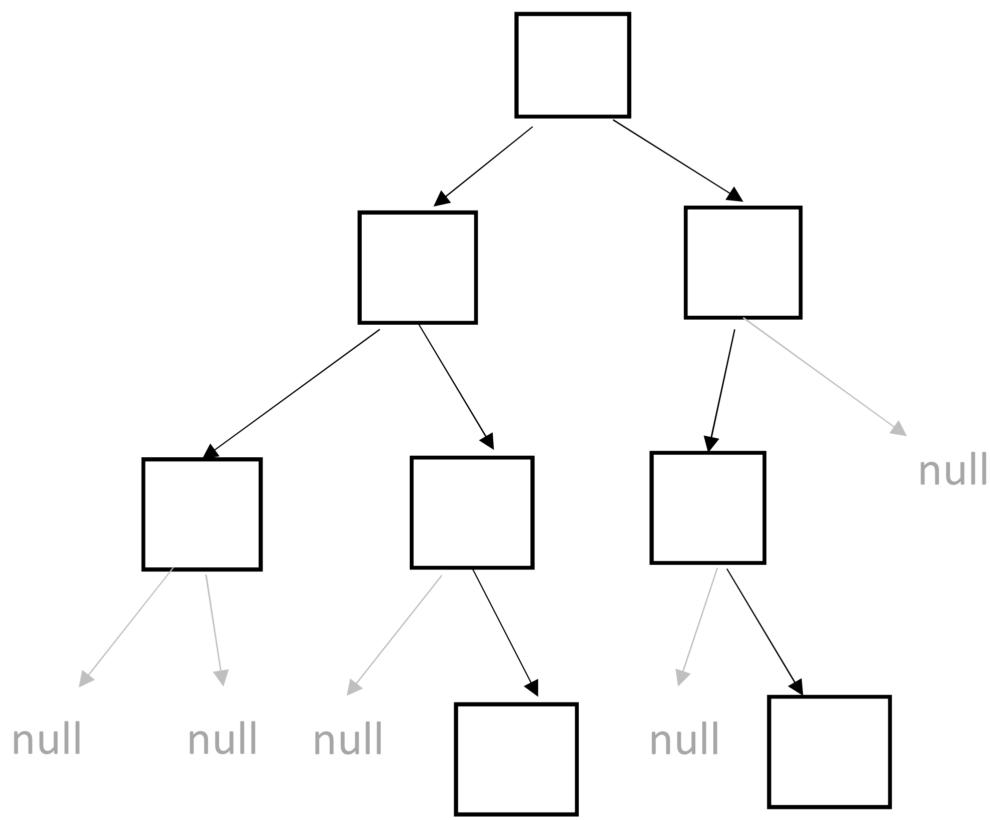
COMP 250

Lecture 25

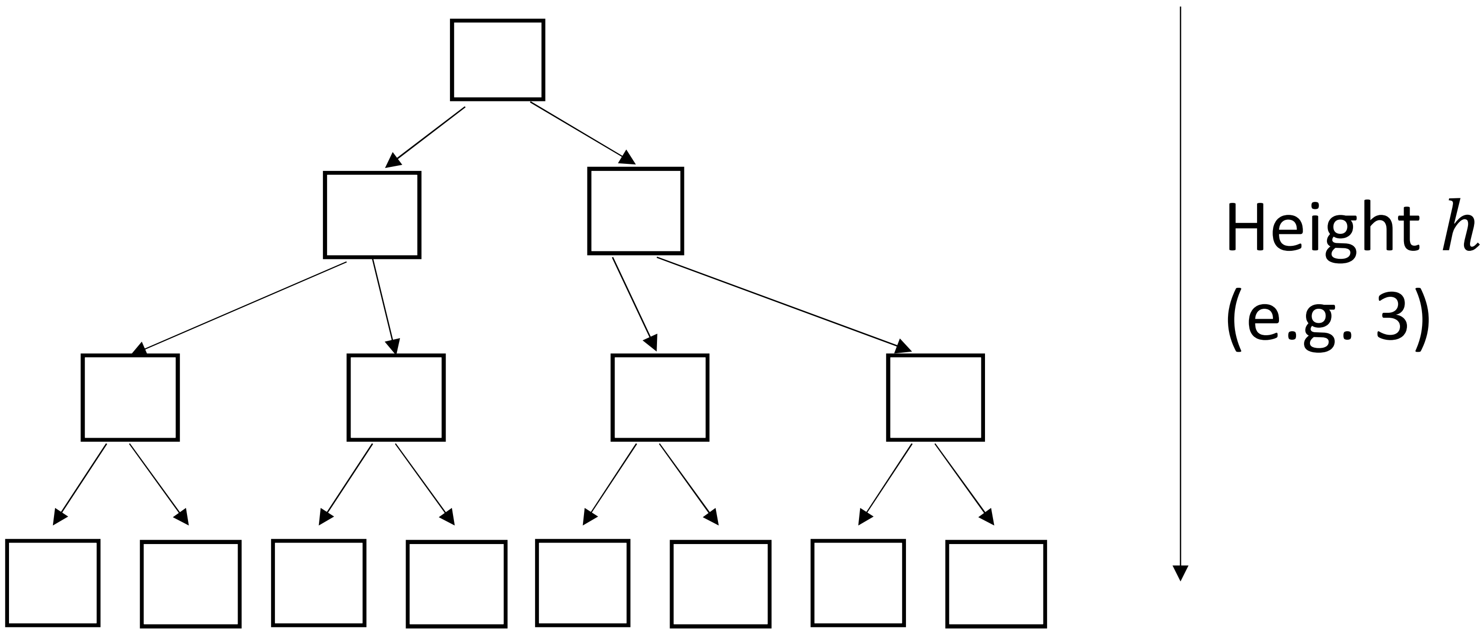
binary trees, expression trees

March 10, 2022

Binary tree:
each node has *at most* two children.



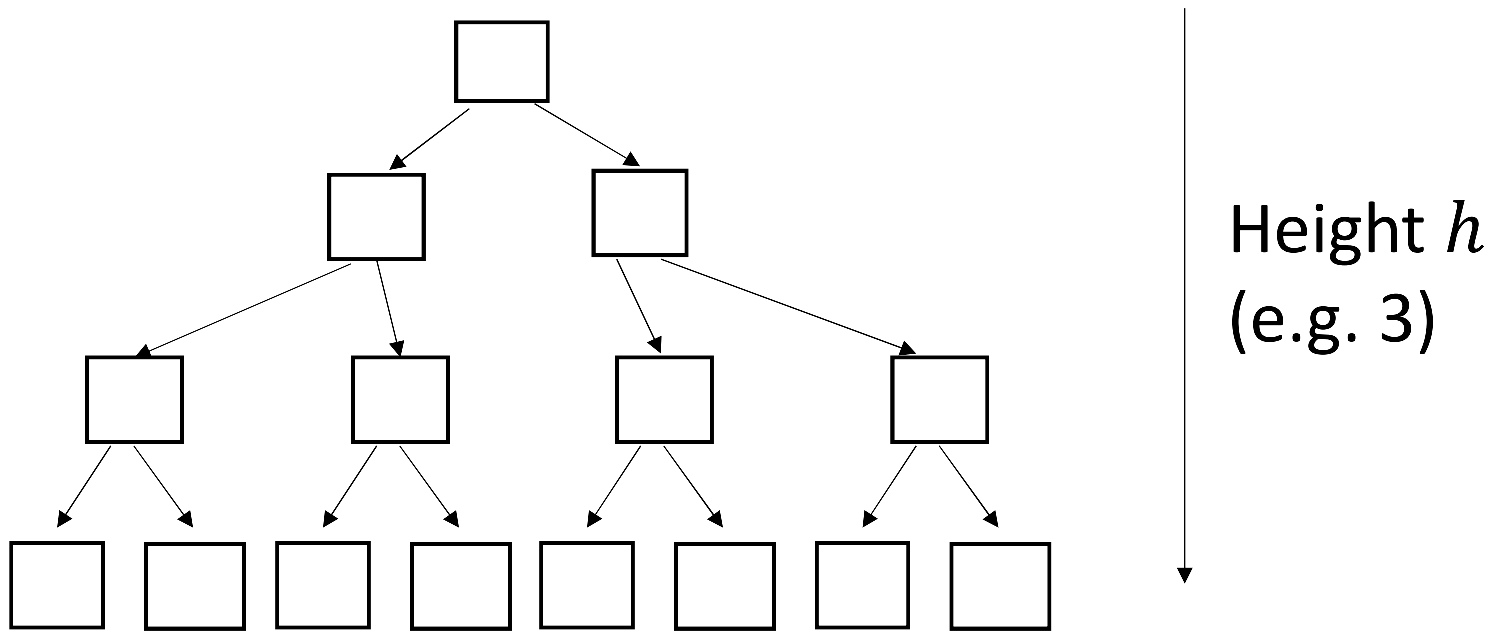
Maximum number of nodes in a binary tree?



$$n = 1 + 2 + 4 + 8 + \cdots 2^h$$

$$= 1 + x + x^2 + x^3 + \cdots x^h = \frac{x^{h+1} - 1}{x - 1}, \quad \text{where } x = 2$$

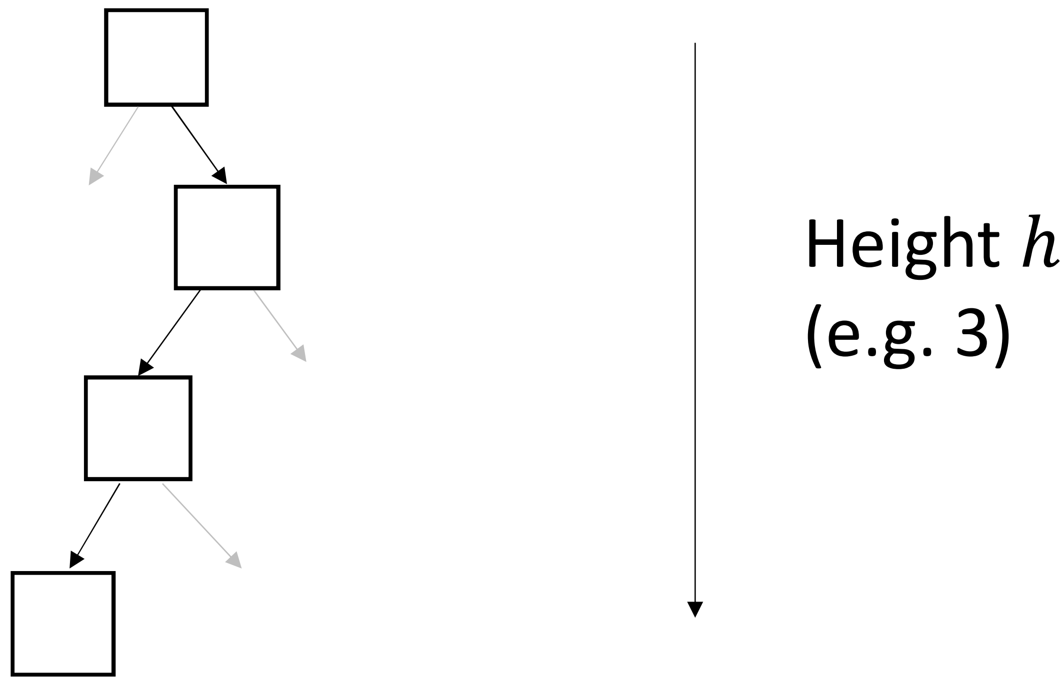
Maximum number of nodes in a binary tree?



$$n = 1 + 2 + 4 + 8 + \cdots 2^h = 2^{h+1} - 1$$

$$= 1 + x + x^2 + x^3 + \cdots x^h = \frac{x^{h+1} - 1}{x - 1}, \text{ where } x = 2$$

Minimum number of nodes in a binary tree?



$$n = h + 1$$

Implementation in Java

```
class  BinaryTree<T>{
    BTNode<T>   root;
       :

    class  BTNode<T>{
        T                   e;
        BTNode<T>     leftchild;
        BTNode<T>     rightchild;
        :
    }
}
```

Recall : depth first tree traversal

```
//   pre-order

depthFirst(root){
      visit root
      for each child of root
            depthFirst( child )
}
```

```
//   post-order

depthFirst(root){
      for each child of root
            depthFirst( child )
      visit root
}
```

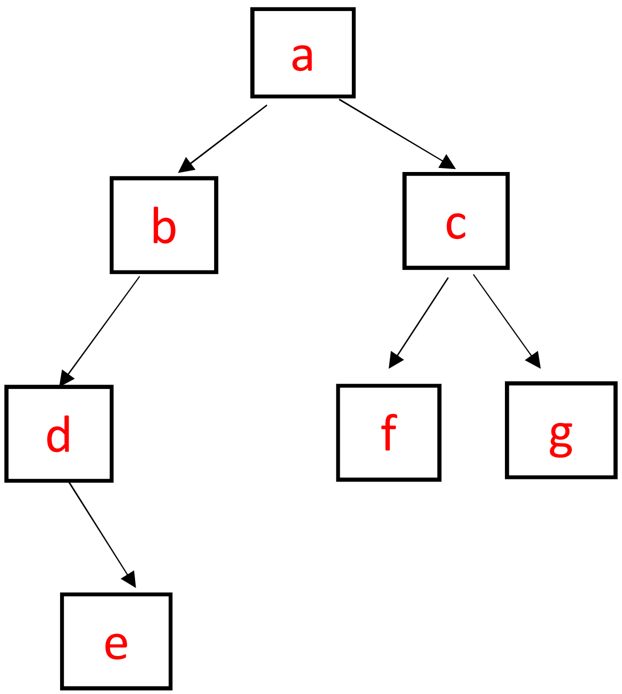
We write these slightly differently for a binary tree (next slide).

```
preorderBT (root){
   if (root is not null){
       visit root
       preorderBT( root.left )
       preorderBT( root.right )
   }
}
```

```
postorderBT (root){
   if (root is not null){
       postorderBT(root.left)
       postorderBT(root.right)
       visit root
   }
}
```

```
inorderBT (root){
   if (root is not null){
       inorderBT(root.left)
       visit root
       inorderBT(root.right)
   }
}
```

Example



Pre order: a b d e c f g

In order: d e b a f c g

Post order: e d b f g c a

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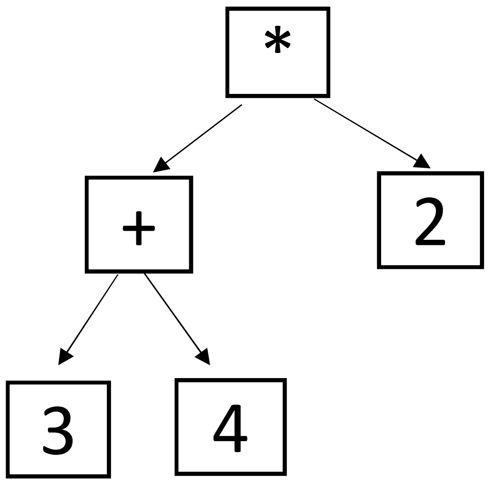
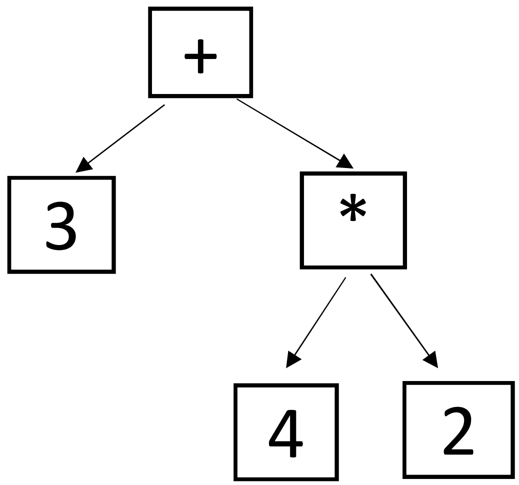
Expression Tree

We often write expressions such as **3 + 4 * 2 .**
We can write and evaluate such expressions using trees.
There are two ways to do so for this example.

(3 + 4) * 2



3 + (4 * 2)



My Windows calculator says
3 + 4 * 2 = 14.

(3 + 4) * 2 = 14.

Whereas….
if I google "3+4*2", I get 11.

3 + (4*2) = 11.

We can make expressions using binary operators +, -, *, /, ^

e.g. **a – b / c + d * e ^ f ^ g**

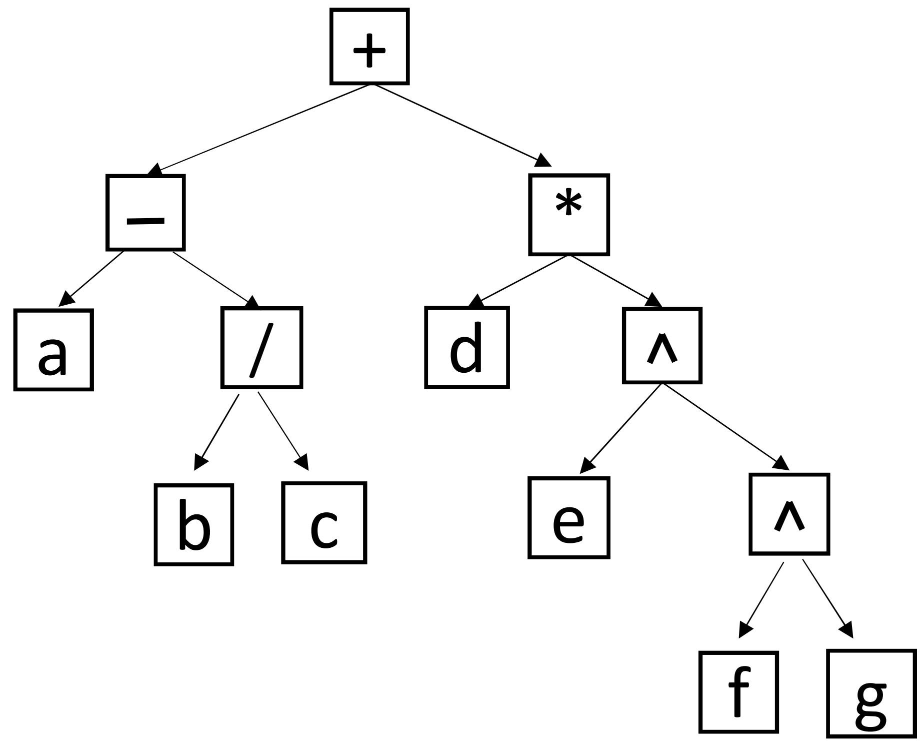
^ is exponentiation: e ^ f ^ g means e ^ (f ^ g)

Operator precedence ordering makes brackets unnecessary.

(a – (b / c)) + (d * (e ^ (f ^ g)))

We don't consider unary operators e.g. 3 + -4 = 3 + (-4)

If we traverse an expression tree,
and *print out* the node label,
what expression is printed out?

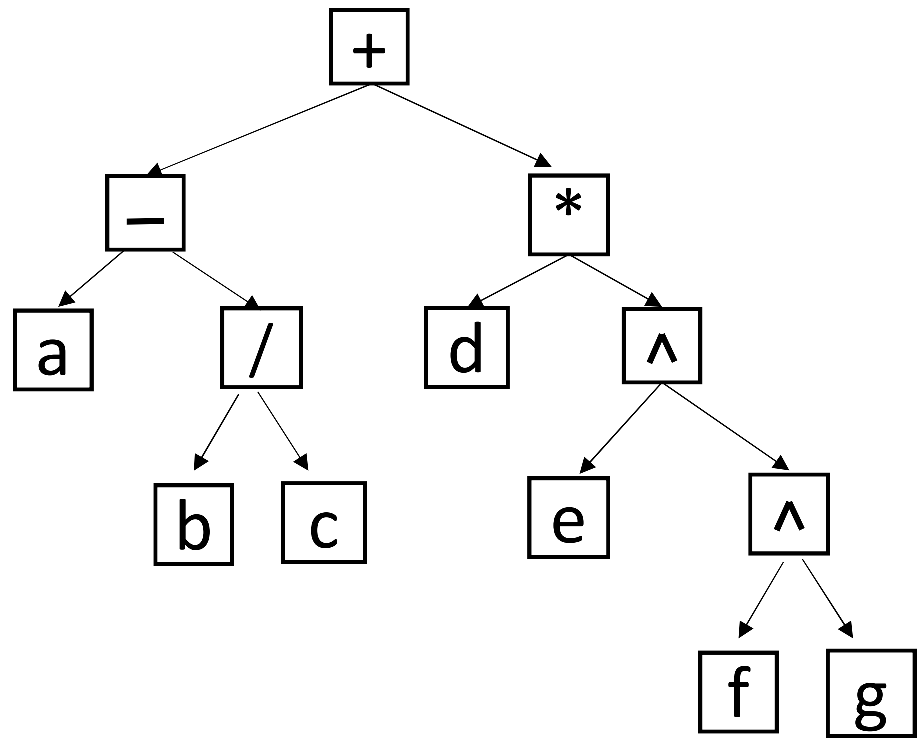


preorder traversal gives : + – a / b c * d ^ e ^ f g

inorder traversal gives : a – b / c + d * e ^ f ^ g

postorder traversal gives : a b c / - d e f g ^ ^ * +

If we traverse an expression tree, and *print out* the node label, what expression is printed out?



"pre-fix" expression: + – a / b c * d ^ e ^ f g

"in-fix" expression : a – b / c + d * e ^ f ^ g

"post-fix" expression : a b c / - d e f g ^ ^ * +

ASIDE: "Formal language" for *prefix* expressions

```
baseExp  =  a | b | c | d   .... | z

op       =  + | - | * | / | ^

preExp   =  baseExp | op preExp preExp
```

where | means "or".

This gives you a hint of how programming languages are formally defined. e.g. *COMP 330 Theory of Computation.*

ASIDE: "Formal language" for expressions

baseExp	=	a | b | c | d .... etc
op	=	+ | - | * | / | ^
preExp	=	baseExp | op preExp preExp
inExp	=	baseExp | inExp op inExp
postExp	=	baseExp | postExp postExp op

Use only one.

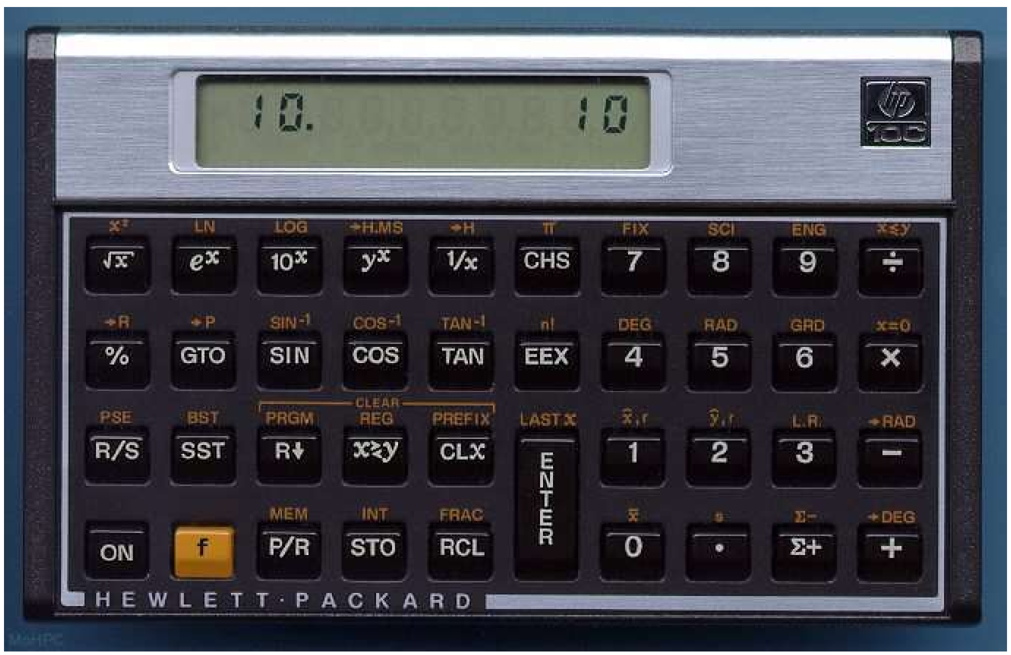
Prefix expressions are called “Polish Notation” .
(after Polish logician Jan Lucasewicz 1920’s)

Postfix expressions are called “Reverse Polish notation” (RPN)

Prefix expressions are called “Polish Notation”
(after Polish logician Jan Lucasewicz 1920’s)

Postfix expressions are called “Reverse Polish notation” (RPN)

Some calculators (esp. Hewlett Packard) require users to input expressions using RPN.



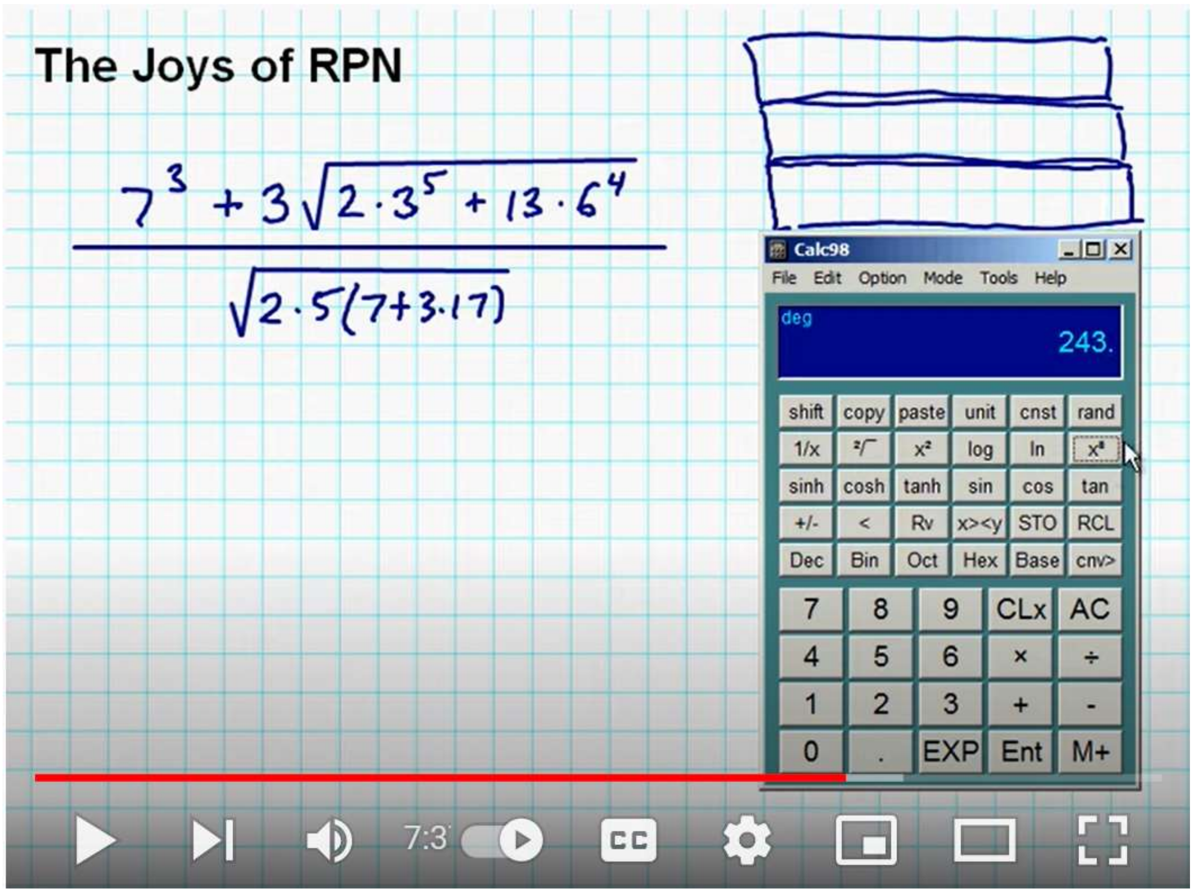
Calculate 3 + 4 * 2

which is **3 4 2 * +** in RPN

3 <enter>
4 <enter>
2
* → yields 8
+ → yields 11

No “=“ symbol on keyboard.

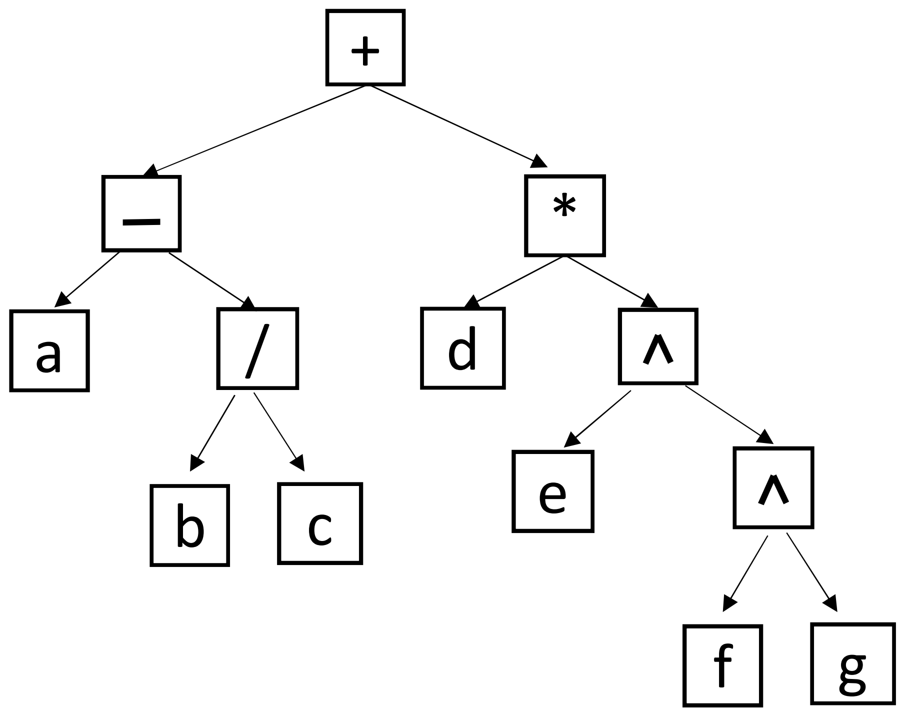
There are lots of youtube videos showing how to use RPN calculators, e.g. this video.



On the previous slide, where we had 3+4*2, ...
with a real RPN calculator, you would first do the 4 * 2 and then add 3.

Suppose we are given an expression tree.
How can we evaluate the corresponding expression ?

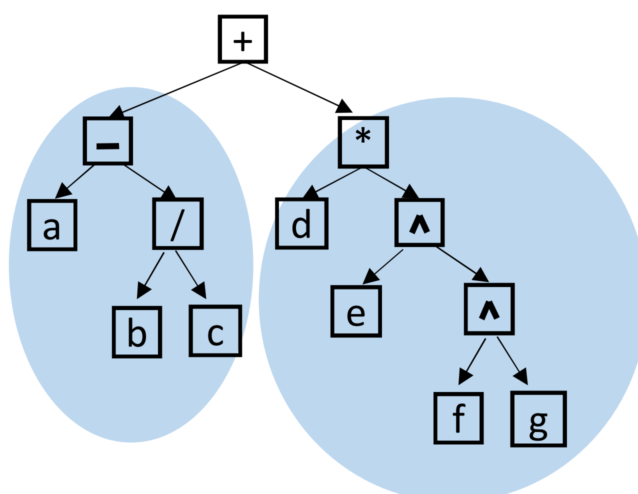
Hint: traverse the tree. But how?



Here we assume the leaves have known values.

Use a **postorder traversal**:

```
evalExpressionTree(root){
    if (root is a leaf)     //  root is a number
        return value
    else{
                  //   root is an operator
    }
}
```



Evaluate left and right subtrees and then combine results.

Use a **postorder traversal**:

```
evalExpressionTree(root){
   if (root is a leaf)      //  root is a number
        return value
   else{                    //  root is an operator
          op    =  root.element
          firstOperand      =   evalExpressionTree ( root.leftchild )
          secondOperand   =   evalExpressionTree( root.rightchild)
          return evaluate(op, firstOperand, secondOperand)
   }
}
```

It is postorder because we need to evaluate the children before we can evaluate the node.

Suppose we are just given a postfix expression.

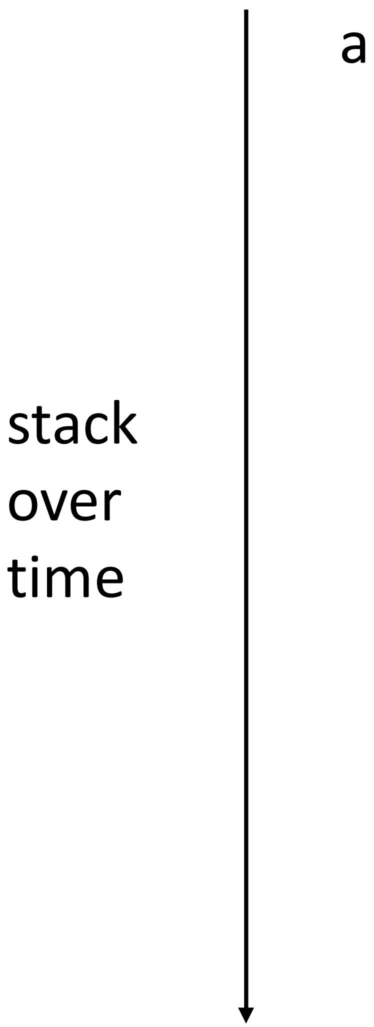
How can we evaluate it?   Data structure ? Algorithm?

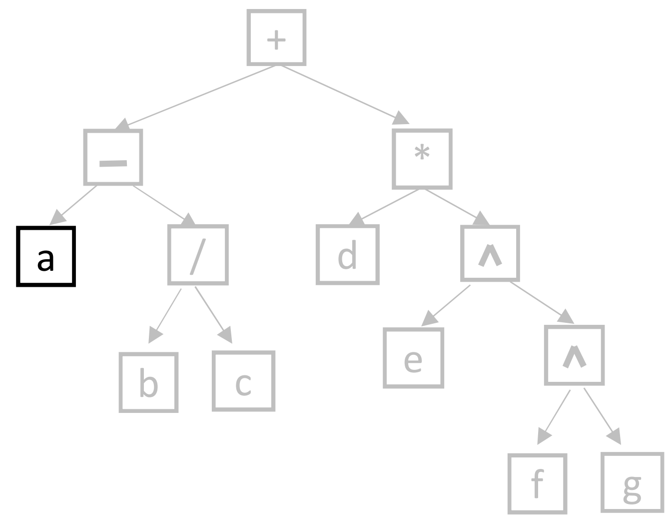
e.g.   a b c / - d e f g ^ ^ * +

Read symbols from left to right.   Use a stack. (Next slides)

Example:

a b c / - d e f g ^ ^ * +





This expression tree is not given. It is shown here so that you can visualize the expression more easily.

Example:

<u>a b</u> c / - d e f g ^ ^ * +

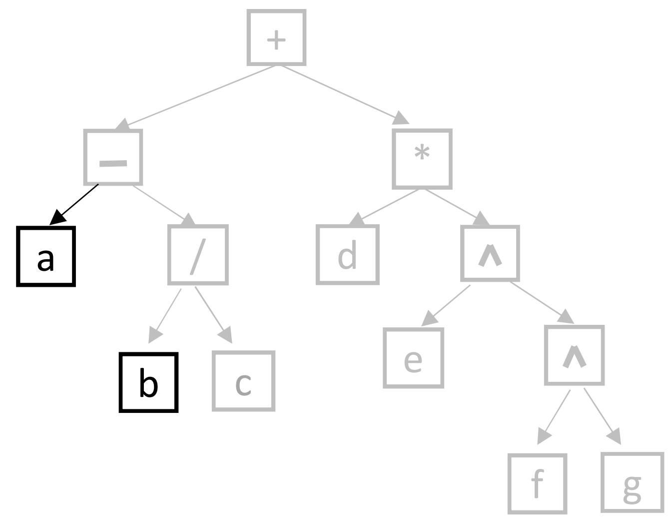
a

a b

↑

top

stack
over
time



This expression tree is not given. It is shown here so that you can visualize the expression more easily.

Example:

a b c / - d e f g ^ ^ * +

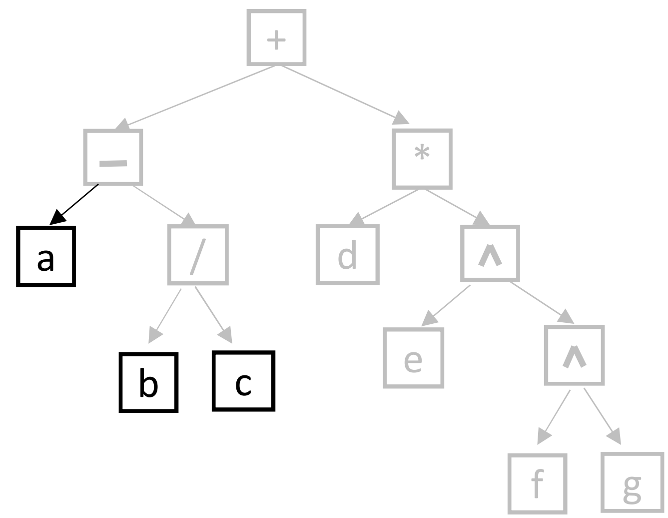
stack over time

a

a b

a b c

top



a b c / - d e f g ^ ^ * +

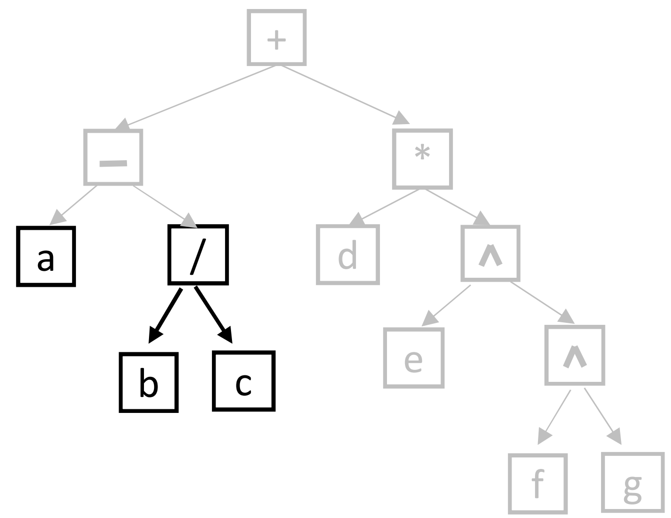
stack over time

a

a b

a b c

a (b c /)



We don't push the operator onto the stack. Instead we pop value twice, evaluate, and push the result.

a b c / - d e f g ^ ^ * +

stack

over

time

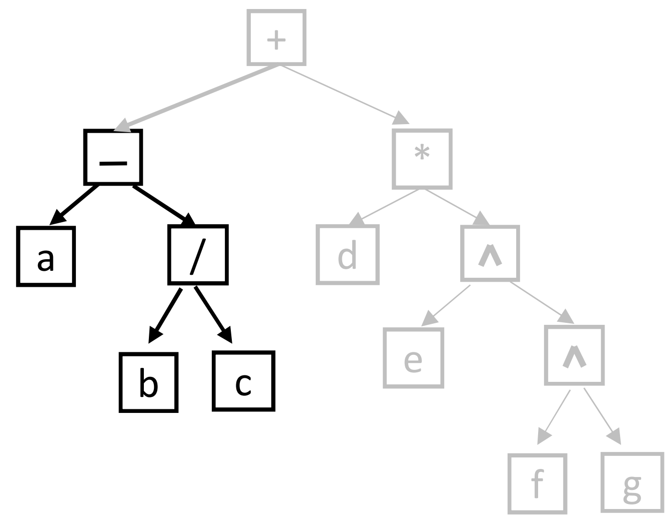
a

a b

a b c

a (b c /)

(a (b c /) -)



Now there is one
value on the stack.

a b c / - d e f g ^ ^ * +

stack over time

a

a b

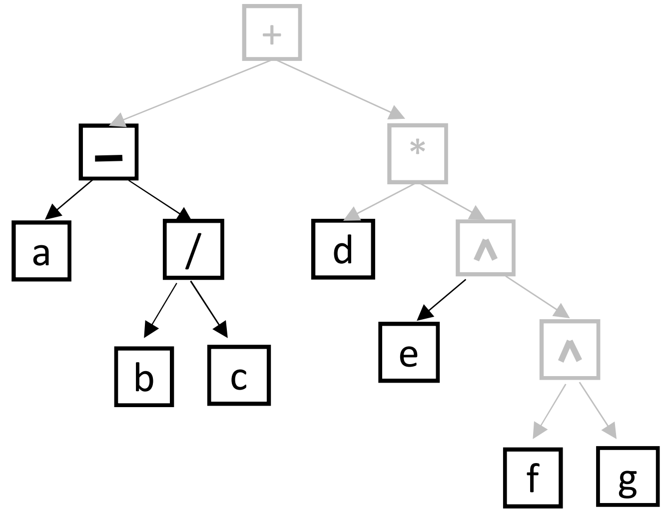
a b c

a (b c /)

(a (b c /) -)

:

(a (b c /) -) d e f g



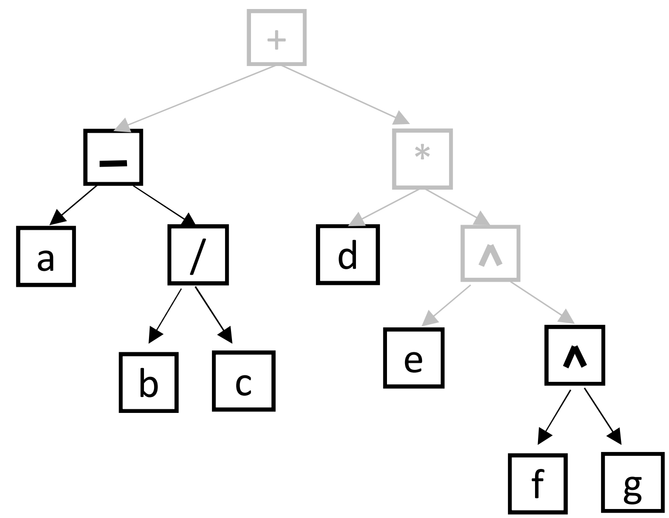
Now there are five values on the stack.

a b c / - d e f g ^ ^ * +

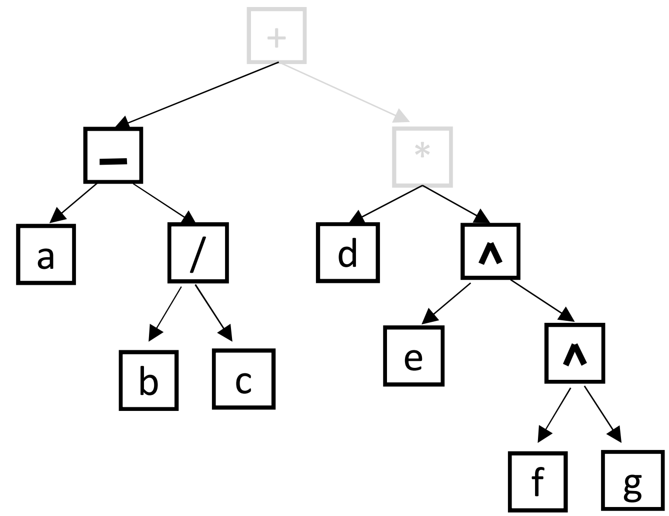
stack
over
time

a
a b
a b c
a (b c /)
(a (b c /) -)
:
(a (b c /) -) d e f g
(a (b c /) -) d e (f g ^)

Now there are four
values on the stack.



a b c / - d e f g ^ ^ * +



stack over time

a

a b

a b c

a (b c /)

(a (b c /) -)

:

(a (b c /) -) d e f g

(a (b c /) -) d e (f g ^)

(a (b c /) -) d (e (f g ^) ^)

Now there are three values on the stack.

a b c / - d e f g ^ ^ * +

stack over time

a

a b

a b c

a (b c /)

(a (b c /) -)

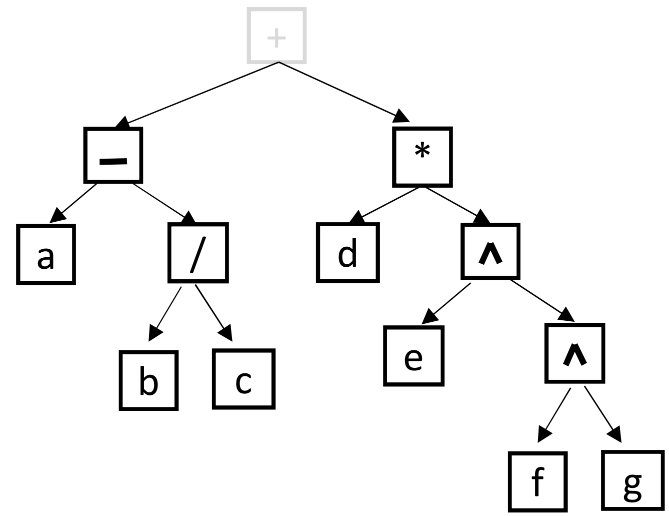
:

(a (b c /) -) d e f g

(a (b c /) -) d e (f g ^)

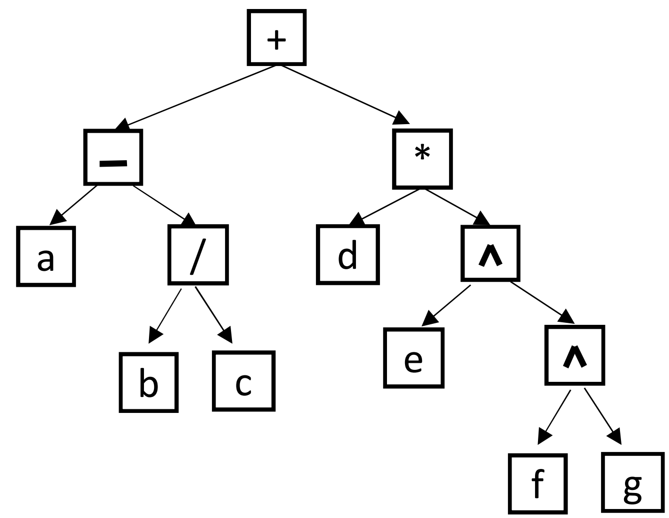
(a (b c /) -) d (e (f g ^) ^)

(a (b c /) -) (d (e (f g ^) ^) *)



Now there are two values on the stack.

a b c / - d e f g ^ ^ * +



stack over time

a
a b
a b c
a (b c /)
(a (b c /) -)
:
(a (b c /) -) d e f g
(a (b c /) -) d e (f g ^)
(a (b c /) -) d (e (f g ^) ^)
(a (b c /) -) (d (e (f g ^) ^) *)
((a (b c /) -) (d (e (f g ^) ^) *) +)

One value on the stack (the result). *Note this corresponded to a postorder traversal of an expression tree.*

Algorithm: Use a stack to evaluate a postfix expression

Let expression be a list of “tokens”.

```
s = empty stack
cur =  first token of expression list
while (cur != null){
    if ( cur  is a base expression )        //  value i.e.  variable or number
        s.push( cur )
    else{                                   //  cur is an operator

    }
    cur = cur.next
}
```

Algorithm: Use a stack to evaluate a postfix expression

Let expression be a list of “tokens”.

```
s = empty stack
cur =  first token of expression list
while (cur != null){
    if ( cur  is a base expression )
        s.push( cur )
    else{                                        //  cur is an operator
        operand2  =  s.pop()
        operand1  =  s.pop()
        op  = cur        //   not necessary (but easier to read)
        s.push( evaluate( op,  operand1, operand2 )   )
    }
    cur = next token in expression list
}     //  terminates with result on the stack
```

ASIDE

As we just saw, **postfix expressions** *without brackets* are easy to evaluate.

A similar algorithm works for **pre-fix expressions**. Read the expression from right to left and swap order of two operands when evaluating.

Infix expressions (with or without brackets) are trickier to evaluate, since you need to incorporate precedence ordering rules for the different operands. You can convert infix to postfix using the following:

https://en.wikipedia.org/wiki/Shunting-yard_algorithm

As you know, the Java language expects expressions to be infix.

The Java compiler converts infix expressions to postfix. At runtime, the JVM then uses a stack to evaluate the postfix expression (much simpler and faster).

Coming up…

Lectures

Mon. March 14

Binary Search Trees

Wed & Fri. March 16 & 18

Heaps

Tutorial + Assessments

Assignment 3

due Wed. March 16

Quiz 4 (lectures 20-25)

Fri. March 18