Questions

1. Use mathematical induction to prove that, for any $n \geq 1$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$

2. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

3. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Answers

1. The base case is easy. Substitute $n = 1$ and we get $1 = 1$ which is true.

   For the induction step, we *hypothesize* that

$$\sum_{i=0}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

   for $k \geq 0$, and we want to show it follows from this hypothesis that

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}.$$

   Take the left side of the last equation, and rewrite it:

$$\begin{aligned}
\sum_{i=0}^{k} x^i &= \sum_{i=0}^{k-1} x^i + x^k \\
&= \frac{x^k - 1}{x - 1} + x^k, \quad \text{by induction hypothesis} \\
&= \frac{x^k - 1}{x - 1} + x^k(\frac{x - 1}{x - 1}) \\
&= \frac{x^{k+1} - 1}{x - 1}
\end{aligned}$$

   which is what we wanted to show.

2. The base case of $n_0 = 1$ is obvious, since there is only a single term on the left hand side, i.e. $1 = 1^2$. The induction hypothesis is the statement $P(k)$:

$$P(k) \equiv \text{"}\ 1 + 3 + 5 + \cdots + (2k - 1) = k^2\ \text{"}$$

   To prove the induction step, we show that if $P(k)$ is true, then $P(k + 1)$ must also be true. As usual, we take the left side of the equation of $P(k + 1)$:

$$\begin{aligned}
\sum_{i=1}^{k+1} (2i - 1) &= 2(k + 1) - 1 + \sum_{i=1}^{k} (2i - 1) \\
&= 2(k + 1) - 1 + k^2, \quad \text{by the induction hypothesis} \\
&= 2k + 1 + k^2 \\
&= (k + 1)^2.
\end{aligned}$$

   Thus, the induction step is also proved, and so we're done.

3. The base case is trivially obvious since $1^3 = 1^2$.

   To prove the induction step, we write

$$
\begin{aligned}
& 1 + 2^3 + 3^3 + \cdots + k + (k+1)^3 \\
&= (1 + 2 + \cdots + k)^2 + (k+1)^3 \quad \text{by the induction hypothesis} \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= (\frac{k^2}{4} + (k+1)) * (k+1)^2 \\
&= \frac{1}{4}(k^2 + 4k + 4) * (k+1)^2 \\
&= \frac{1}{4}(k+2)^2 * (k+1)^2 \\
&= \{\frac{1}{2}(k+2)(k+1)\}^2 \\
&= (1 + 2 + 3 + \cdots + k + (k+1))^2.
\end{aligned}
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