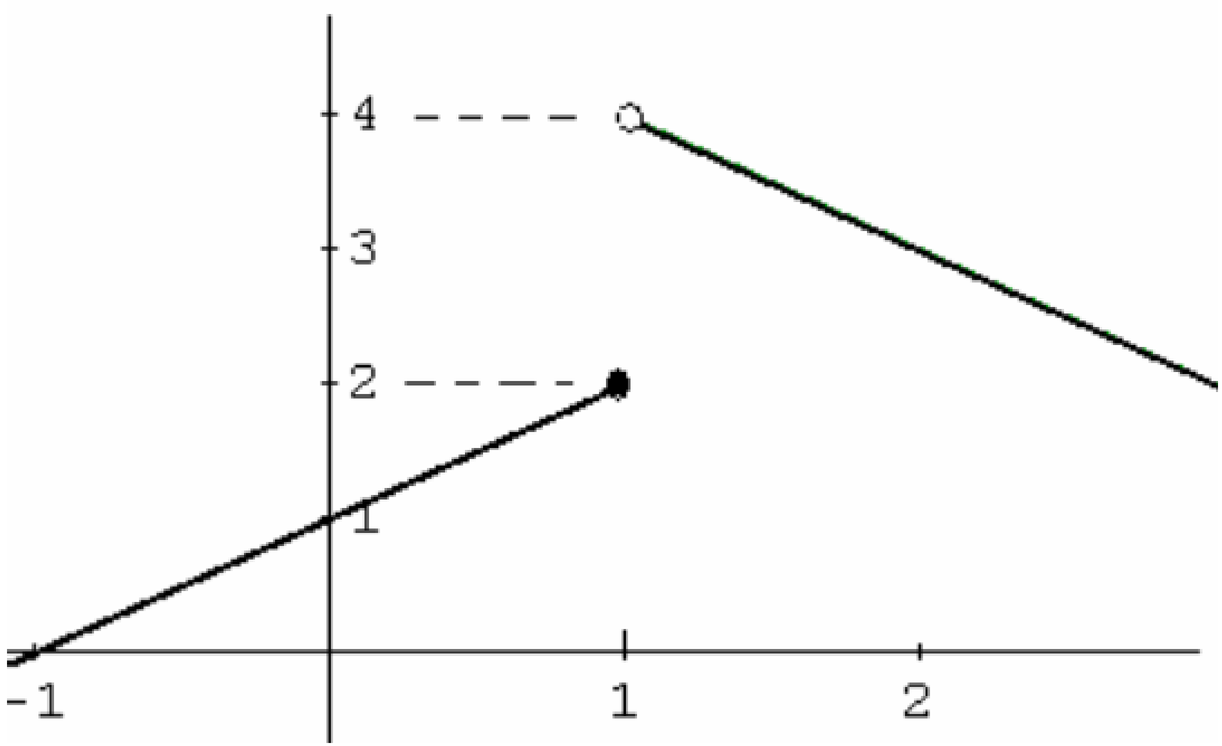
COMP 250

Lecture 35

big O

April 4, 2022

Recall Calculus 1:
Limit of a continuous function



Limit of a sequence

$$\lim_{n\to\infty} \frac{1}{1+\ n} = 0$$

$$\lim_{n\to\infty} \frac{2n^2}{n^2+n-5} = 2$$

What is a "limit" of a sequence ?

Informal definition:

A sequence $t(n)$ has a limit t_∞ if $t(n)$ becomes arbitrarily close to t_∞ as $n \to \infty$.

Formal definition : (ASIDE)

A sequence $t(n)$ has a limit t_∞ if, for any $\varepsilon > 0$, there exists an n_0 such that for any $n \geq n_0$,

$$| \, t(n) - t_\infty \, | < \varepsilon.$$

Informal definition of big O

Let $t(n)$ be a function that describes the time or number of steps for some algorithm to run for an input size n.

Let $g(n)$ be some other function that we compare $t(n)$ to.

$g(n)$ is typically a simple function such as $log_2 n$, n, n^2, ... , 2^n, , etc.

We say informally that $\boldsymbol{t(n)}$ **is O($\boldsymbol{g(n)}$)** if $g(n)$ is the dominant term in $t(n)$, as n becomes large i.e. *asymptotic* behavior.

Towards a Formal Definition of Big O

Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.

We say $t(n)$ is ***asymptotically bounded above*** by $g(n)$

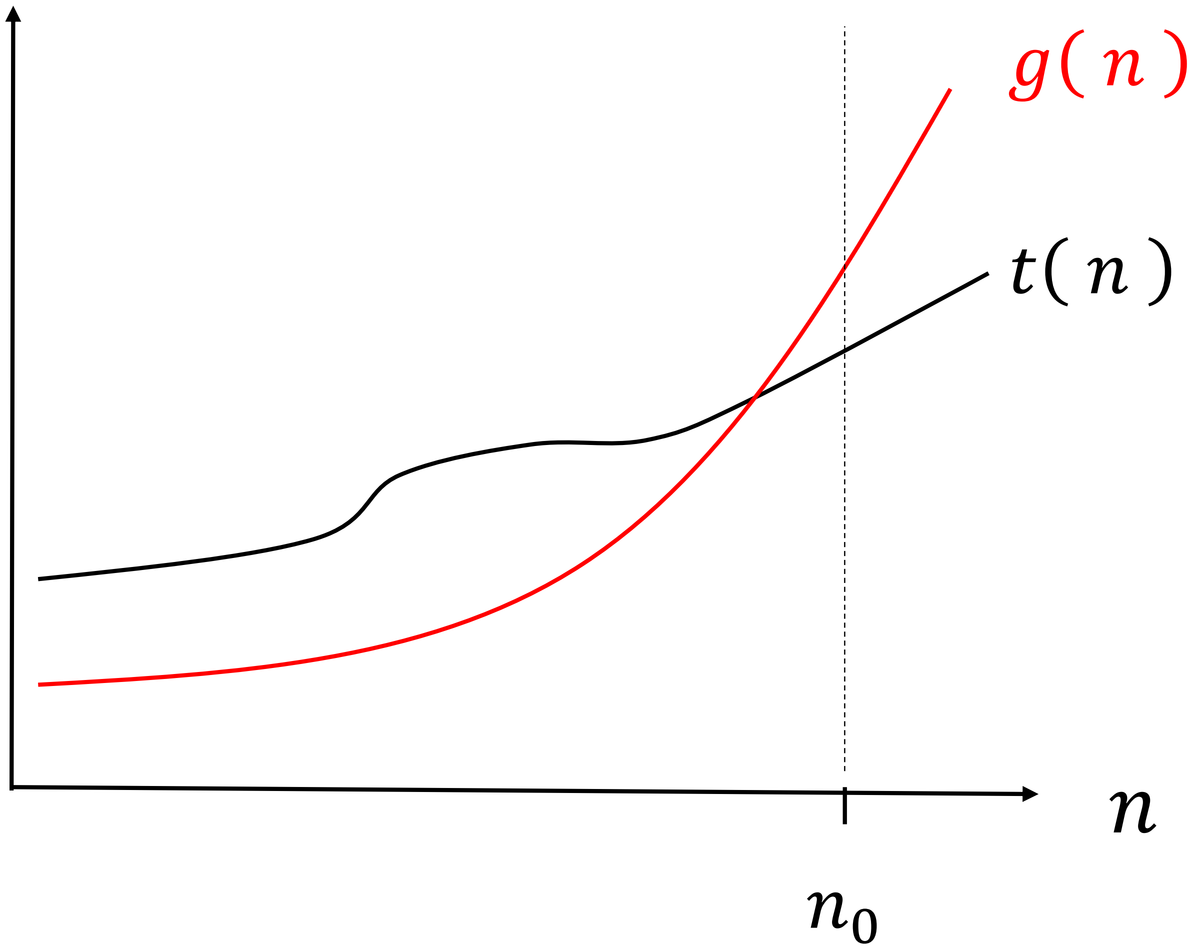
if there exist a constant n_0 such that, for all $n \geq n_0$,

$$t(n) \leq g(n).$$

This is not yet a formal definition *of big O.*

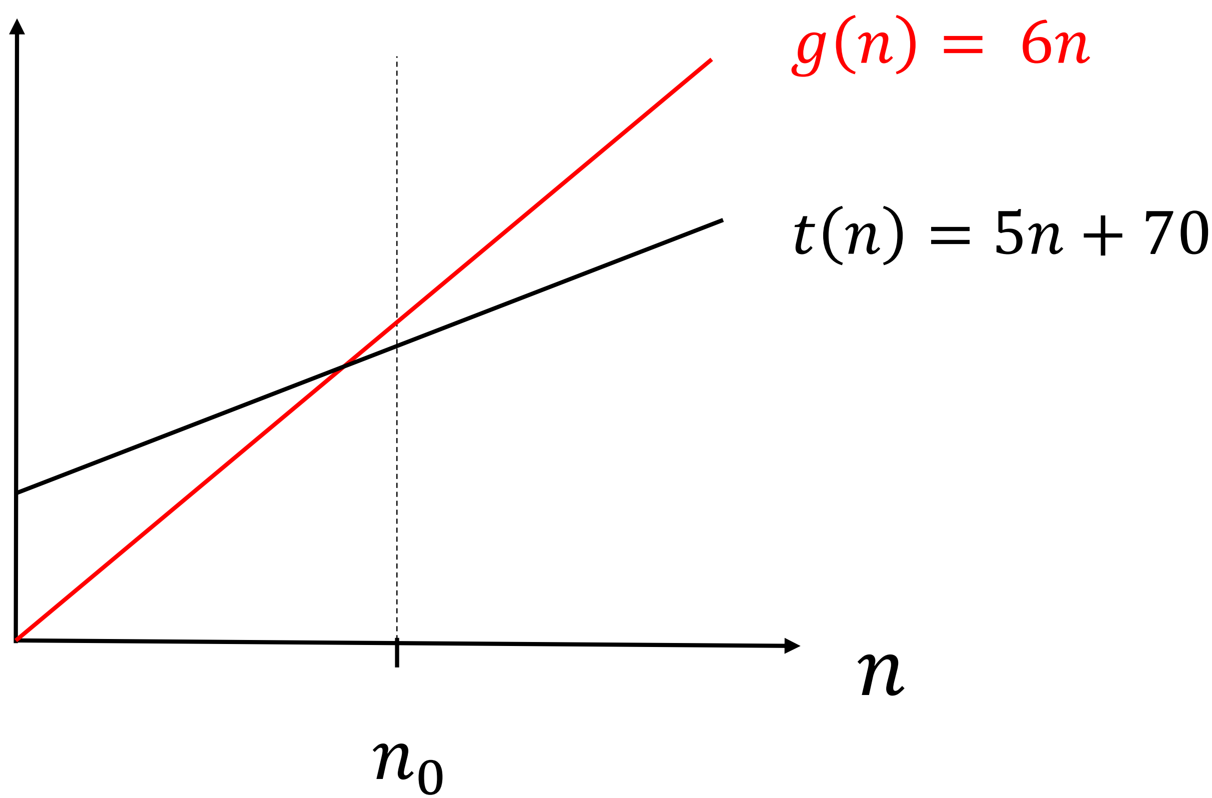
How to visualize ?

"… there exists n_0 such that, for all $n \geq n_0$, $t(n) \leq g(n)$"



Example

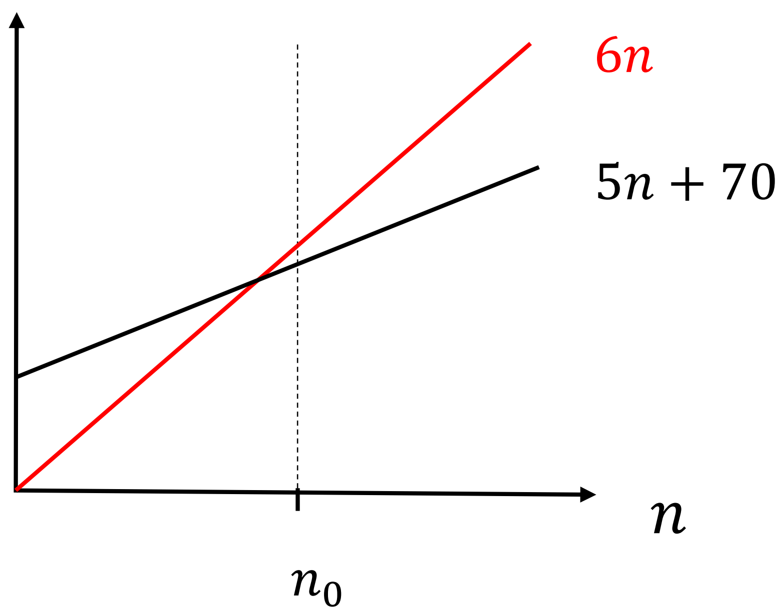
We say $t(n)$ is *asymptotically bounded above* by $g(n)$ if there exist a constant n_0 such that, for all $n \geq n_0$, $t(n) \leq g(n)$.



Claim: $5n + 70$ is asymptotically bounded above by $6n$.

Proof:

(State definition) We want to show there exists an n_0 such that, for all $n \geq n_0$, $5n + 70 \leq 6n$.



Claim: $5n + 70$ is asymptotically bounded above by $6n$.

Proof:

(State definition) We want to show there exists an n_0 such that, for all $n \geq n_0$, $5n + 70 \leq 6n$.

$$
\begin{aligned}
& 5n + 70 \leq 6n \\
\Leftrightarrow \quad & 70 \leq n
\end{aligned}
$$

So we could use $n_0 = 70$.

Symbol "$\Leftrightarrow$" means "if and only if" i.e. logical equivalence

The formal definition of big O is similar to the definition "asymptotically bounded above by" that we just saw.

The formal definition of big O *allows* us to compare the function $t(n)$ with simpler functions , $g(n)$, such as

$log_2 n, \quad n, \quad n^2, \quad \ldots, \quad 2^n$, etc.

Formal Definition of Big O

Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.

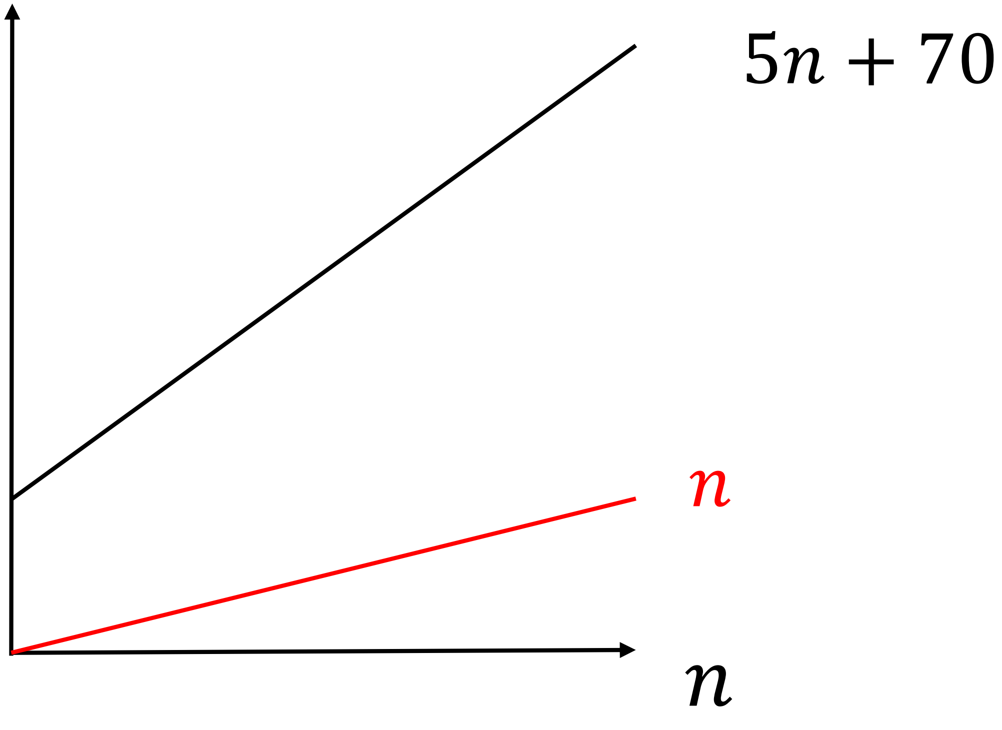
$t(n)$ is $O(\ g(n)\ )$ if there exist *two* positive constants

n_0 and c such that, for all $n \geq n_0$,

$$t(\ n\ ) \leq\ \ c\ \ g(\ n\ ).$$

$g(n)$ typically will be a simple function, but this is not required in the definition.

Claim: $5\,n + 70$ is $O(\,n\,)$.



Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\le\; ?$$

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \ge n_0$,

$$t(\,n\,) \le \; c\; g(\,n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\leq\; 5\,n + 70n, \quad \text{if } n \geq 1$$

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\,n\,) \leq \; c\;\; g(\,n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\le\; 5\,n + 70n, \quad \text{if } n \ge \underbrace{1}_{n_0}$$

$$= \underbrace{75}_{c}\,n$$

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \ge n_0$,

$$t(\,n\,) \le c\;g(\,n\,).$$

✓

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\leq \qquad ?$$

We can come up with a tighter bound for c by using a larger n_0.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + \mathbf{70} \;\leq\; 5\,n + \boldsymbol{6n}, \qquad \text{if } n \geq 12$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\le\; 5\,n + 6n, \qquad \text{if } n \ge 12$$

$$= \; 11\,n$$

So take $c = 11$, $n_0 = 12$. ✓

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \ge n_0$, $t(\,n\,) \le c\;g(\,n\,)$.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 3:

$$5\,n + 70 \;\leq \qquad\qquad ?$$

We can come up with a tighter bound for c by using a larger n_0.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 3:

$$5\,n + 70 \;\leq\; 5\,n + n, \qquad\qquad n \geq 70$$

Claim: $5\,n + 70$ is $O(\,n\,)$.

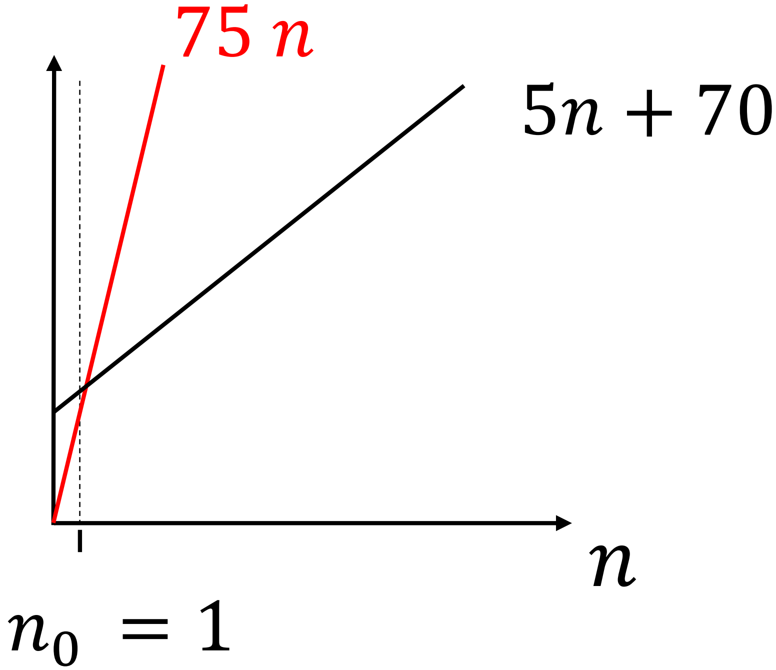
Proof 3:

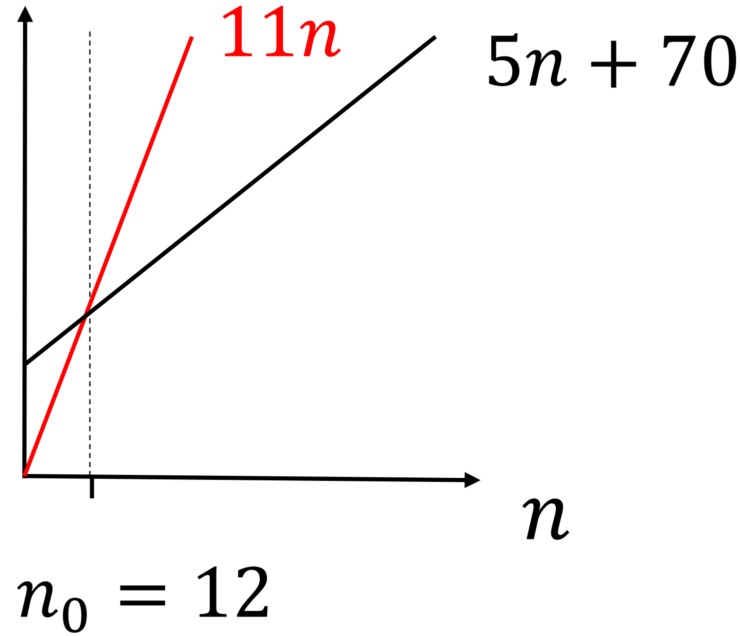
$$5\,n + 70 \;\le\; 5\,n + n, \qquad\qquad n \ge 70$$

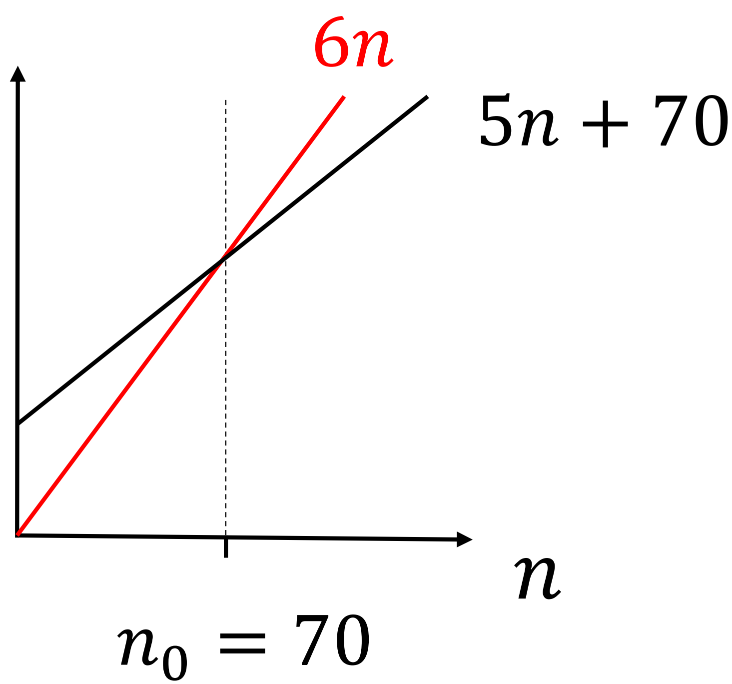
$$=\; 6\,n$$

So take $c = 6$, $n_0 = 70$.

We say $t(n)$ is $O(\,g(n)\,)$ if there exist two positive constants n_0 and c such that, for all $n \ge n_0$, $t(\,n\,) \le c\;g(\,n\,)$.







So, different combinations of n and c will satisfy the definition that $t(n)$ is O($g(n)$).

Claim: $8\,n^2 - 17n + 46$ is $O\left(n^2\right)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

We want to bound this by c n^2 for some c.

Claim: $8\,n^2 - 17n + 46$ is $O\left(n^2\right)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

$$\leq \quad 8\,n^2 \quad + \quad 46\,n^2, \qquad n \geq 1$$

Claim: $8\,n^2 - 17n + 46$ is $O\left(n^2\right)$.

Proof (1):

$$8\,n^2 - 17n + 46$$

$$\leq \quad 8\,n^2 \quad + 46\,n^2, \qquad n \geq 1$$

$$\leq \; 54\,n^2$$

So take $c = 54,\; n_0 = 1.$ ✓

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (2):

$$8\,n^2 - 17n + 46$$

Can we bound this by c n^2 for some smaller c?

Claim: $8\,n^2 - 17n + 46$ is $O(n^2)$.

Proof (2):

$$8\,n^2 - 17n + 46$$

$$\leq \; 8\,n^2 \;, \qquad\qquad n \geq 3$$

i.e. $-17 * 3 + 46 < 0$

So take $c = 8,\; n_0 = 3.$ ✓

What does $O(1)$ mean?

$t(n)$ is $O(1)$ if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(n) \leq c.$$

So it just implies that $t(n)$ is bounded.

Note: we assume $t(n)$ is defined only on $n \geq 0$.

We don't write $O(3n)$, $O(5 \, log_2 n)$, etc.

Instead, write $O(n)$, $O(log_2 n)$, etc.

Why? The point of the formal definition of big O is that it allows you to *avoid dealing with these constant factors*.

“Tight Bounds”

Big O is about *upper* bounds.

If $t(n)$ is $O(n)$, then is $t(n)$ also $O\left(n^2\right)$?

According to the formal definition, yes, since $n < n^2$.

When we ask for “tight bounds” on $t(n)$, we want the simple function $\mathrm{g}(n)$ with the *smallest* growth rate.

(More on this next lecture.)

Incorrect Proofs

In MATH 240 (for CS) or MATH 235 (for Math/CS), you will learn how to *write* proofs.

Here are some typical mistakes that one might make.

Claim: $5n + 70$ is $O(n)$.

Incorrect Proof:

$$
\begin{aligned}
5n + 70 &\leq cn \\
5n + 70n &\leq cn, \quad n \geq 1 \\
75n &\leq cn
\end{aligned}
$$

Thus, $c = 75, \quad n_0 = 1$ works.

Q: Why is this *proof* incorrect ?

A: Because we don't know how lines are logically related.

Another Example of an Incorrect Proof

Claim: for all $n > 0$, $2n^2 \leq (n+1)^2$.

Proof:

$$
\begin{aligned}
2n^2 &\leq (n+1)^2 \\
&\leq (n+n)^2, \quad \text{when } n > 0 \\
&= 4n^2
\end{aligned}
$$

Since $2n^2 \leq 4n^2$, we are done.

Unfortunately, the claim is false! (Take $n = 3$)

Claim: for all $n > 0$, $2n^2 \leq (n+1)^2$.

Proof:

It is incorrect to assume what you are trying to prove.

$$\begin{aligned} 2n^2 &\leq (n+1)^2 \\ &\leq (n+n)^2, \quad \text{when } n > 0 \\ &= 4n^2 \end{aligned}$$

Since $2n^2 \leq 4n^2$, we are done.

Coming up...

Lectures

Wed : April 6

big Omega, big Theta

Fri : April 8

best and worst cases

Assessments

Quiz 5 today.

Assignment 4 due Wed. April 6.