COMP 250

lecture 19

mathematical induction

The course so far has been mostly object orientated programming (in Java).

From here on, it will be mostly data structures, algorithms, and computational complexity (mathematical analysis).

Many of the algorithms will use *recursion*. Recursion is closely related to mathematical induction. We will cover these topics next.

18. **Induction**
19. **Recursion**
20. Binary Search
21. Mergesort & Quicksort
22. Trees
23. Tree traversal
24. Binary trees
25. Binary search trees
26. Heaps 1
27. Heaps 2
28. Hashing 1 (maps)
29. Hashing 2
30. Graphs 1
31. Graphs 2
32. Big O 1
33. Big O 2
34. Big O 3
35. Recurrences 1
36. Recurrences 2

Recall lecture 0 slide 11:

Math Prerequisites

CEGEP level math (Cal 1) :

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n\,(n+1)}{2}$$

$$1 + x + x^2 + x^3 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

$$\log(ab) = \log a + \log b$$

How to prove the following *statement* ?

For all $n \geq 1$,

$$1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}.$$

By "proof", we mean a formal logical argument that convincingly shows the statement is true.

You've probably seen before the proof shown on the next two slides.

Consider the sum: $1 + 2 + \ldots + (n-1) + n$.

If n is even, then we can group into $\frac{n}{2}$ pairs :

$$1 + 2 + \cdots + \frac{n}{2} + (\frac{n}{2} + 1) \ldots + (n-1) + n$$

Each pair adds up to $n + 1$ and there are $\frac{n}{2}$ pairs .

Adding them up gives $\frac{n}{2}(n + 1)$.

If n is odd, then, n-1 is even. So,

$$1 + 2 + \ldots + (n-1) + n$$

$$= \left(\frac{n-1}{2}\right) n + n$$

$$= \left(\frac{n-1}{2} + 1\right) n$$

$= \left(\frac{n+1}{2}\right) n$ which is the same formula as before.

Mathematical Induction

Consider a statement of the form:

"For all $n \geq n_0$, $P(n)$ is true" where n_0 is some constant, and $P(n)$ is some proposition that has a value true or false which may depend on n.

Mathematical induction is a *general technique* for proving such a statement.

Note that the statement about infinitely many n's!

"For all $n \geq n_0$, $P(n)$ is true."

For our previous example:

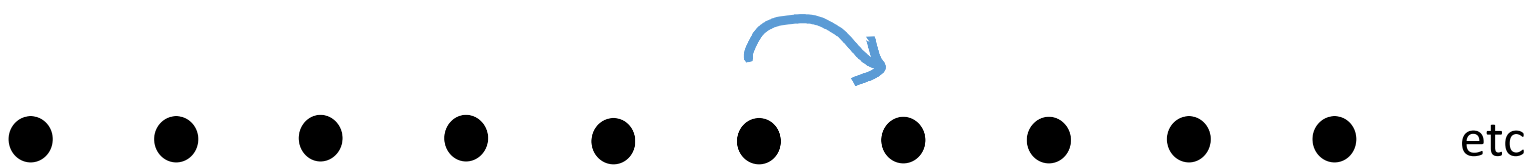
For all $n \geq 1$,

$$1 + 2 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$$ is true.

Note in mathematics, one typically does not write the "is true" part of the statement. I am writing it here to emphasize that $P(n)$ can be thought of as a boolean valued expression. Also note we are not treating $n \geq n_0$ as a Boolean valued expression but rather as short hand for "n greater than n_0".

To prove a statement "For all $n \geq n_0, \quad P(n)$ is true"
we will use the following analogous concept:

Suppose you have an *infinite* sequence of stepping stones numbered 0, 1, ... and you know how to jump from any stone to the next one, and you start on stone 0 (or more generally n_0). Then you will be able to reach all stones (eventually).



start here (or more generally start at stone n_0)

you know how to jump from one stone to the next

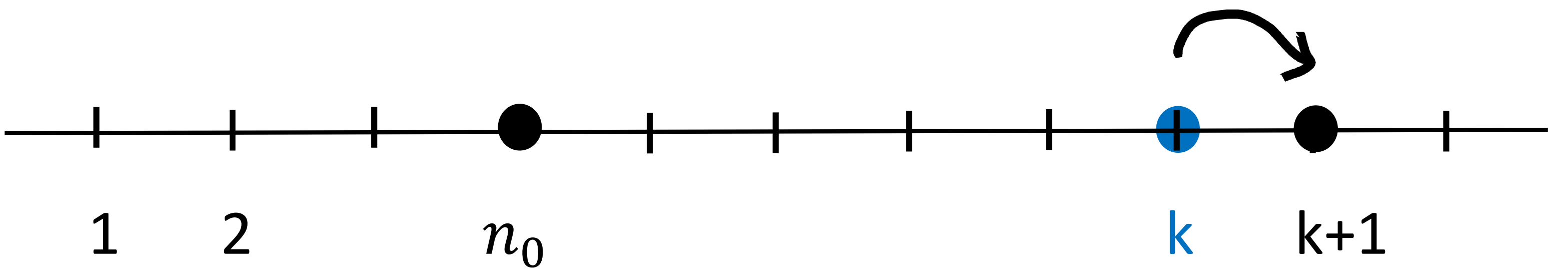
To prove a statement "For all $n \geq n_0$, $P(n)$ is true" using *mathematical induction*, we show:

<u>Base case:</u>

$P(n_0)$ is true.

<u>Induction step:</u>

For any $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is also true.



Base case:

$P(n_0)$ is true.

Induction step:

For any $k \geq n_0$, if $P(k)$ is true then $P(k+1)$ is also true.

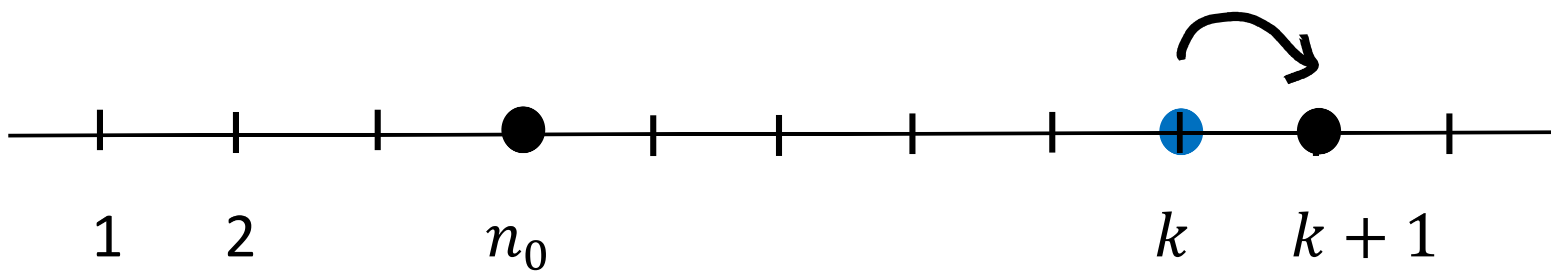
The statement "$P(k)$ is true" is called the *induction hypothesis*.

Base case:

$P(n_0)$ is true.

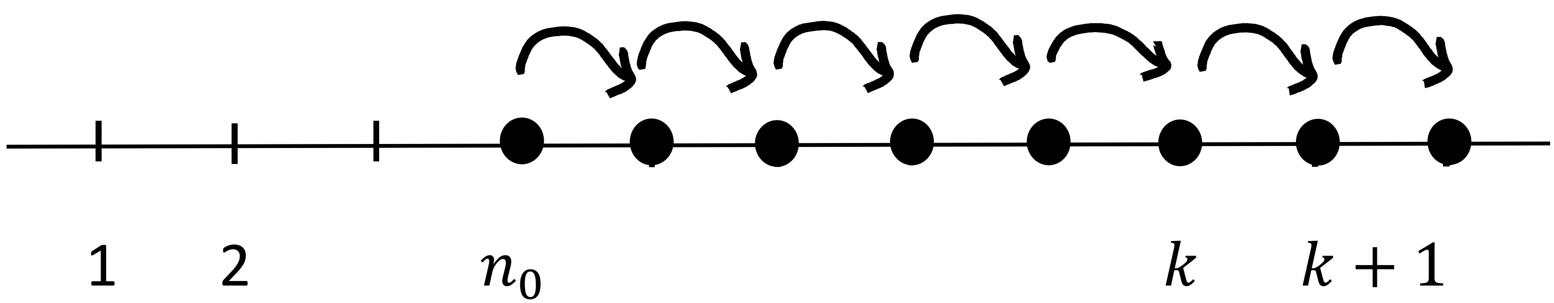
Induction step:

For any $k \geq n_0$, if $P(k)$ is true then $P(k+1)$ is true.



If we can prove the base case and induction step (both are true), then we can conclude:

For any $n \geq n_0$, $P(n)$ is true.



Example 1

Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ \ldots. + (n-1) + n = \frac{n(n+1)}{2}$$

Proof (base case, $n = 1$):

?

Example 1

Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}$$

Proof (base case, $n = 1$):

$$1 = \frac{1(1+1)}{2}$$ (this is true)

The *induction hypothesis*: $P(k)$ is true:

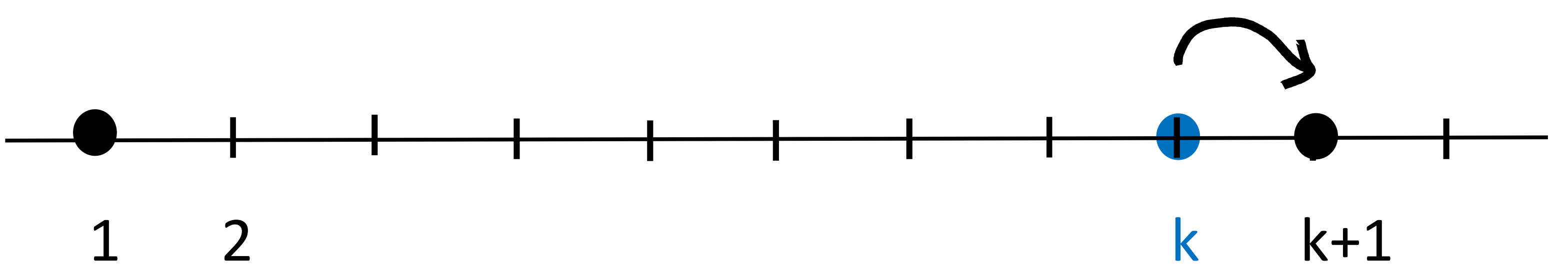
$$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$$

Base case:

P(1) is true.

Induction step:

For any $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.



Proof of Induction Step:

$$(1 + 2 + 3 + \ ....+\ k) \ + k + 1$$

$= \quad \frac{k(k+1)}{2} \ + \ k + 1$ by the induction hypothesis

=

Proof of Induction Step:

$$(1 + 2 + 3 + \ldots. + k) + k + 1$$

$= \frac{k(k+1)}{2} + k + 1$ by the induction hypothesis

$= \left(\frac{k}{2} + 1\right)(k + 1)$

$= \frac{1}{2}(k + 2)(k + 1)$

Thus, if $P(k)$ *is true* then $P(k+1)$ *is true*.

Ambiguities & Possible Confusion

We are using "$a = b$" in two different ways.

- a boolean valued statement, which may be either true or false

- "a equals b" : for example, when we do an algebraic manipulation such as $k(k+1) = k^2 + k$ as previous slide

  When I wrote "$k(k+1) = k^2 + k$" on the previous slide, I was *not* saying this is a Boolean expression that is either true of false. Rather I was saying it is true!

It should be clear from the context which usage I mean.

There was a similar ambiguity in Quiz 1 (Fall 2021). How to interpret the expression “38 % x == 2” below?

Let 38 % x == 2. What could be the value of x ? Select all that apply.

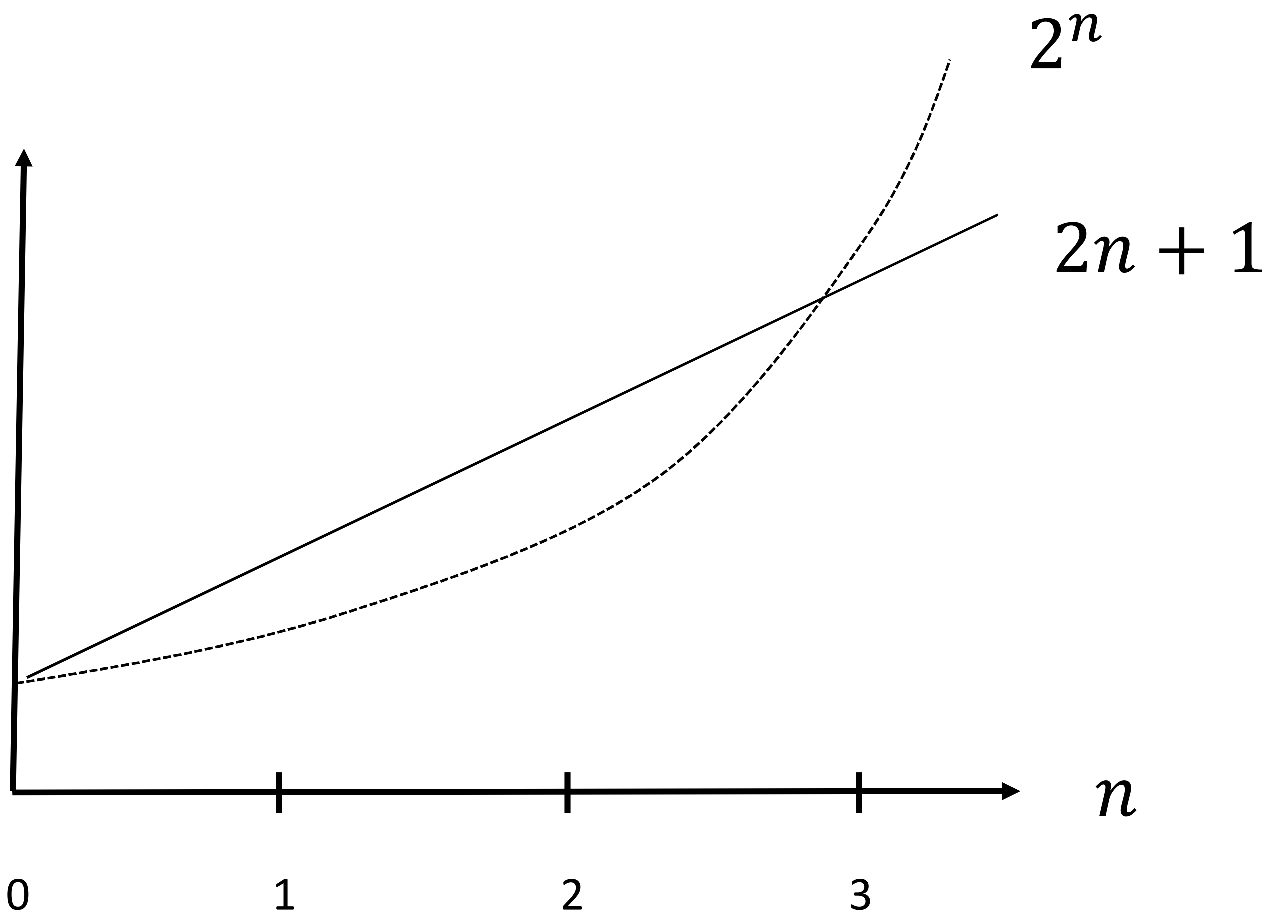
- [ ] 12
- [ ] 14
- [ ] 16
- [ ] 18

Some students interpreted it as a Java expression (boolean value) and claimed that the sentence didn’t makes sense. They said that I should have written “38 % x equals 2”. (BTW, I gave 0.5 pts to students who interpreted it this way and checked all the boxes.)

Mathematical Induction: Example 2

Prove the following statement:

For all $n \geq 3$, $2n + 1 < 2^n$.



Statement: For all $n \geq 3$, $2n + 1 < 2^n$ is true.

Note: $P(n)$ is false for n = 1, 2.

But that is not a problem. Why not?

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof (base case, n = 3):

?

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof (base case, n = 3):

2*3 + 1 < 8 (true)

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k+1)$ is true.

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k + 1)$ is true.

$$2(k + 1) + 1 = ?$$

(The left side is $2n + 1$ where $n = k + 1$.)

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k+1)$ is true.

$$2(k+1) + 1 = 2k + 2 + 1$$

trivial calculation
(i.e. not boolean
expression that may be
either true or false)

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k+1)$ is true.

$$2(k+1) + 1 = 2k + 2 + 1$$

$$< 2^k + 2$$

by induction hypothesis (where $k \geq 3$)

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

For any $k \geq 3$,

We want to show: if $P(k)$ is true, then $P(k+1)$ is true.

$$2(k+1) + 1 = 2k + 2 + 1$$

$$< 2^k + 2$$

by induction hypothesis (where $k \geq 3$)

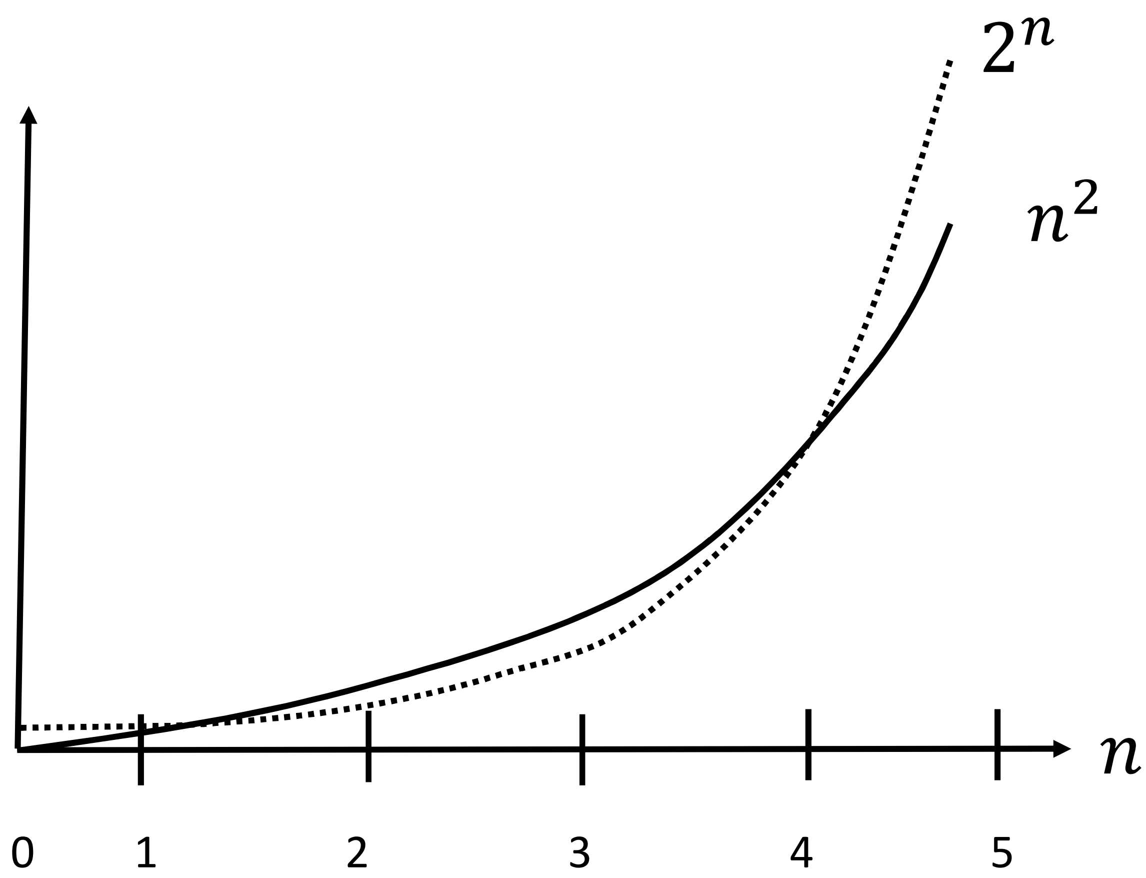
$$< 2^k + 2^k,$$

This inequality is also true for k ≥ 2 but we don't care because we are trying to prove for k ≥ 3.

$$= 2^{k+1}$$

Example 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.



Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case $(n = 5)$:

Induction step:

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case $(n = 5)$:

25 < 32

Induction step:

What do we hypothesize ? (if ____):

What do we want to show ? (then____)

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case $(n = 5)$:

25 < 32

Induction step:

What do we hypothesize ? (if _____):

$k^2 < 2^k$, $where\ k \geq 5$

What do we want to show ? (then_____)

$(k+1)^2 < 2^{k+1}$

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

25 < 32

Induction step: Let $k \geq 5$

$$(k+1)^2 = k^2 + 2k + 1$$

trivial calculation

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

25 < 32

Induction step: Let $k \geq 5$

$$(k+1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case ($n = 5$):

25 < 32

Induction step: Let $k \geq 5$

$$(k+1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

by Example 2 (for $k \geq 3$, so it also holds for $k \geq 5$)

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

which is what we wanted to show.

Example 4 : Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Statement: For all $n \geq 0$, $F(n) < 2^n$

Let's prove it by mathematical induction.

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

For all $n \geq 0$, $F(n) < 2^n$ is true.

Base case(s):

$n = 0$: $0 < 2^0$ yes, this is true

$n = 1$: $1 < 2^1$ yes, this is true

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

For all $n \geq 0$, $F(n) < 2^n$ is true.

Induction step:

$$F(k+1) = F(k) + F(k-1)$$

$$< 2^k + 2^{k-1}$$

by induction hypothesis (applied twice, $k > 0$)

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

For all $n \geq 0$, $F(n) < 2^n$

Induction step:

$$F(k+1) = F(k) + F(k-1)$$

$$< 2^k + 2^{k-1}$$

by induction hypothesis (applied twice, $k > 0$)

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

which is what we wanted to show.

Exercises

1. Use mathematical induction to prove that, for any $n \geq 1$

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}.$$

2. Use mathematical induction to prove that, for all $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$