and vector, along $\mathbf{f}$, of length equal to the projection of $\mathbf{x}$ on $\mathbf{f}$, respectively. A purely geometric construction to obtain the circle whose diameter is $2\mathbf{x}\cdot\mathbf{f}/|\mathbf{f}|$ is shown in Fig. 3. Note how the length of $\mathbf{x}\times\mathbf{a}/|\mathbf{a}|$ is transferred to the vertical side of the right triangle on base OX and, with an arc centred on X, cuts the diameter $2|\mathbf{f}|$ of the circular section of the cylinder.

Now Fig. 4 shows, via similar triangles, the proportionality between the cylinder radius $|\mathbf{f}|$ and the smaller, construction circle radius to be

$$\frac{\mathbf{x}\cdot\mathbf{f}/|\mathbf{f}|}{|\mathbf{f}|} = \frac{|\mathbf{x}\times\mathbf{a}|}{|\mathbf{a}|\,|\mathbf{f}|}\,.$$

Therefore one may write

$$\frac{2\mathbf{x}\cdot\mathbf{f}}{|\mathbf{f}|} = \frac{|\mathbf{x}\times\mathbf{a}|\,|\mathbf{x}\times\mathbf{a}|}{|\mathbf{a}|\,|\mathbf{a}|\,|\mathbf{f}|}$$

which is seen to be Eq. (1).

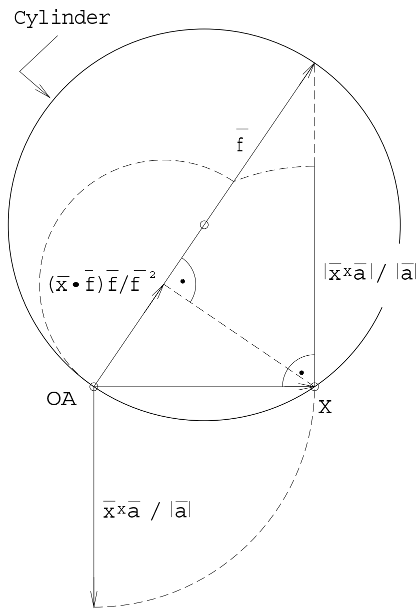

Figure 3: Cylinder and vectors pertaining to Eq. (1)

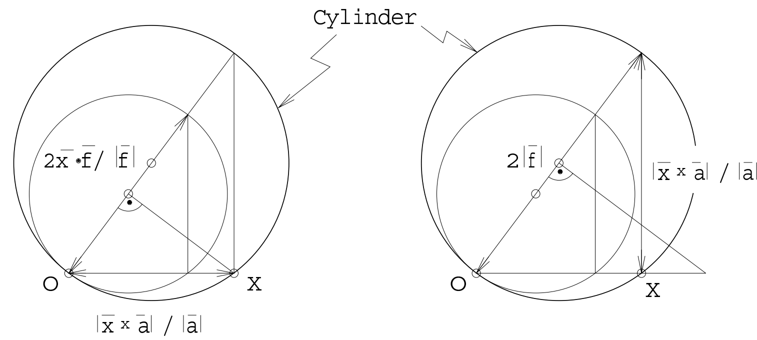

Figure 4: Scaling via similar triangles

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