Coronado
Systems

# Introduction to Feedback Control Systems

## Course Notes

 REV. 0 , APRIL 2000

# 1 Introduction to Feedback Control Systems

Feedback control was used by the Egyptians in a water clock more than 2000 years ago. The same principle allowed James Watt to invent the governor which regulated the speed of steam engines in the 19th century.

But it was only in the 1930's that a theory of feedback control was first developed by Black and Nyquist at Bell Labs. They were studying feedback as a means to linearize repeater amplifiers for telephone lines, but they had problems with what they called "singing". This was simply the onset of closed-loop instability when the feedback gain was set too high, transforming the amplifier into an oscillator.

Many new applications of feedback control were developed during World War II, such as radar-based anti-aircraft gun control, rocket flight control (the German V2's), pilotless reconnaissance flights over Germany, etc.

Nowadays, feedback control is an enabling technology in most industries. For example:

- Aerospace
  -autopilots
  -open-loop unstable jet fighters
  -rockets

- Process control
  -electric arc furnaces
  -continuous blending processes
  -nuclear reactors
  -chemical reactors

- Electronics
  -Op-amp circuits
  -Phase-lock loops
  -hard disk drives

- Manufacturing Automation
  -Robotics
  -CNC mills

A feedback control system is a system whose output is controlled using its measurement as a feedback signal. This feedback signal is compared with a *reference signal* to generate an *error signal* which is filtered by a *controller* to produce the system's control input. We will concentrate on continuous-time linear time-invariant (LTI) feedback systems. Thus, the Laplace transform and the state-space framework will be our main tools for analysis and design. The block diagram below illustrates a general feedback system.

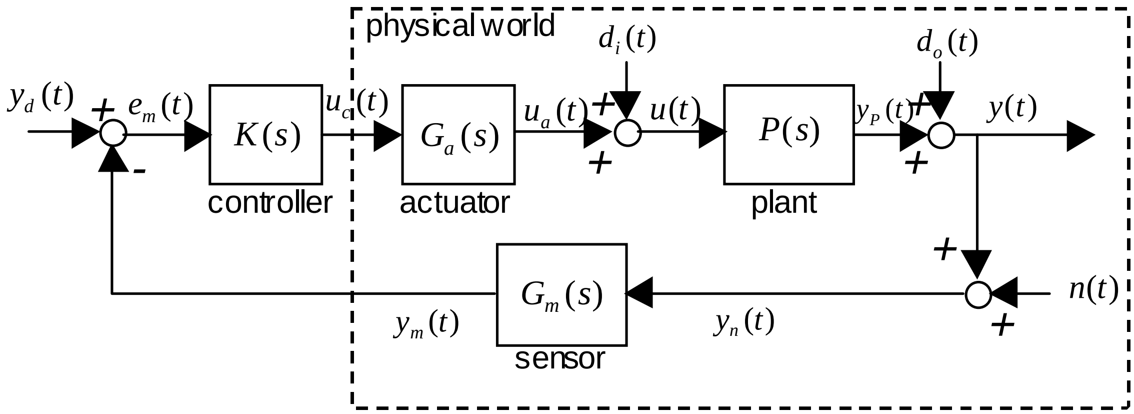



Figure 1: Typical feedback control system

The controlled system is called the *plant*, and its LTI model is the transfer function $P(s)$. The disturbed output of the plant is $y(t)$ and its noisy measurement is $y_m(t)$, corrupted by the measurement noise $n(t)$. The error between the desired output $y_d(t)$ (or reference) and $y_m(t)$ is the *measured error*, denoted as $e_m(t)$. The actual error between the plant output and the reference is $e(t) := y_d(t) - y(t)$. The *output disturbance* is the signal $d_o(t)$ and the output measurement noise is $n(t)$. The feedback measurement sensor dynamics are modeled by $G_m(s)$. The *actuator* (e.g., a valve) modeled by $G_a(s)$ is the device that translates a control signal from the controller $K(s)$ into an action on the plant input. The *input disturbance* signal $d_i(t)$ (e.g., a friction force) disturbs the control signal from the actuator to the plant input.

In many cases, we will assume that the actuator and sensor are perfect (meaning $G_a(s) = G_m(s) = 1$), and that measurement noise can be neglected so that $n(t) = 0$. This will simplify the analysis.

## 1.1 Objectives of feedback control

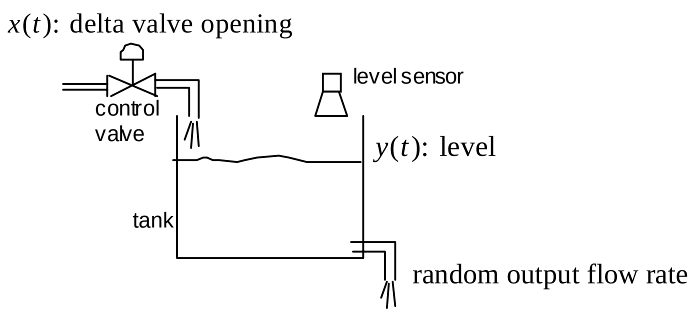
The main objectives of feedback control is to ensure that variables of interest in a process or a system, thought of as the *output signals,* either

- track reference trajectories (called *tracking* or *servo*), or
- are maintained close to their setpoints (called *regulation*).

## 1.2 Need for feedback

Why do we need feedback anyway? Fundamentally, for three reasons:

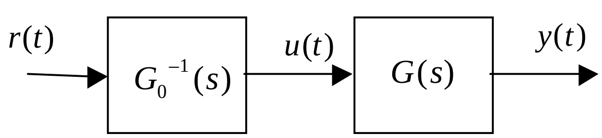
1. To counteract disturbance signals affecting the output.
   For example, suppose that we would like to maintain the level of liquid in a process tank close to 2m. But an uncontrolled drain valve produces unpredictable disturbances in the output flow rate. Then, one would need to use a level sensor to measure the level of liquid and to control the input feed valve to counteract the effect of the disturbance created by the drain valve.




2. To improve system performance in the presence of model uncertainty.
   Open-loop control is often used on processes or machines for which a model (mathematical, or a knowledge base) is available. The idea behind open-loop control is to specify a desired reference output signal, and to invert the model to compute what input signal is needed to get the desired output. Then the plant is driven with this computed input signal. A major problem with this technique is caused by uncertainty in the plant model. Such uncertainty may lead to unacceptable discrepancies between the desired output signal and the actual.

   For example, consider the open-loop control setup below where the physical plant has a transfer function $G(s) = G_0(s) + \Delta(s)$ (unknown to us), for which we have the nominal model $G_0(s)$. Assume this nominal model is invertible. The error between the actual plant output and the reference in the Laplace domain is given by:

$$\begin{aligned}\hat{y}(s) - \hat{r}(s) &= \left[G_0(s) + \Delta(s)\right]G_0^{-1}(s)\hat{r}(s) - \hat{r}(s) \\ &= \Delta(s)G_0^{-1}(s)\hat{r}(s)\end{aligned}$$




3. To stabilize an unstable plant.
   Many industrial processes are open-loop unstable. For example, for a fixed inlet valve position, the level of liquid in a tank will rise until the tank overflows. Other example of open-loop unstable plants are hydraulic rams, aircraft, nuclear reactors. Feedback control is then necessary to stabilize these plants.

## 1.3 Control system technology: actuators, sensors, controllers

Control theory can be applied to many types of plants, but there may be limitations imposed by the available technology of actuators, sensors, and controllers.

Actuators are devices that allow the manipulation of *control variables* in a process. For example, in a typical home temperature control system, electric baseboard heaters and heat pumps are the actuators. Typical actuators for process control include valves, electric motors, heaters,

Sensors are devices that can measure the output variables to be controlled. Typical sensors include thermocouples for temperature measurements, differential pressure transmitters, radar range sensors, magnetic flowmeters.

Controllers use feedback measurements of the output variables to compute and issue control signals to the actuators. Controllers used to be mechanical devices (combinations of governors and lever arms) or analog circuits, until the advent of the microprocessor. Nowadays, controllers are mostly implemented using industrial computers or PLC's (programmable logic controllers).

## 1.4 Some applications

### 1.4.1 Water level regulator for a toilet tank

A ubiquitous application of feedback control that everyone has in her/his home is the toilet! (necessity the mother of invention…) Let's see how the toilet works. First, we identify the components of the water tank level control system in Figure 2. The actuator is the pin valve, the sensor if the floater, and the controller is the lever arm connecting the floater and to the valve's pin. After the toilet is flushed, the valve is fully opened and the water level starts to rise. When the level gets close to the desired level, the floater pushes on the pin to slowly close the valve until it closes completely when the level is at the setpoint.

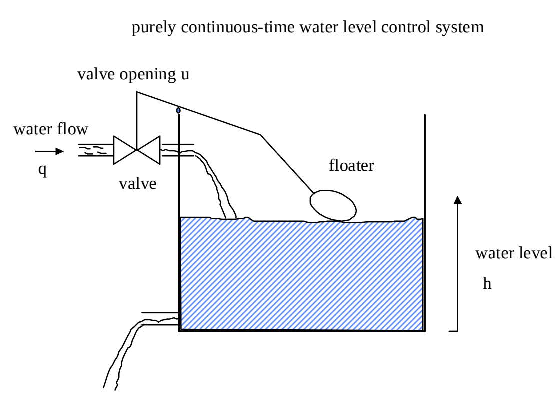



Figure 2: Toilet tank level control system

### 1.4.2 Single-link robot

A classical technique to control the position of an inertial load (e.g., a robot link) driven by a permanent-magnet DC motor is to vary the armature current based on a potentiometer measurement of the load angle.

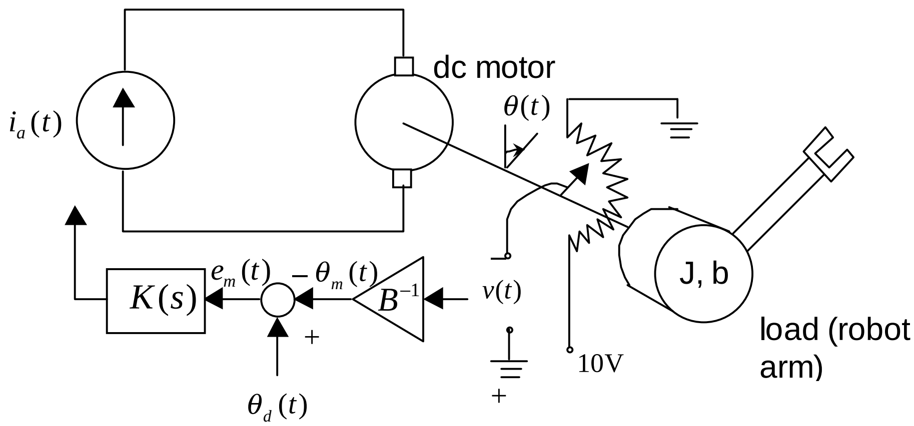



Figure 3: Single-link robot with DC motor

Let's identify the components of this control system. The plant is the load. The actuator is the DC motor, the sensor is the potentiometer, and the controller $K(s)$ could be an op-amp circuit driving a voltage-to-current power amplifier. Note that a similar setup can be used for conveyor speed control, except that the potentiometer angle sensor would be replaced with an angular velocity sensor.

### 1.4.3 Air pressure control in a vessel

Constant air or gas pressure control is important for many processes. Consider the air pressure control system shown in  below.

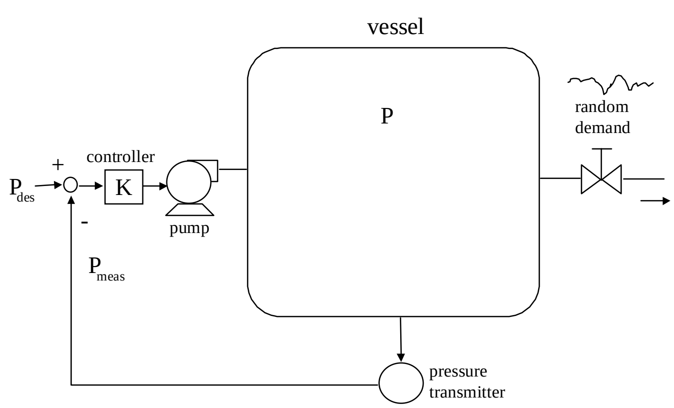



Figure 4: Air pressure control system

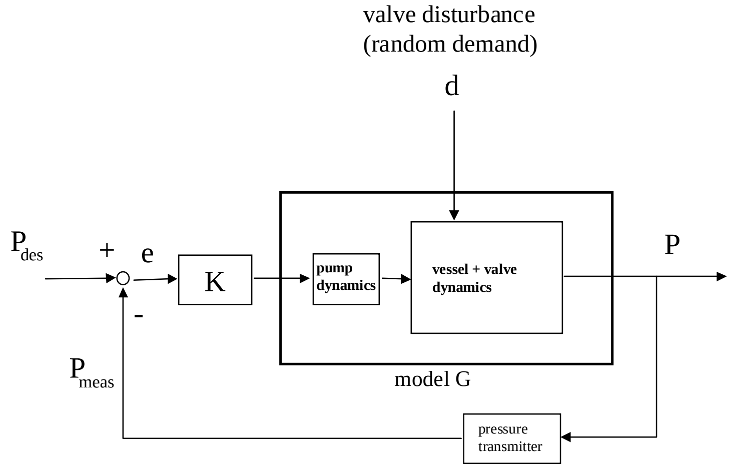



Figure 5: Block diagram of pressure control system

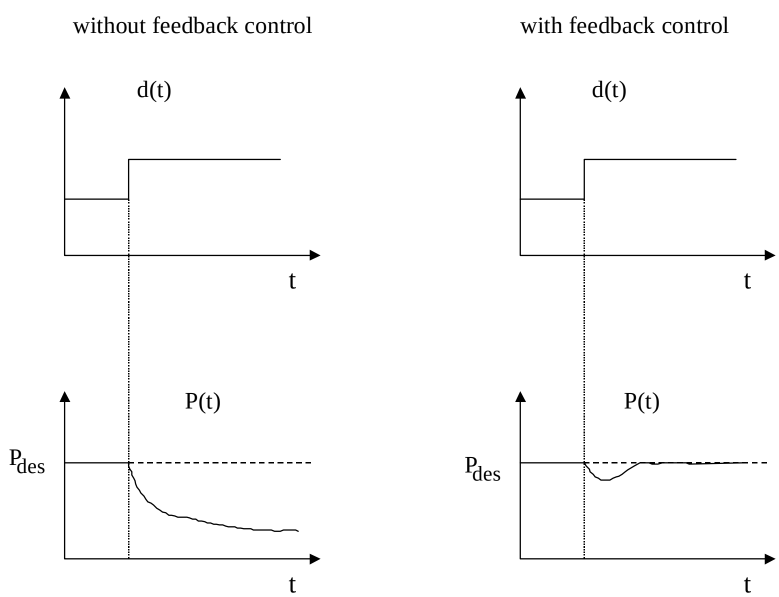



Figure 6: Effect of an input disturbance without and with feedback control

# 2 State-space and transfer matrix models of linear time-invariant differential systems

We will study linear time-invariant differential feedback systems, i.e. systems for which the plant, actuator, sensor and controller models are linear constant-coefficient differential equations. The theory of nonlinear, time-varying control systems is much less developed, and much more complicated. Fortunately, LTI control theory can be successfully used for most nonlinear processes by linearizing the dynamics around an operating point. But first, we will review some key aspects of basic signals and systems theory.

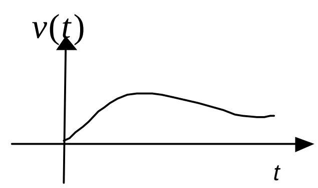
## 2.1 Signals and systems

### 2.1.1 Mathematical functions as signal models

Signals are functions of time that represent the evolution of variables such as a furnace temperature, the speed of a car, a motor shaft position, or a voltage. There are two types of signals: *continuous-time* signals and *discrete-time* signals.

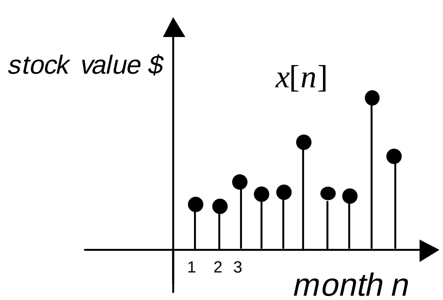
Continuous-time signals are functions of a continuous variable (time).

*Example:* The speed of a car $v(t)$




Discrete-time signals are functions of a discrete variable, i.e., they are defined only for integer values of the independent variable (time steps).

*Example*: The value of a stock at the end of each month



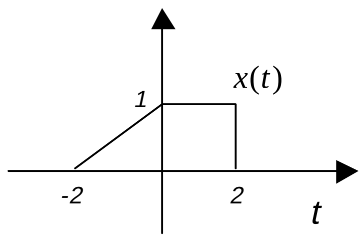

Real continuous-time and discrete-time signals are functions of time mapping their *domain* $T$ (time interval) into their *range* $\mathcal{R} \subseteq \mathbb{R}$ (subset of the real numbers). This is expressed as $x : T \to \mathcal{R}$. If the range $\mathcal{R}$ is subset of the set of all real n-vectors $\mathbb{R}^n$, then $x$ is a vector-valued signal. Note that we often use the notation $x(t)$ to designate a continuous-time signal (not just the value of $x$ at time t), and $x[n]$ to designate a discrete-time signal.

For the car speed example above, the domain of $v(t)$ could be $T = [0, +\infty)$ seconds, assuming the car keeps on running forever (!), and the range is $\mathcal{R} = [0, +\infty) \subset \mathbb{R}$, the set of all positive speeds in km/h.

For the stock trend, the domain of $x[n]$ is $T = 1, 2, 3, \ldots$, and the range is $\mathcal{R} = [0, \infty) \subset \mathbb{R}$.

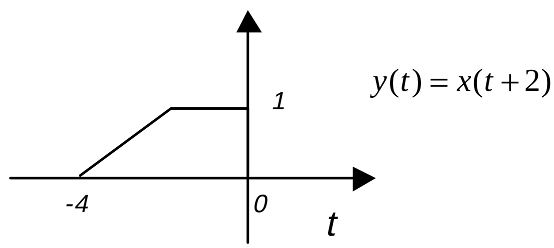
### 2.1.2 Signal time shift

Consider the following signal $x(t)$

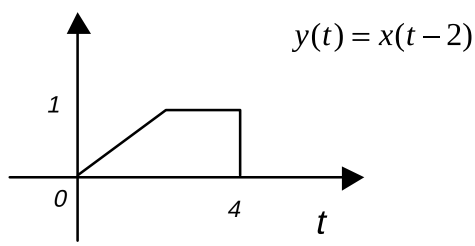



A time shift delays or advances the signal in time by a time interval $T$ (or $N$ for a discrete-time signal).

Time advance of 2 seconds:




Time delay of 2 seconds:




### 2.1.3 Periodic signals

A continuous-time signal $x(t)$ is periodic if there exists a time $T$ for which:

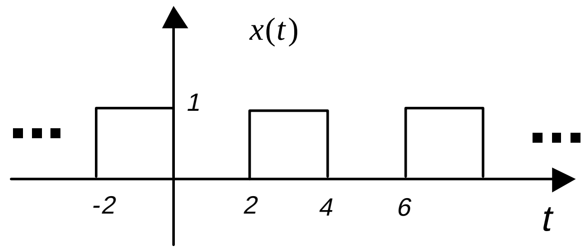
$$x(t) = x(t + T) \tag{1.1}$$

A discrete-time signal $x[n]$ is periodic if there exists an integer $N$ for which:

$$x[n] = x[n + N] \tag{1.2}$$

The smallest such $T$ or $N$ is called the *fundamental period* of the signal.

*Examples: Periodic square wave*



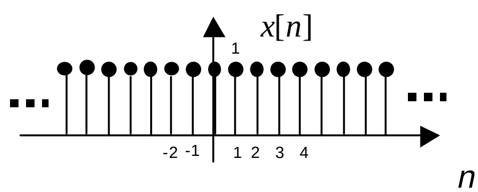

The fundamental period of this square wave signal is $T = 4$, but 8, 12 and 16 are also periods of the signal.

*Complex exponential* $x(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$:

$$x(t+T) = e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T} \qquad (1.3)$$

The right-hand side is equal to $x(t) = e^{j\omega_0 t}$ if $T = \dfrac{2\pi k}{\omega_0}, k = \pm 1, \pm 2, \ldots$ so these are all periods of the complex exponential. The fundamental period is $T = \dfrac{2\pi}{\omega_0}$.

*Discrete-time signal* $x[n] = 1$ is periodic with fundamental period $N = 1$



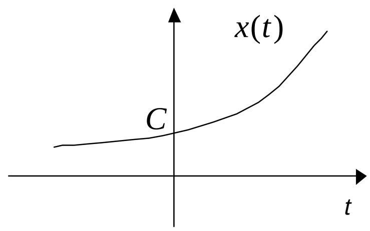

## 2.2 Exponential signals

Exponential signals are extremely important in signals and systems analysis because they are *eigenfunctions* of linear time-invariant systems (i.e., they are invariant in form when applied as an input to an LTI system.)

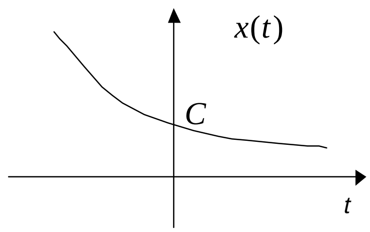
### 2.2.1 Real exponential signals

$$x(t) = Ce^{at}, \quad C, a \text{ real} \qquad (1.4)$$

Case a=0: We simply get the constant signal $x(t) = C$.
Case a>0: The exponential tends to infinity as $t \to \infty$ (here C>0).




Case a<0: The exponential tends to zero as $t \to \infty$ (here C>0).




### 2.2.2 Complex Exponential Signals

$$x(t) = Ce^{at}$$
$$C, a \text{ complex}, \quad C = |C|e^{j\theta}, \quad a = r + j\omega_0 \tag{1.5}$$

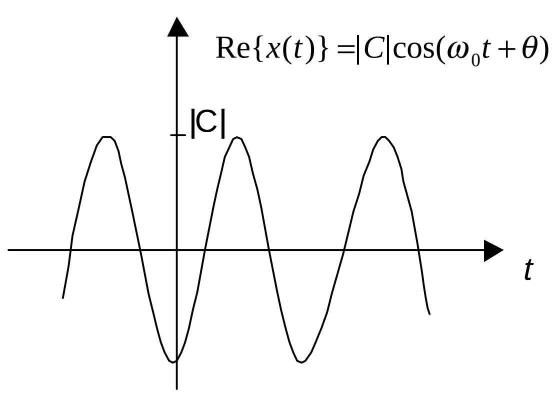
$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)} \tag{1.6}$$

Using Euler's relation, we get

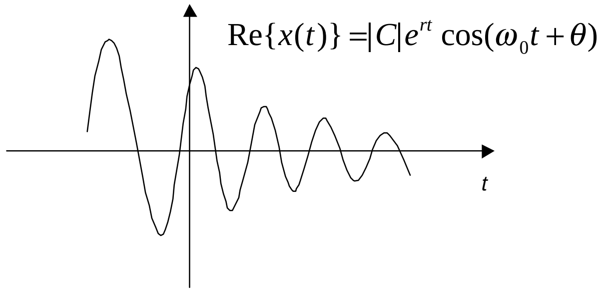
$$x(t) = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta) \tag{1.7}$$

For $r=0$, we obtain a complex periodic signal of period $T = \frac{2\pi}{\omega_0}$ whose real and imaginary parts are sinusoidal

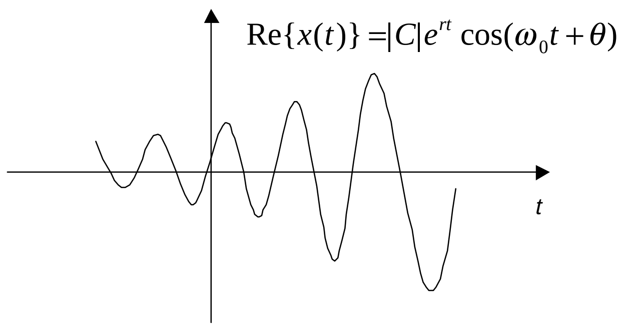
$$x(t) = |C|e^{j\theta}e^{j\omega_0 t} = |C|\cos(\omega_0 t + \theta) + j|C|\sin(\omega_0 t + \theta) \tag{1.8}$$



For $r<0$, we get a complex periodic signal multiplied by a decaying exponential whose real and imaginary parts are "damped sinusoids" (examples: response of RLC circuits, mass-spring-damper systems (car suspension))




For $r>0$, we get a complex periodic signal multiplied by a growing exponential (example: response of an unstable feedback system)




### 2.2.3 Finite-energy and finite-power signals

The power dissipated in a resistor is

$$p(t) = v(t)i(t) = \frac{v^2(t)}{R}$$

and the *total energy* dissipated during a time interval $[t_1, t_2]$ is

$$E = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt$$

The *average power* dissipated over that interval is just

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt$$

Analogously, the total energy and average power over $[t_1, t_2]$ of an arbitrary signal are defined as follows.

$$E := \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

$$P := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt \qquad (1.9)$$

The total energy and average power of a signal defined over $-\infty < t < \infty$ are defined as:

$$E_\infty := \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|x(t)|^2\,dt \tag{1.10}$$

$$P_\infty := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt \tag{1.11}$$

Class of <u>Finite-Energy Signals</u>: signals for which $E_\infty < \infty$.

Class of <u>Finite-Power Signals</u>: signals for which $P_\infty < \infty$.

Examples: $x(t) = 4$ has infinite energy but an average power of 16:

$$P_\infty := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}4^2\,dt = \lim_{T\to\infty}\frac{4^2}{2T}2T = 16 \tag{1.12}$$

The average power of periodic signals can be calculated on one period only. For

$x(t) = Ce^{j\omega t}$, it is

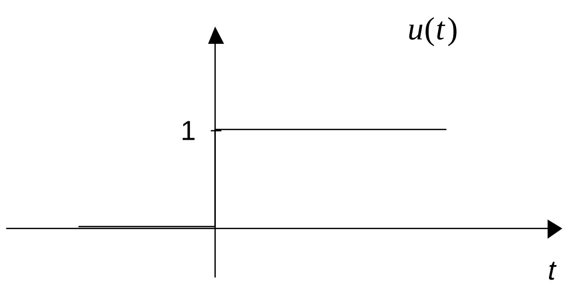
$$P_\infty := \frac{1}{T}\int_0^T |Ce^{j\omega t}|^2\,dt = \frac{|C|^2}{T}\int_0^T dt = \frac{|C|^2}{T}[T-0] = |C|^2 \tag{1.13}$$

Note that $e^{j\omega_0 t}$ has unity power.

### 2.2.4 Continuous-time impulse and step functions

The continuous-time *unit step* function $u(t)$ is defined as follows:

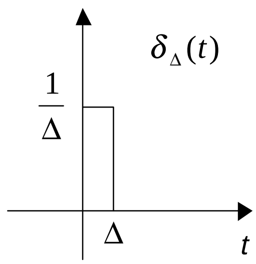
$$u(t) := \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.14}$$



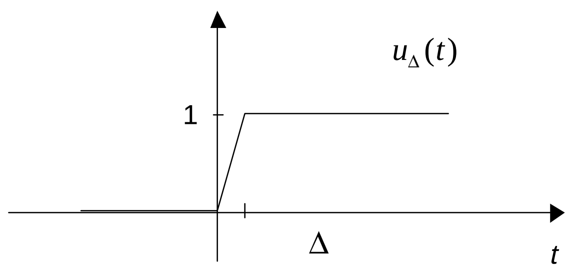

Note that since $u(t)$ is discontinuous at the origin, it can't be formally differentiated.

The *unit impulse,* a generalized function, can be defined as follows. Consider a pulse function of unit area:

$$\delta_\Delta(t) := \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{1.15}$$




The integral of this pulse is an approximation to the unit step:




As $\Delta$ tends to 0, The pulse $\delta_\Delta(t)$ gets taller and thinner, but keeps its unit area, while $u_\Delta(t)$ approaches a unit step function. At the limit,

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t) \tag{1.16}$$

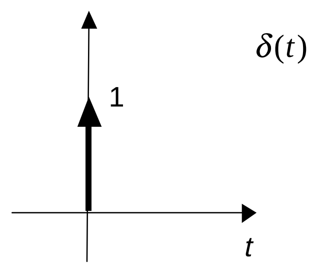
$$u(t) := \lim_{\Delta \to 0} u_\Delta(t) \tag{1.17}$$

Note that $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$, and in this sense we can write $\delta(t) = \frac{d}{dt} u(t)$ at the limit.

Conversely, we have the important relationship:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \,. \tag{1.18}$$

Graphically, $\delta(t)$ is represented by an arrow "pointing to infinity" at t=0 with its length equal to its area.




2.2.4.1 Sampling property of the impulse function

The pulse function $\delta_\Delta(t)$ can be made narrow enough so that $x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$, and at the limit, for an impulse at time $t_0$:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \tag{1.19}$$

so that

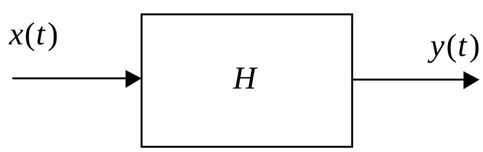
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0) \tag{1.20}$$

This last equation is often cited as the correct definition of an impulse for $t_0 = 0$, since it implicitly defines the impulse through what it does to any continuous functions under the integral sign, rather than using a limiting argument pointwise.

## 2.3 Linear time-invariant (LTI) differential systems

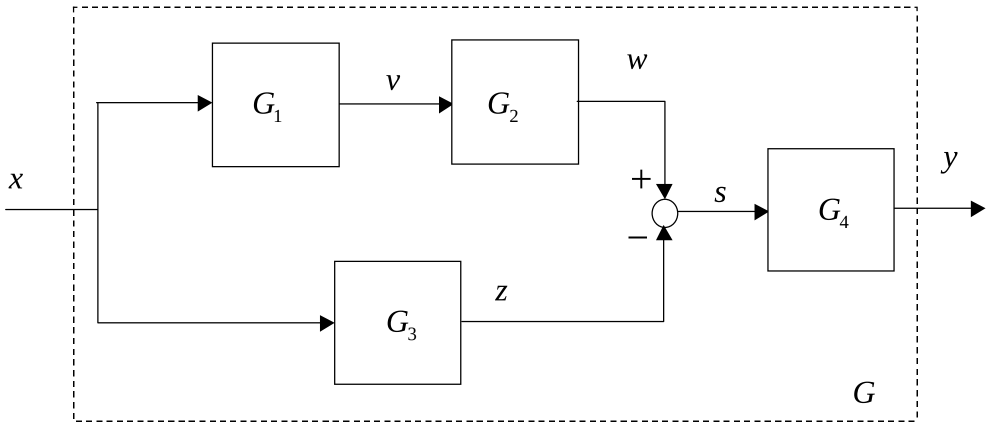
### 2.3.1 Input-Output System Models

A *system* is a mathematical relationship between an input signal and an output signal. It is represented by the abstract operator equation $y = Hx$, where $H$ is an operator that may represent anything from a simple gain to a complicated nonlinear differential equation.

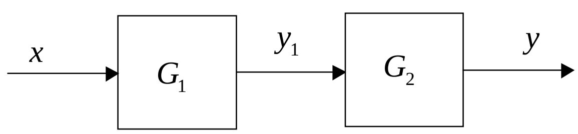


### 2.3.2 System Block Diagrams

Systems may be interconnections of other systems.

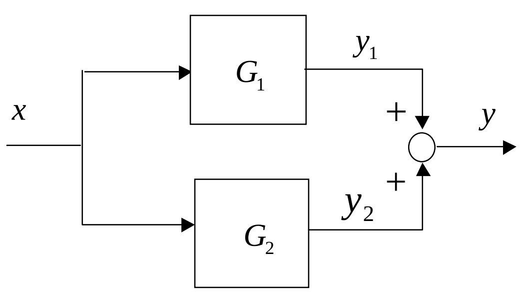



#### 2.3.2.1 Cascade Interconnection



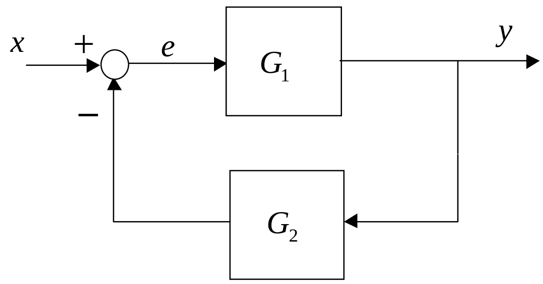

$$y = G_2 G_1 x \quad (1.21)$$

#### 2.3.2.2 Parallel Interconnection



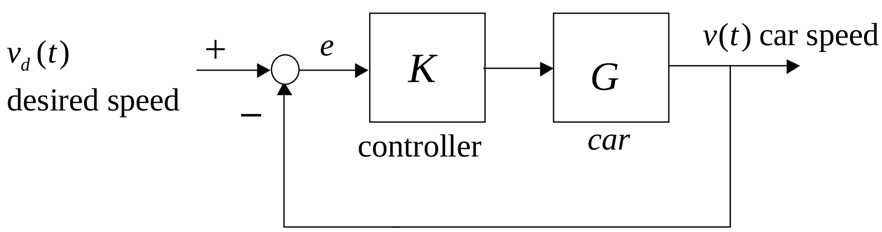

$$y = G_1 x + G_2 x \quad (1.22)$$

#### 2.3.2.3 Feedback Interconnection




$$e = x - G_2 y$$
$$y = G_1 e \qquad (1.23)$$

*Example: car cruise control system*



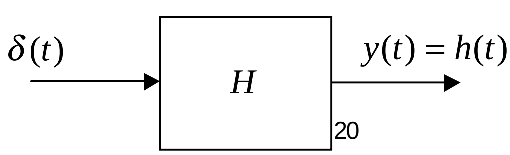

$$e = v_d - v$$
$$v = GKe \qquad (1.24)$$

### 2.3.3 Linear time-invariant (LTI) systems

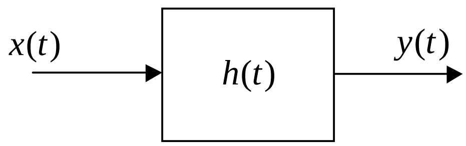
Consider a system $G$ and any two input signals $x_1$, $x_2$ such that $y_1 = Gx_1, y_2 = Gx_2$. System $G$ is said to be *linear* if for $x = ax_1 + bx_2$, $a, b \in \mathbb{C}$, $y := Gx = ay_1 + by_2$ (linear superposition).

Now suppose $y(t) = Gx(t)$. System $G$ is said to be *time-invariant* if for any $T$, $y(t-T) = Gx(t-T)$. In other words, if we time shift the input signal by $T$, then the output signal is simply the original output signal time-shifted by $T$.

A linear time-invariant (LTI) system is uniquely represented by its *impulse response* $h(t)$. The class of LTI systems has a very rich theory and a broad range of applications in engineering. The impulse response of a system $H$ is simply its response to a unit impulse as shown in the block diagram below.



Now because an LTI system is completely and uniquely represented by its impulse response $h(t)$, it is often depicted as follows:




The relationship between the input signal and the output signal is given by the classical convolution equation between $x(t)$ and $h(t)$:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \, . \tag{1.25}$$

This convolution integral is often difficult to compute. This is why we will make use of the Laplace transform and transfer functions.

### 2.3.4 Causal LTI differential systems

Differential LTI systems constitute an extremely important subset of LTI systems in engineering. They are used for circuit analysis, filter design, controller design, process modeling, etc.

This is the subset of LTI systems for which the input and output signals are related *implicitly* through a *linear constant coefficient differential equation.*

*Example:* First-order differential equation relating the input $x(t)$ to the output $y(t)$

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{1.26}$$

This equation can represent the velocity of a car $y(t)$ subjected to a friction force proportional to its speed, and in which $x(t)$ would be the force applied on the car by the engine.

Given the input signal $x(t)$, we have to solve the differential equation to obtain the output signal (the response) of the system.

In general, an $N^{th}$*-order linear constant coefficient differential equation* has the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{1.27}$$

which can be expanded to

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t) \qquad (1.28)$$

To find a solution to a differential equation of this form, we need more information than what the equation provides. In fact, we need N initial conditions (or auxiliary conditions) on the output variable and its derivatives to be able to calculate a solution.

Recall that the complete solution to Equation (1.27) is the sum of the *homogeneous solution* of the differential equation (a solution with the input signal set to zero), and of a *particular solution* (a function that satisfies the differential equation).

Forced response of the system = particular solution (usually has the form of the input signal)

Natural response of the system = homogeneous solution (depends on initial conditions and forced response)

Example: Consider the system described by the linear constant coefficient differential equation of (1.26) . We will calculate the solution (or output) of this system to the input signal $x(t) = Ke^{3t}u(t)$ where $K$ is a real number. As stated above, the solution is composed of a homogeneous response (natural response), and a particular solution (forced response) of the system:

$$y(t) = y_h(t) + y_p(t) \qquad (1.29)$$

where the particular solution satisfies (1.26), and the homogeneous solution $y_h(t)$ satisfies:

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0\,. \qquad (1.30)$$

For the particular solution (forced response) for $t > 0$, we look at a signal $y_p(t)$ of the same form as $x(t)$ for $t > 0$: $y_p(t) = Ye^{3t}$. Substituting the exponentials for $x(t)$ and $y_p(t)$ in Equation (1.26), we get

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}\,, \qquad (1.31)$$

which yields $Y = K/5$ and

$$y_p(t) = \frac{K}{5}e^{3t},\ t > 0\,. \qquad (1.32)$$

Now we want to determine $y_h(t)$, the natural response of the system. We hypothesize a solution of the form of an exponential: $y_h(t) = Ae^{st}$. Substituting this exponential in (1.30), we get

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s+2) = 0\,,$$

which holds for $s = -2$. Note that the polynomial $s+2$ is the *characteristic polynomial* of the system (more generally it is defined as $p(s) := \sum_{k=0}^{N} a_k s^k$.) Also with this value for $s$, $Ae^{-2t}$ is a solution to the homogeneous equation (1.30) for any choice of $A$.

Combining the natural response and the forced response, we find the solution to the differential equation (1.26):

$$y(t) = y_h(t) + y_p(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0. \tag{1.33}$$

Now because we haven't specified an initial condition on $y(t)$, this response is not completely determined as the value of $A$ is still unknown.

*For causal LTI systems defined by linear constant coefficient differential equations, the initial conditions are always* $y(0) = \frac{dy(0)}{dt} = \cdots = \frac{d^{N-1}y(0)}{dt^{N-1}} = 0$, *what's called "initial rest"*. That is, if at least one initial condition is nonzero, then strictly speaking the system is nonlinear. For our example above, initial rest implies that $y(0) = 0$, so that

$$y(0) = A + \frac{K}{5} = 0 \tag{1.34}$$

and we get $A = -\frac{K}{5}$. Thus for $t > 0$, the solution (output signal) is:

$$y(t) = \frac{K}{5}(e^{3t} - e^{-2t}), \quad t > 0.$$

*Example: Room heating process* (taken from Skogestad and Postlethwaite, 1996)

Suppose that we want to model a room heating process.

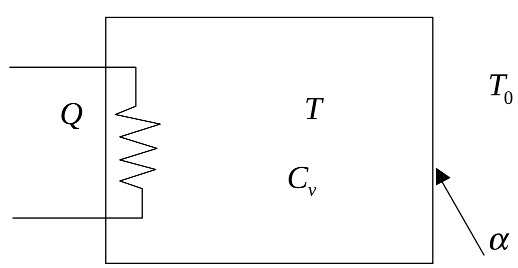



Figure 7: Room heating process

The system is defined from the heat rate input (in Watts) applied by a heater to the temperature at a specific location in the room. An energy balance dictates that the change of energy in the room must equal the net flow of energy per unit of time into the room:

$$\frac{d}{dt}\left(C_V T\right) = Q + \alpha\left(T_0 - T\right) \tag{1.35}$$

where $T\,[K]$ is the room's temperature, $C_V\,[J/K]$ is the heat capacity of the room, $\alpha\left(T_0 - T\right)[W]$ is the total heat loss through the walls, and $Q\,[W]$ is the heat rate input.

Suppose that the *operating point* is $Q^* = 2000W$ with a difference between indoor and outdoor temperatures of $T^* - T_0^* = 20K$. Then, the steady-state energy balance yields $\alpha^* = \dfrac{Q^*}{T^* - T_0^*} = 100\dfrac{W}{K}$. Assume that the room heat capacity is a constant $C_V = 100\,kJ/K$.

Now we can write Equation (1.35) in terms of deviations from the operating point. Let

$$\begin{aligned} \Delta T(t) &:= T(t) - T^* \\ \Delta Q(t) &:= Q(t) - Q^* \end{aligned} \tag{1.36}$$

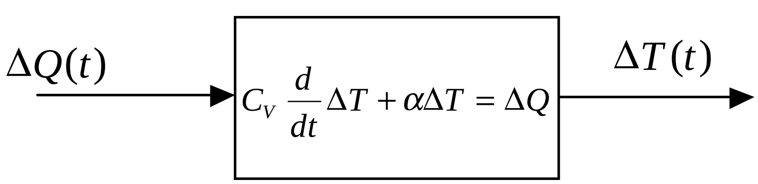
and assume that the outdoor temperature remains constant at $T_0^*$. Then Equation (1.35) becomes

$$C_V \frac{d}{dt}\Delta T(t) + \alpha \Delta T(t) = \Delta Q(t), \tag{1.37}$$

numerically:

$$100000\frac{d}{dt}\Delta T(t) + 100\Delta T(t) = \Delta Q(t). \tag{1.38}$$

This is an LTI differential system with the variation in heat being the input signal and the temperature deviation being the output signal:




## 2.4 State-space models of LTI differential systems

For a system described by an *N*th-order causal, LTI differential (difference) equation, it is always possible to find a set of *N* first-order differential (difference) equations and an output equation describing the same input-output relationship. These *N* first-order differential (difference) equations are called the *state equations* of the system. The *states* are the *N* variables seen as outputs of the state equations.

A general m-input, p-output state-space system, often simply denoted as $(A,B,C,D)$, has the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}, \qquad (1.39)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, and the state-space matrices have dimensions $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times p}$.

- The matrices $A,B,C,D$ are called *state-space matrices*
- The first equation in (1.39) is called the *state equation* as it described the evolution of the vector of state variables.
- The second equation in (1.39) is called the *output equation* as it described the evolution of the output vector as a linear combination of the state variables and the inputs.

For a given LTI system described by a differential equation, there are an infinite number of state-space realizations. The concept of state is directly applicable to certain types of real systems such as linear circuits and mechanical systems. The state variables in a circuit are the capacitor charges (or equivalently the voltages since $q = Cv$) and the inductor currents. In a mechanical system, the state variables are generally the position and velocity of a body.

A word on notation: for state-space systems, the general input signal is usually written as $u(t)$ (not to be confused with the unit step) instead of $x(t)$, as the latter is used for the vector of state variables.

*Example: Mixing tank process*

Consider the continuous mixing process shown in Figure 8 with the liquid in the tank being the system. We are interested in the level and temperature of the liquid for subsequent control purposes. The assumptions are:

- unsteady-state, variable volume
- liquid phase well-mixed
- no phase change
- inert (no chemical reactions)
- well-insulated
- negligible kinetic energy and potential energy changes
- negligible pressure changes (Pin = P = Pout)
- negligible work done by the mixer
- constant liquid density and heat capacity

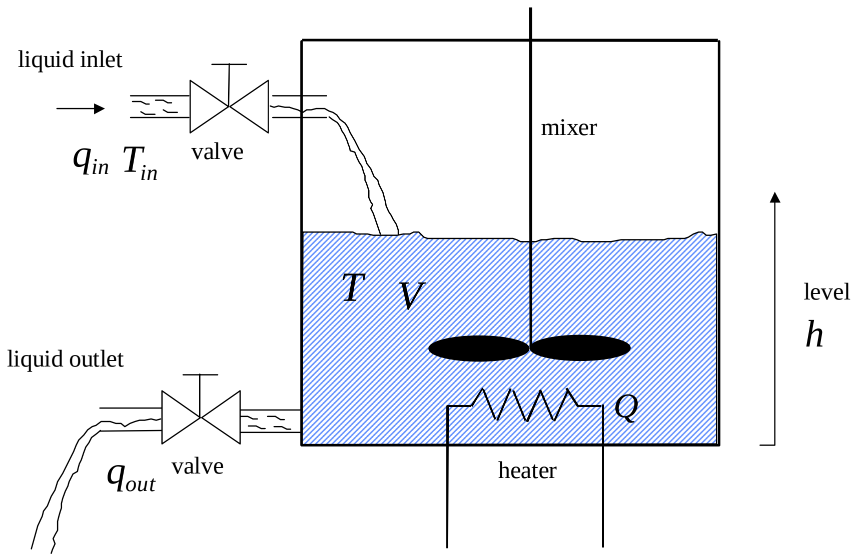



Figure 8: Continuous mixing tank process

Under the above assumptions, mass and energy balances on this process yield one linear and one nonlinear first-order state equations given by:

$$\frac{dV(t)}{dt} = q_{in}(t) - q_{out}(t) \tag{1.40}$$

$$\frac{dT(t)}{dt} = \frac{q_{in}(t)}{V(t)}\left(T_{in}(t) - T(t)\right) + \frac{Q(t)}{C_p \rho V(t)} \tag{1.41}$$

where $V(t)[m^3]$ is the total volume of liquid in the tank, $q_{in}(t)[\frac{m^3}{s}]$ is the inlet flow rate into the tank, $q_{out}(t)[\frac{m^3}{s}]$ is the outlet flow rate, $T(t)[K]$ is the liquid temperature in the tank, $T_{in}(t)[K]$ is the liquid temperature at the inlet valve, $Q(t)[W]$ is the input heat from the heater, $C_p[\frac{J}{kgK}]$ is the heat capacity of the liquid in the tank, and $\rho[\frac{kg}{m^3}]$ is the liquid density.

Let the state vector be defined as $x(t) := \begin{bmatrix} V(t) & T(t) \end{bmatrix}^T$ and suppose that we decide to use the inlet flow rate and the heat as input signals so that $u(t) := \begin{bmatrix} q_{in}(t) & Q(t) \end{bmatrix}^T$. The output signals are chosen to be the liquid level and temperature: $y(t) := \left[h(t) \quad T(t)\right]^T = \left[\frac{1}{a}V(t) \quad T(t)\right]^T$ where $a[m^2]$ is the cross-sectional area of the tank. Then the nonlinear state and output equations can be written in a matrix-vector form as

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t),u(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{1.42}$$

where the nonlinear function $f(x(t),u(t))$ is nothing but the right-hand sides of (1.40) and (1.41) stacked up in a vector, and the 2 by 2 C matrix is given by $C = \begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}$.

Now we need to linearize (1.42) around an operating point to get a *linear* state equation. Note that the output equation is already linear. Let

$$\begin{aligned} \Delta x(t) &:= x(t) - x^* = \left[V(t) - V^* \quad T(t) - T^*\right]^T \\ \Delta u(t) &:= u(t) - u^* = \left[q_{in}(t) - q_{in}{}^* \quad Q(t) - Q^*\right]^T \\ \Delta y(t) &:= y(t) - y^* = \left[h(t) - h^* \quad T(t) - T^*\right]^T \end{aligned} \tag{1.43}$$

Suppose that the liquid is water for which $\rho = 1000\frac{kg}{m^3}$ and $C_p = 4187\frac{J}{kgK}$ and we have the following operating point:

| | |
|---|---|
| $T^* = 323K$ | $T_{in}{}^* = 293K$ |
| $V^* = 1m^3$ | $h^* = 1m$ |
| $a = 1m^2$ | $Q^* = 1000W$ |
| $q_{in}{}^* = 0.001\frac{m^3}{s}$ | $q_{out}{}^* = 0.001\frac{m^3}{s}$ |

The linearized state-space equations are given by

$$\begin{aligned} \frac{d\Delta x(t)}{dt} &= \underbrace{\left.\frac{\partial f(x,u)}{\partial x}\right|_*}_{A} \Delta x(t) + \underbrace{\left.\frac{\partial f(x,u)}{\partial u}\right|_*}_{B} \Delta u(t) \\ \Delta y(t) &= C\Delta x(t) \end{aligned} \tag{1.44}$$

Thus, matrices $A$ and $B$ is given by:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_* = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} \\ \frac{\partial f_2(x,u)}{\partial x_2} & \frac{\partial f_2(x,u)}{\partial x_2} \end{bmatrix}_*$$
$$= \begin{bmatrix} 0 & 0 \\ -\frac{C_p \rho q_{in}^* \left(T_{in}^* - T^*\right) + Q^*}{C_p \rho V^{*2}} & -\frac{q_{in}^*}{V^*} \end{bmatrix} \tag{1.45}$$
$$= \begin{bmatrix} 0 & 0 \\ 0.0298 & -0.001 \end{bmatrix}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_* = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial u_1} & \frac{\partial f_1(x,u)}{\partial u_2} \\ \frac{\partial f_2(x,u)}{\partial u_1} & \frac{\partial f_2(x,u)}{\partial u_2} \end{bmatrix}_*$$
$$= \begin{bmatrix} 1 & 0 \\ \frac{\left(T_{in}^* - T^*\right)}{V^*} & \frac{1}{C_p \rho V^*} \end{bmatrix} \tag{1.46}$$
$$= \begin{bmatrix} 1 & 0 \\ -30 & 2.388\times 10^{-7} \end{bmatrix}$$

and finally the LTI differential system for the mixing tank is given by:

$$\begin{aligned} \frac{d\Delta x(t)}{dt} &= \begin{bmatrix} 0 & 0 \\ 0.0298 & -0.001 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 1 & 0 \\ -30 & 2.388\times 10^{-7} \end{bmatrix} \Delta u(t) \\ \Delta y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta x(t) \end{aligned} \tag{1.47}$$

Note that $D = 0_{2\times 2}$ for this plant.

## 2.5 Transfer function models of LTI differential systems

Along with state-space models of LTI differential systems, another very useful representation of such systems is the transfer function. First let's very briefly review the Laplace transform and the Fourier transform for continuous-time signals.

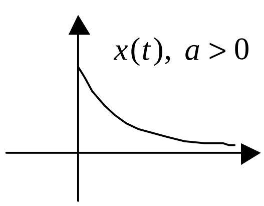
### 2.5.1 The Laplace Transform

The Laplace transform of a signal $x(t)$ is defined as follows:

$$X(s) := \int_{-\infty}^{+\infty} x(t) e^{-st} dt \quad (1.48)$$

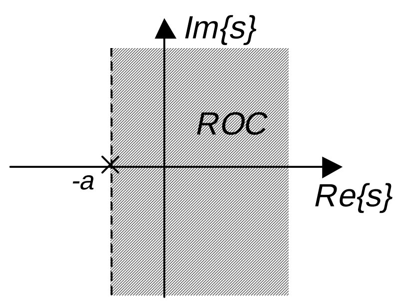
where $s$ is a complex variable. Given $x(t)$, this integral may or may not converge, depending on the value of $\sigma$ (the real part of $s$).

Example: Find the Laplace transform of $x(t) = e^{-at}u(t),\ a \text{ real}$.




$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st} dt \\ &= \int_{0}^{+\infty} e^{-(s+a)t} dt \\ &= \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a \end{aligned} \quad (1.49)$$

This Laplace transform converges only for values of $s$ in the open half-plane to the right of $s = -a$. This half-plane is the *region of convergence* (ROC) of the Laplace transform. It is represented as follows:




*The ROC is an integral part of a Laplace transform.* Without it, you can't tell what the corresponding time-domain signal is.

### 2.5.2 Inverse Laplace transform

The inverse Laplace transform is in general given by

$$x(t):=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds \tag{1.50}$$

This contour integral is rarely used because we are mostly dealing with linear systems and standard signals whose Laplace transforms are found in tables of Laplace transform pairs. Thus, we will mainly use the partial fraction expansion technique to find the continuous-time signal corresponding to a Laplace transform.

For example, assuming no multiple-order poles in the set of poles $\{p_k\}_{k=1}^{m}$ of the rational transform $X(s)$, and assuming that the order of the denominator polynomial is greater than the order of the numerator polynomial, we can expand $X(s)$ in the form

$$X(s)=\sum_{k=1}^{m}\frac{A_k}{s-p_k} \tag{1.51}$$

From the ROC of $X(s)$, the ROC of each of the individual terms in (1.51) can be found, and then the inverse transform of each of these terms can be determined from a table of Laplace transform pairs. If the ROC is to the right of the pole at $s=p_i$, then the inverse transform of this term is $A_i e^{p_i t}u(t)$, a right-sided signal. If, on the other hand, the ROC is to the left of the pole at $s=p_i$, then the inverse transform of this term is $-A_i e^{p_i t}u(-t)$, a left-sided signal. Adding the inverse transforms of the individual terms in (1.51) yields the inverse transform of $X(s)$.

### 2.5.3 Convolution property of the Laplace transform

If $x_1(t)\overset{\mathcal{L}}{\leftrightarrow}X_1(s),\ \ s\in \mathrm{ROC}_1$ and $x_2(t)\overset{\mathcal{L}}{\leftrightarrow}X_2(s),\ \ s\in \mathrm{ROC}_2$, then

$$\int_{-\infty}^{+\infty}x_1(\tau)x_2(t-\tau)d\tau\overset{\mathcal{L}}{\leftrightarrow}X_1(s)X_2(s),\ \ s\in\mathrm{ROC}\supseteq\mathrm{ROC}_1\cap\mathrm{ROC}_2. \tag{1.52}$$

This is of course an extremely useful property for LTI system analysis. Note that the resulting ROC includes the intersection of the two original ROC's, but it may be larger, e.g., when a pole-zero cancellation occurs.

Example: The response of the LTI system with $h(t)=\left[e^{-2t}+e^{-t}\right]u(t)$ to the input $x(t)=-e^{-2t}u(t)+\delta(t)$ is given by the inverse Laplace transform of $Y(s)$:

$$\begin{aligned}h(t)&\overset{\mathcal{L}}{\leftrightarrow}H(s)=\frac{2s+3}{(s+2)(s+1)},\ \ \mathrm{Re}\{s\}>-1\\ x(t)&\overset{\mathcal{L}}{\leftrightarrow}X(s)=\frac{-1}{s+2}+1=\frac{s+1}{s+2},\ \ \mathrm{Re}\{s\}>-2\end{aligned} \tag{1.53}$$

and

$$Y(s) = H(s)X(s) = \frac{(2s+3)}{(s+2)(s+1)}\frac{(s+1)}{(s+2)},\ \{s\!:\mathrm{Re}\{s\} > -2\} \cap \{s\!:\mathrm{Re}\{s\} > -1\} = \mathrm{Re}\{s\} > -1$$
$$= \frac{(2s+3)}{(s+2)^2},\ \mathrm{Re}\{s\} > -2 \tag{1.54}$$

Expanding this transform into partial fractions, we get

$$Y(s) = \frac{(2s+3)}{(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2},\ \mathrm{Re}\{s\} > -2\,. \tag{1.55}$$

We find the factor B first,

$$\left.\frac{(2s+3)}{1}\right|_{s=-2} = -1 = B\,, \tag{1.56}$$

and factor A is given by

$$\left.\frac{2s+3}{s+2}\right|_{s=+\infty} = 2 = A\,. \tag{1.57}$$

Therefore, using Table 9.2 of Laplace transform pairs in ["Signals and Systems, 2nd Edition" A.V. Oppenheim, A.S. Willsky, and H. Nawab, Prentice Hall, 1997], we obtain

$$y(t) = \left[2e^{-2t} - te^{-2t}\right]u(t) \tag{1.58}$$

### 2.5.4 Time shifting property of the Laplace transform

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s),\ \ s \in \mathrm{ROC}$ , then

$$x(t-t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s),\ \ s \in \mathrm{ROC} \tag{1.59}$$

### 2.5.5 The Fourier transform

The *Fourier transform* or *spectrum* of a signal $x(t)$ is defined as:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \tag{1.60}$$

The *inverse Fourier transform* is given by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{1.61}$$

The Fourier transform describes uniquely the frequency contents of a signal. If the ROC of the Laplace transform $X(s)$ contains the imaginary axis, then the Fourier transform of $x(t)$ exists is given by $X(j\omega) = X(s)\big|_{s=j\omega}$.

### 2.5.6 Transfer functions of LTI systems

We've seen above that the convolution property makes the Laplace transform useful to obtain the response of an LTI system to an arbitrary input (with a Laplace transform). Specifically, the Laplace transform of the output of an LTI system with impulse response $h(t)$ is simply given by

$$Y(s) = H(s)X(s), \quad \mathrm{ROC_Y} \supseteq \mathrm{ROC_H} \cap \mathrm{ROC_X}. \tag{1.62}$$

Also recall that the frequency response of the system is $H(j\omega) = H(s)\big|_{s=j\omega}$.

The Laplace transform $H(s)$ of the impulse response of an LTI system is called the *transfer function*.

#### 2.5.6.1 Stability

Bounded-input bounded-output stability of a continuous-time LTI system is equivalent to its impulse response being absolutely integrable, in which case its Fourier transform converges. Also, the stability of an LTI *differential* system is equivalent to having all the zeros of its characteristic polynomial having a negative real part.

The stability condition for a *causal* LTI system with a *rational* transfer function is stated below. Note that a large class of causal differential LTI systems have rational transfer functions.

*A causal system with rational transfer function $H(s)$ is stable if and only if all of its poles are in the left-half of the s-plane (i.e., all of the poles have negative real parts.)*

### 2.5.7 Transfer functions of LTI differential systems

We have seen that the transfer function of an LTI system is the Laplace transform of its impulse response. For a *differential* LTI system, like the frequency response, the transfer function can be readily written by inspecting the LTI differential equation. Recall that $s$ is the transfer function of the differentiation operator and $s^{-1}$ represents the integration operator.

Consider the general form of an LTI differential system:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{1.63}$$

We use the differentiation and linearity properties of the Laplace transform to obtain the transfer function $H(s) = Y(s)/X(s)$.

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s), \tag{1.64}$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}. \tag{1.65}$$

Note that we haven't specified an ROC yet. This means that differential equation (1.16) can have many different impulse responses, i.e., it is not a complete specification of the LTI system. If we know that the differential system is causal, then the ROC is the right half-plane to the right of the rightmost pole in the s-plane. The impulse response is then uniquely defined.

### 2.5.8 Transfer functions of LTI state-space systems

Consider the general causal continuous-time state-space system initially at rest:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}. \tag{1.66}$$

Taking the Laplace transform on both sides of (1.66), we obtain:

$$\begin{aligned}sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned} \tag{1.67}$$

Solving for the Laplace transform of the output, we get

$$Y(s)=\left[C(sI_n - A)^{-1}B + D\right]U(s). \tag{1.68}$$

The transfer function of the system is therefore given by:

$$H(s)=U(s)\mapsto Y(s)=C(sI_n - A)^{-1}B + D. \tag{1.69}$$

### 2.5.9 Poles and zeros of the transfer function

Let $N(s):=\sum_{k=0}^{M} b_k s^k$ be the numerator polynomial of the transfer function $H(s)$ in Equation (1.65) and let $D(s):=\sum_{k=0}^{N} a_k s^k$ be its denominator polynomial. Then,

The poles of $H(s)$ are the N roots of the *characteristic equation* $D(s)=0$ (or equivalently, the N zeros of the characteristic polynomial),

The zeros of $H(s)$ are the M roots of equation $N(s)=0$.

*Example:* Find the transfer function poles of the following causal differential LTI system.

$$\frac{d^2y(t)}{dt^2}+\omega_c\sqrt{2}\frac{dy(ct)}{dt}+\omega_c^2 y(t)=\frac{d^2x(t)}{dt^2}+\omega_c\sqrt{2}\frac{dx(t)}{dt} \tag{1.70}$$

Taking the Laplace transform on both sides, we get

$$s^2Y(s)+\omega_c\sqrt{2}sY(s)+\omega_c^2Y(s)=s^2X(s)+\omega_c\sqrt{2}sX(s)\,, \tag{1.71}$$

which yields

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s^2+\omega_c\sqrt{2}s}{s^2+\omega_c\sqrt{2}s+\omega_c^2} \tag{1.72}$$

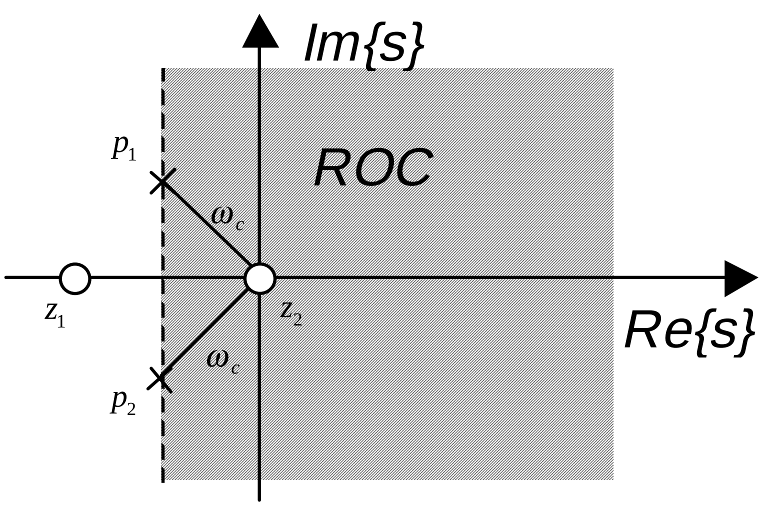
The poles are the roots of $s^2+\omega_c\sqrt{2}s+\omega_c^2=0$. Identifying the coefficients with the standard second-order denominator $s^2+2\varsigma\omega_n s+\omega_n^2$, one finds the damping ratio $\varsigma=\sqrt{2}/2=0.707$ and the natural frequency $\omega_n=\omega_c$. The poles of $H(s)$ are then

$$\begin{aligned} p_1&=-\varsigma\omega_n+\omega_n\sqrt{\varsigma^2-1}=-\frac{\sqrt{2}}{2}\omega_c+j\frac{\sqrt{2}}{2}\omega_c=\omega_c e^{j\frac{3\pi}{4}},\\ p_2&=-\varsigma\omega_n-\omega_n\sqrt{\varsigma^2-1}=-\frac{\sqrt{2}}{2}\omega_c-j\frac{\sqrt{2}}{2}\omega_c=\omega_c e^{-j\frac{3\pi}{4}} \end{aligned} \tag{1.73}$$

and its zeros are

$$z_1=-\omega_c\sqrt{2},\qquad z_2=0\,.$$

The poles and zeros of a transfer function are often denoted with the symbols 'X' and 'O' respectively in on a *pole-zero plot* in the s-plane.



### 2.5.10 System stability

The stability of an LTI differential system is directly related to the poles of the transfer function and its region of convergence. An LTI system (including a *differential* LTI system) is stable if and only if the ROC of its transfer function includes the $j\omega$-axis. Assume there is no pole-zero cancellation in the closed right half-plane[1] when $H(s)=N(s)/D(s)$ is formed. Then,

*A causal LTI differential system is stable if and only if all of the poles of its transfer function lie in the open left half-plane.*

Note that for the case where a zero cancels out an unstable pole (call it $p_0$) in the transfer function, the corresponding differential LTI system is considered to be unstable. The reason is that any nonzero initial condition would cause the output to either grow unbounded (case $\mathrm{Re}\{p_0\}>0$), oscillate forever (case $p_0$ imaginary), or settle down to a nonzero value (case $p_0=0$).

For the example in 2.5.9 above, the two complex conjugate poles are in the open left half-plane, so the system is stable.

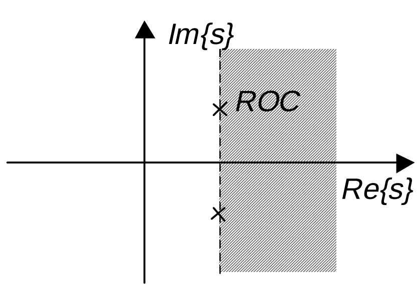
*Example: Second-order system*

$$H(s)=\frac{A\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}, \tag{1.74}$$

where $\omega_n$ is called the *undamped natural frequency* and $\varsigma$ (zeta) is called the *damping ratio*. Its two poles are real for $\varsigma\geq 1$ and complex (conjugate of each other) if $\varsigma<1$.

$$\begin{aligned} p_1 &= -\varsigma\omega_n+\omega_n\sqrt{\varsigma^2-1} \\ p_2 &= -\varsigma\omega_n-\omega_n\sqrt{\varsigma^2-1} \end{aligned} \tag{1.75}$$

In the case where $\varsigma<0$ and $\omega_n>0$, the poles are complex and have a positive real part $-\varsigma\omega_n$. The system is therefore unstable. The ROC is depicted below.




[1] It is customary to refer to $\mathrm{Re}\{s\}\geq 0$ as *the* right half-plane (or to $\mathrm{Re}\{s\}>0$ as *the* open right half-plane), and to $\mathrm{Re}\{s\}\leq 0$ as *the* left half-plane (or to $\mathrm{Re}\{s\}<0$ as *the* open left half-plane.)

*Example:* Suppose we know that if the input of a differential LTI system is

$$x(t) = e^{-3t}u(t)\,, \tag{1.76}$$

then the output is

$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t)\,. \tag{1.77}$$

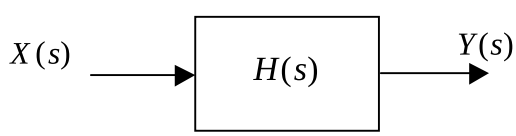
We can deduce the transfer function as follows. First take the Laplace transforms of the input and output signals:

$$X(s) = \frac{1}{s+3}, \quad \mathrm{Re}\{s\} > -3 \tag{1.78}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1\,. \tag{1.79}$$

Then, the transfer function is simply

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{1}{(s+1)(s+2)}}{\dfrac{1}{s+3}} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2} \tag{1.80}$$




The ROC of $H(s)$ is $\mathrm{Re}\{s\} > -1$, and it follows that the LTI system is stable.

Furthermore, a linear constant coefficient differential equation representing the system can be derived from the transfer function of Equation (1.26):

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t) \tag{1.81}$$

## 2.6 Frequency response of LTI differential systems

The *frequency response* of an LTI system is its phasor response to a complex exponential $e^{j\omega t}$. That is, if $x(t) = e^{j\omega t}$, then the output signal is given by the convolution integral (1.25):

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{j\omega(t-\tau)}d\tau \quad . \\ &= e^{j\omega t} \underbrace{\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau}_{H(j\omega)} \end{aligned} \tag{1.82}$$

Therefore, the frequency response of an LTI system is just the Fourier transform of its impulse response.

$$H(j\omega) := \int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau = H(s)\big|_{s=j\omega} \tag{1.83}$$

The physical interpretation of the frequency response is the level of attenuation and phase shift the system introduces in the output when the input is a sinusoidal signal of frequency $\omega$. For example suppose that the input signal to an LTI system is $x(t) = \sin(\omega t)$. Then the output signal is given by:

$$y(t) = |H(j\omega)|\sin\left[\omega t - \angle H(j\omega)\right]. \tag{1.84}$$

Some mechanical systems have a resonance frequency $\omega_R$, which means that the amplitude of the frequency response at this frequency $|H(j\omega_R)|$ is large.

We already know how to solve for the response of an LTI differential system which, in general, is given by a sum of a forced response and a natural response. However, it is often easier to use a Laplace or Fourier transform approach, provided the system is known to be *stable*.

Consider the stable LTI system defined by an N$^{th}$-order linear constant-coefficient differential equation initially at rest:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{1.85}$$

Assume that $X(j\omega), Y(j\omega)$ denote the Fourier transforms of the input $x(t)$ and the output $y(t)$ respectively. Recall that differentiation in the time domain is equivalent to a multiplication of the Fourier transform by $j\omega$. Thus, if we take the Fourier transform of both sides of Equation (1.85), we get

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega). \tag{1.86}$$

Now since $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$, the frequency response of the system is given by:

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{\sum_{k=0}^{M}b_k(j\omega)^k}{\sum_{k=0}^{N}a_k(j\omega)^k} \tag{1.87}$$

*Example:* The frequency response of the second-order LTI differential system

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}-x(t) \tag{1.88}$$

is calculated as follows:

$$[(j\omega)^2+3j\omega+2]Y(j\omega)=(j\omega-1)X(j\omega) \tag{1.89}$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega-1}{(j\omega)^2+3j\omega+2} \tag{1.90}$$

### 2.6.1 Bode plots

It is often convenient to use a logarithmic scale to plot the magnitude of a frequency response. One reason is that frequency responses can have a wide dynamic range covering many orders of magnitude. Some frequencies may be amplified by a factor 1000 while others may be attenuated by a factor $10^{-4}$. Another reason is that using a log scale, we can *add* rather than multiply the magnitudes of cascaded Fourier transforms, which is easier to do graphically:

$$\log|Y(j\omega)|=\log|H(j\omega)|+\log|X(j\omega)| \tag{1.91}$$

It is customary to use *decibels (dB)* as log units. The *bel* (named after Alexander Graham Bell) was defined as a power amplification of $|H(j\omega)|^2=10$ for a system. The *decibel* is one tenth of a bel. Therefore for the system with power gain of 10 at frequency $\omega$,

$$10\frac{dB}{bel}\times\log_{10}|H(j\omega)|^2\,bel=10\log_{10}10\,dB=10\,dB \tag{1.92}$$

But to measure the actual linear gain (not the power gain) of a system, we use the identity

$$10\log_{10}|H(j\omega)|^2\,dB=20\log_{10}|H(j\omega)|\,dB \tag{1.93}$$

Thus, a magnitude plot of $|H(j\omega)|$ is represented as $20\log_{10}|H(j\omega)|\,dB$ using a linear scale in dB units. Here is a table of gains expressed in dB

| Gain | Gain (dB) |
|---|---|
| **0** | $-\infty$ dB |
| **0.01** | -40 dB |
| **0.1** | -20 dB |
| **1** | 0 dB |
| **10** | 20 dB |
| **100** | 40 dB |
| **1000** | 60 dB |

It is also convenient to use a log scale for the frequency as features of a frequency response can be spread over a wide frequency band.

Bode Plot

A *Bode plot* is the combination of a magnitude and phase plots using log scales for the magnitude and the frequency, and a linear scales (radians or degrees) for the phase. Only positive frequencies are normally considered. As stated above, the Bode plot is quite useful since the overall frequency response of cascaded systems is simply the graphical addition of the Bode plots of the individual systems. In particular, this property is used to hand sketch a Bode plot of a rational transfer function in pole-zero form by considering each first-order factor corresponding to a pole or a zero to be an individual system with its own Bode plot.

*First-Order Example:* Consider again the first-order system with transfer function

$$H(s) = \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -2 \tag{1.94}$$

which has the frequency response

$$H(j\omega) = \frac{1}{j\omega + 2}. \tag{1.95}$$

It is convenient to write it as the product of a gain and a first-order transfer function with unity gain at dc:

$$H(j\omega) = \frac{1}{2}\frac{1}{j\omega/2 + 1}. \tag{1.96}$$

The *break frequency* is 2 radians/s. The Bode magnitude plot is the graph of

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{1}{2}\right| + 20\log_{10}\left|\frac{1}{\frac{j\omega}{2}+1}\right| dB$$
$$= -20\log_{10}2 - 20\log_{10}\left|\tfrac{j\omega}{2}+1\right| dB \,. \qquad (1.97)$$
$$= -6\,dB - 20\log_{10}\left|\tfrac{j\omega}{2}+1\right| dB$$

Note that for low frequencies ($\omega << 2$),

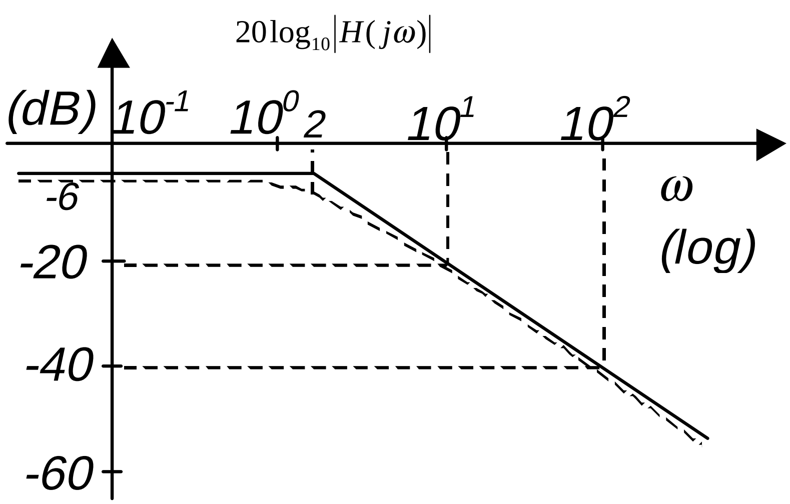
$$20\log_{10}|H(j\omega)| \approx -6\,dB - 20\log_{10}|1|\,dB = -6\,dB \,.$$

For high frequencies ($\omega >> 2$),

$$20\log_{10}|H(j\omega)| \approx -6\,dB - 20\log_{10}\left|\frac{\omega}{2}\right| dB$$
$$= -6\,dB - 20\log_{10}|\omega|\,dB + 20\log_{10}2\,dB$$
$$= -20\log_{10}|\omega|\,dB$$

For $\omega = 10$, we get -20 dB, for $\omega = 100$, we get -40 dB, etc. The slope of this asymptote is therefore -20dB/decade.

With the asymptotes meeting at the break frequency 2 radians/s, we can sketch the magnitude Bode plot as follows (dashed line: actual magnitude):



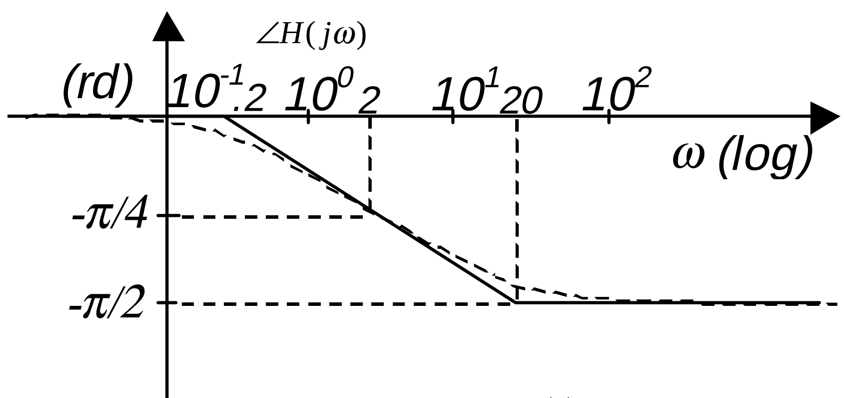
The Bode phase plot is the graph of

$$\angle H(j\omega) = \angle \frac{1}{2}\frac{1}{j\omega/2+1} = \angle \frac{1}{j\omega/2+1} = \arctan\left(\frac{-\frac{\omega}{2}}{1}\right). \quad (1.98)$$

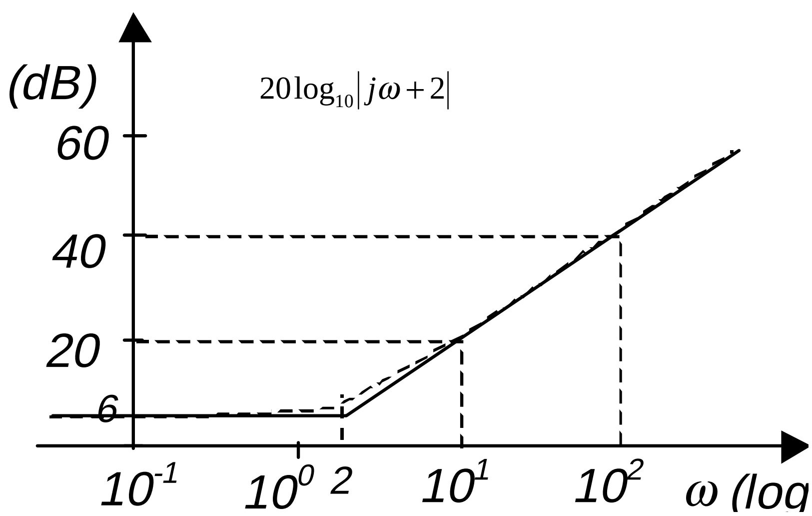
We know that the phase is 0 at $\omega = 0$ and $-\frac{\pi}{2}$ at $\omega = \infty$. A piecewise linear (with log frequency scale) approximation to the phase that helps us sketch it is given by

$$\angle H(j\omega) = \begin{cases} 0, & \omega \le \frac{2}{10} \\ -\frac{\pi}{4}\left[\log_{10}(\frac{\omega}{2})+1\right], & \frac{2}{10} < \omega < 20 \\ -\frac{\pi}{2}, & \omega \ge 20 \end{cases}. \quad (1.99)$$

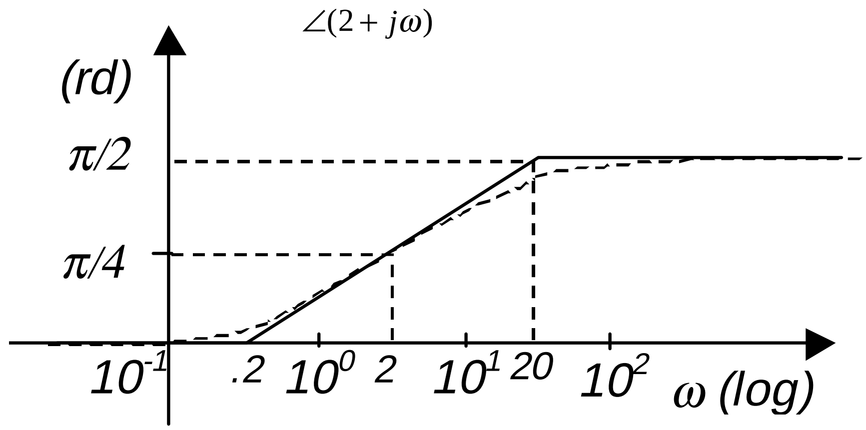
In words, the rule of thumb is: The phase is 0 at $\frac{\omega_b}{10}$, $-\frac{\pi}{4}$ at $\omega_b$, and $-\frac{\pi}{2}$ at $10\omega_b$ where $\omega_b$ is the break frequency. The Bode phase plot is shown below.




Note that the Bode magnitude plot of $s+2$ (the inverse of $H(s)$) is simply the above magnitude plot flipped around the frequency axis:



And likewise for the phase plot




*Second-Order Example:* Consider the second-order stable system with transfer function

$$H(s)=\frac{s+100}{\left(s^2+11s+10\right)}=\frac{s+100}{(s+1)(s+10)}=10\frac{\frac{s}{100}+1}{(s+1)\left(\frac{s}{10}+1\right)},\ \mathrm{Re}\{s\}>-1 \tag{1.100}$$

which has the frequency response

$$H(j\omega)=10\frac{\frac{j\omega}{100}+1}{(j\omega+1)\left(\frac{j\omega}{10}+1\right)}. \tag{1.101}$$

The break frequencies are 1, 10 and 100 radians/s. The Bode magnitude plot is the graph of

$$\begin{aligned}20\log_{10}|H(j\omega)| &= 20\log_{10}|10|+20\log_{10}\left|\tfrac{j\omega}{100}+1\right|+20\log_{10}\left|\frac{1}{\frac{j\omega}{10}+1}\right|+20\log_{10}\left|\frac{1}{j\omega+1}\right| dB \\ &= 20+20\log_{10}\left|\tfrac{j\omega}{100}+1\right|-20\log_{10}\left|\tfrac{j\omega}{10}+1\right|-20\log_{10}|j\omega+1|\ dB\end{aligned} \tag{1.102}$$

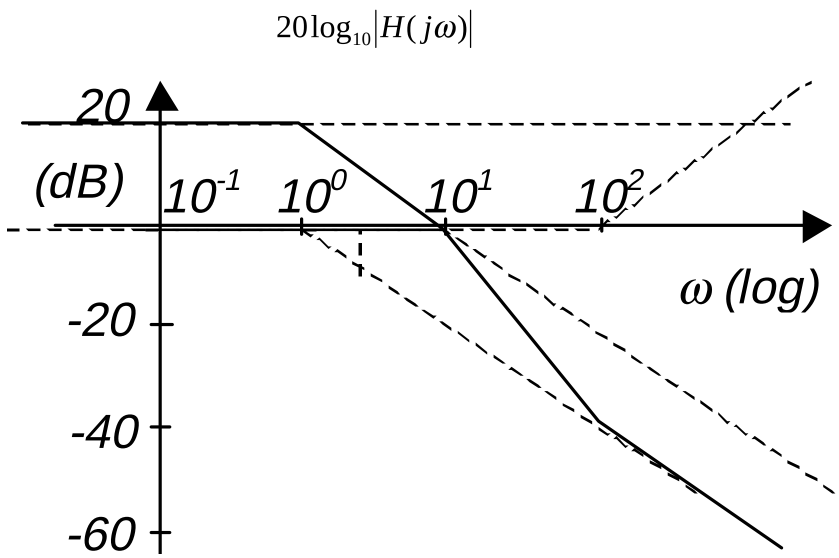
Note that at low frequencies ($\omega << 1$),

$$20\log_{10}|H(j\omega)|\approx 20\,dB+20\log_{10}|1|-20\log_{10}|1|-20\log_{10}|1|\,dB=20\,dB.$$

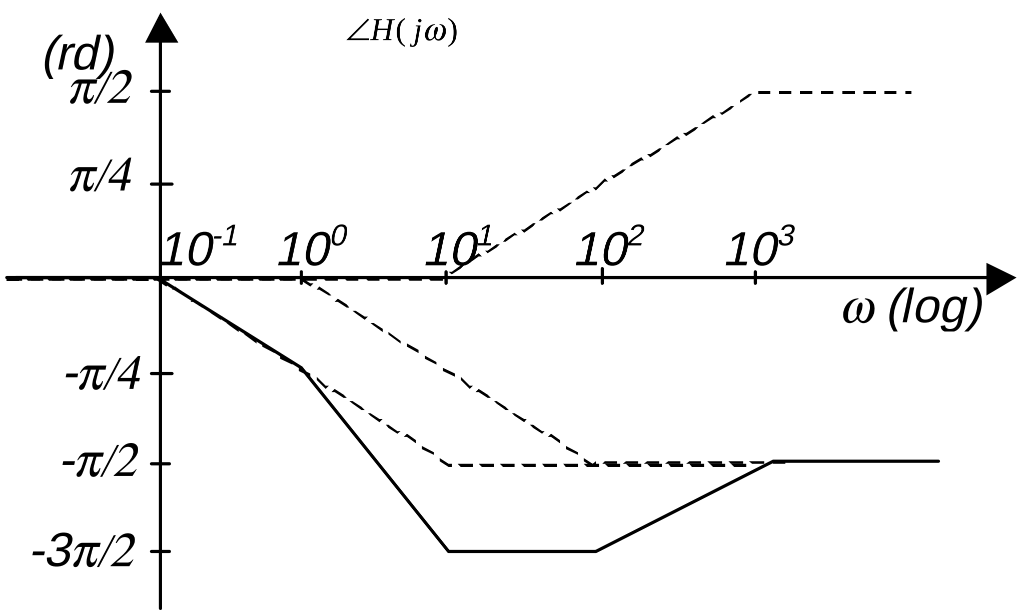
For high frequencies ($\omega >> 100$),

$$\begin{aligned}20\log_{10}|H(j\omega)| &\approx 20+20\log_{10}\left|\frac{\omega}{100}\right|-20\log_{10}\left|\frac{\omega}{10}\right|-20\log_{10}|\omega|\,dB \\ &= 20+20\log_{10}|\omega|-40-20\log_{10}|\omega|+20-20\log_{10}|\omega|\,dB \\ &= -20\log_{10}|\omega|\,dB\end{aligned}$$

We can plot the asymptotes of each first-order term in (1.65) on the same magnitude graph (dashed lines) and then add them together to obtain the Bode magnitude plot.




We proceed in a similar fashion to obtain the Bode phase plot:

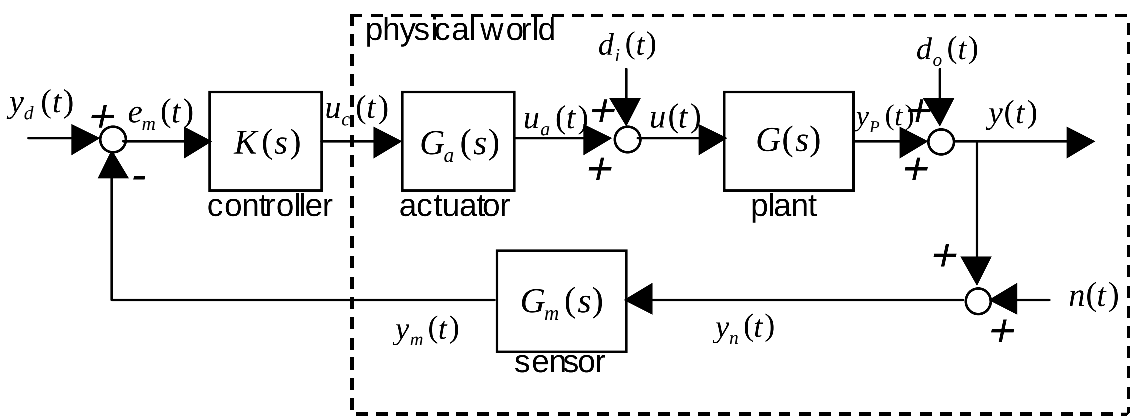


# 3 Feedback interconnections of LTI systems

With all these preliminaries in place, we can start our study of feedback control systems. The first step is to describe feedback interconnections of systems.

## 3.1 Feedback interconnection of LTI systems

As previously seen, the feedback interconnection of systems is conveniently represented by a block diagram. Generally we are interested in computing a closed-loop transfer function and analyzing it. Consider the block diagram of Figure 1, shown here once more for convenience. Note that any or all of these signals may be vector-valued, or stacked together as vectors. In these cases we would have a multi-input multi-output (MIMO) control system, as opposed to a simpler single-input single-output (SISO) control system. MIMO systems have *transfer matrices* instead of scalar transfer functions.




• Figure 9: Typical feedback control system

As an example, let's compute the closed-loop transfer function from the reference signal $y_d(t)$ to the output $y(t)$. Denoting the Laplace transforms of the signals with "hats" and dropping the Laplace variable $s$ to ease the notation, we obtain:

$$\hat{y} = GG_aK\hat{y}_d - GG_aKG_m\hat{y} \tag{1.103}$$

This type of equation with the signal of interest appearing on both sides is typical of feedback systems. Solving for $\hat{y}$, we get:

$$\hat{y} = (I + GG_aKG_m)^{-1} GG_aK\hat{y}_d \tag{1.104}$$

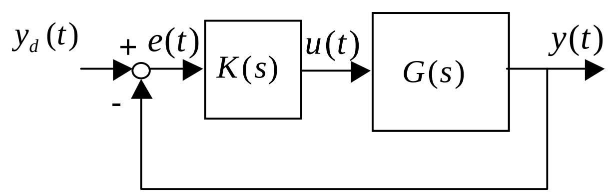
where $I$ is the identity matrix in the MIMO case, or equals 1 in the SISO case. Thus, the closed-loop transfer matrix is given by:

$$T = (I + GG_aKG_m)^{-1} GG_aK\,. \tag{1.105}$$

## 3.2 Feedback configurations for tracking and regulation

### 3.2.1 Tracking Systems

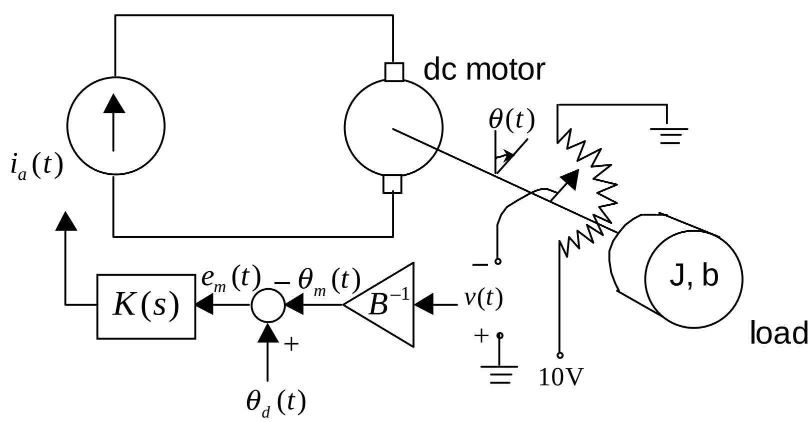
Two types of control systems can be distinguished: tracking (or servo) systems and regulators (disturbance rejection systems). As the name implies, a tracking system controls the plant output so it tracks the reference signal. From the simplified block diagram below (no noise or disturbance), good tracking is obtained when the error signal $e(t) \approx 0$ for all desired outputs $y_d(t)$. Then, $y(t) \approx y_d(t)$.




This diagram represents a general *unity feedback system.* Such a system has no sensor dynamics but a simple unity gain in the feedback path, i.e., the sensor is perfect. Here the actuator dynamics is lumped in the plant transfer function.

*Example*

A classical technique to control the position of an inertial load driven by a permanent-magnet DC motor is to vary the armature current based on a potentiometer measurement of the load angle.




Recall that the plant is the load, the actuator is the DC motor, the sensor is the potentiometer, and the controller $K(s)$ could be an op-amp circuit driving a voltage-to-current power amplifier.

The open-loop dynamics of this system are now described. The torque $\tau(t)$ in Newton-metres applied to the load by the motor is proportional to the armature current in Amps: $\tau(t) = Ai_a(t)$, so that

$$G_a(s) = \frac{\hat{\tau}(s)}{\hat{i}_a(s)} = A \tag{1.106}$$

The plant (or load) is assumed to be an inertia with viscous friction. The equation of movement for the plant is

$$J\frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} = \tau(t)\,, \tag{1.107}$$

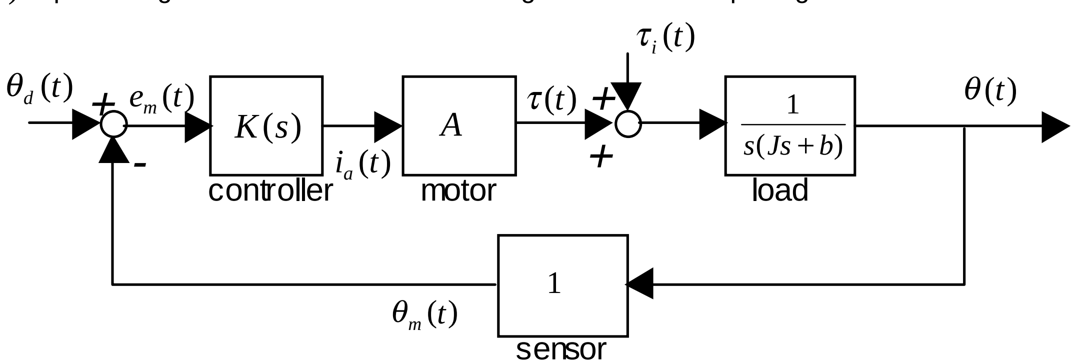
which yields the unstable plant transfer function

$$G(s) = \frac{\hat{\theta}(s)}{\hat{\tau}(s)} = \frac{1}{s(Js+b)}\,. \tag{1.108}$$

The potentiometer can be modeled as a pure gain mapping the load angle in the range $[0, \pi]$ radians to a voltage in the range $[0\text{V}, +10\text{V}]$: $v(t) = B\theta(t) = \frac{10}{\pi}\theta(t)$, thus

$$G_m(s) = \frac{\hat{\theta}_m(s)}{\hat{\theta}(s)} = \frac{\hat{\theta}_m(s)}{\hat{v}(s)} \cdot \frac{\hat{v}(s)}{\hat{\theta}(s)} = B^{-1}B = 1 \tag{1.109}$$

Assume that the measurement noise is negligible, and that there is only an input torque disturbance $\tau_i(t)$ representing unmodeled friction. A block diagram for this example is given below.




This load angle control system is a single-input, single-output unity feedback tracking system. The closed-loop transfer function from the reference to the output, called the *transmission*, is given by

$$T(s) := \frac{\hat{y}(s)}{\hat{y}_d(s)} = \frac{K(s)G(s)}{1+K(s)G(s)}\,. \tag{1.110}$$

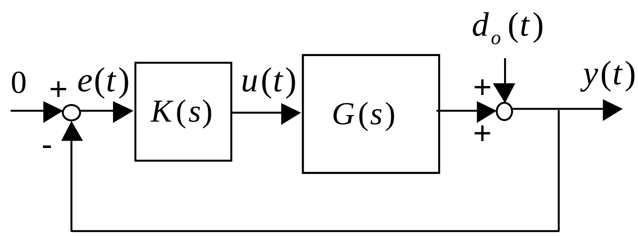
The tracking objective of $y(t) \approx y_d(t)$ translates into the following requirement on the closed-loop transfer function:

$$T(s) \approx 1\,. \tag{1.111}$$

We see that this objective will be obtained for a "large" *loop gain*, i.e. for $|K(s)G(s)| \gg 1$, which, for a given plant, suggests that the magnitude of the controller be made large. However, we will see later that a high-gain controller often leads to instability of the closed-loop transfer function.

### 3.2.2 Regulators

A *regulator* is a control system whose main objective is to reject the effect of disturbances and maintain the output of the plant to a desired constant value (often taken to be 0 without loss of generality). An example is a liquid tank level regulator. The block diagram shown below is for a regulator that must reject the effect of an output disturbance.




The transfer function from the output disturbance to the output, called the *sensitivity*, is obtained from the following loop equation:

$$\hat{y}(s) = -K(s)G(s)\hat{y}(s) + \hat{d}(s) \tag{1.112}$$

which yields (SISO case)

$$S(s) := \frac{\hat{y}(s)}{\hat{d}_o(s)} = \frac{1}{1+K(s)G(s)}. \tag{1.113}$$

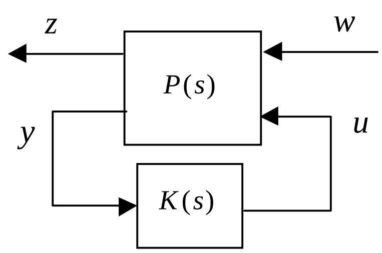
The objective that $y(t) \approx 0$ for expected output disturbance signals translate into the requirement that

$$S(s) \approx 0. \tag{1.114}$$

Again, a high loop gain $|K(s)G(s)| \gg 1$ would appear to be the solution to minimize the sensitivity, but the closed-loop stability constraint often makes this difficult.

## 3.3 Linear fractional transformations (LFT)

A very general way of representing feedback control systems is to use a linear fractional transformation (LFT). Consider the interconnection of transfer matrices $P(s)$ and $K(s)$ shown below where $P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$ and the letters designating the signals are standard.




The lower LFT $\mathcal{F}_L[P(s),K(s)]$ is the closed-loop transfer matrix from $\hat{w}(s)$ to $\hat{z}(s)$:

$$\mathcal{F}_L[P(s),K(s)] := P_{11}(s) + P_{12}(s)K(s)\left[I - P_{22}(s)K(s)\right]^{-1} P_{21}(s) \qquad (1.115)$$

Similarly defined is the upper LFT $\mathcal{F}_U[P(s),K(s)]$ with $K(s)$ connected as a feedback around the upper input and output signals this time:

$$\mathcal{F}_U[P(s),K(s)] := P_{22}(s) + P_{21}(s)K(s)\left[I - P_{11}(s)K(s)\right]^{-1} P_{12}(s) \qquad (1.116)$$

Even though it may not be necessary to use LFT's for simple SISO systems, they are useful for MIMO systems as they provide a unified framework to compute closed-loop transfer matrices. This is an advantage for so-called CACSD (computer-aided control system design.) For example, Matlab's $\mu$-Analysis and Synthesis Toolbox uses such an approach.

Example: The general feedback control system of Figure 9 can be recast into an LFT framework. Let the vectors of input signals and output signals be

$$w(t) := \begin{bmatrix} y_d(t) \\ d_i(t) \\ d_o(t) \\ n(t) \end{bmatrix}, \quad z(t) := y(t) \ . \qquad (1.117)$$

Then the *generalized plant* $P(s)$ is a block-partitioned 2 by 5 transfer matrix given by:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

where

$$P_{11} := w \mapsto z = \begin{bmatrix} 0 & G & 1 & 0 \end{bmatrix},$$
$$P_{12} := u_d \mapsto z = PG_a,$$
$$P_{21} := w \mapsto e_m = \begin{bmatrix} 1 & -G_m G & -G_m & -G_m \end{bmatrix},$$
$$P_{22} := u_d \mapsto e_m = -G_m G G_a$$

The closed-loop transfer matrix mapping the exogenous signals (reference, disturbances and noise) to the output is given by the lower LFT:

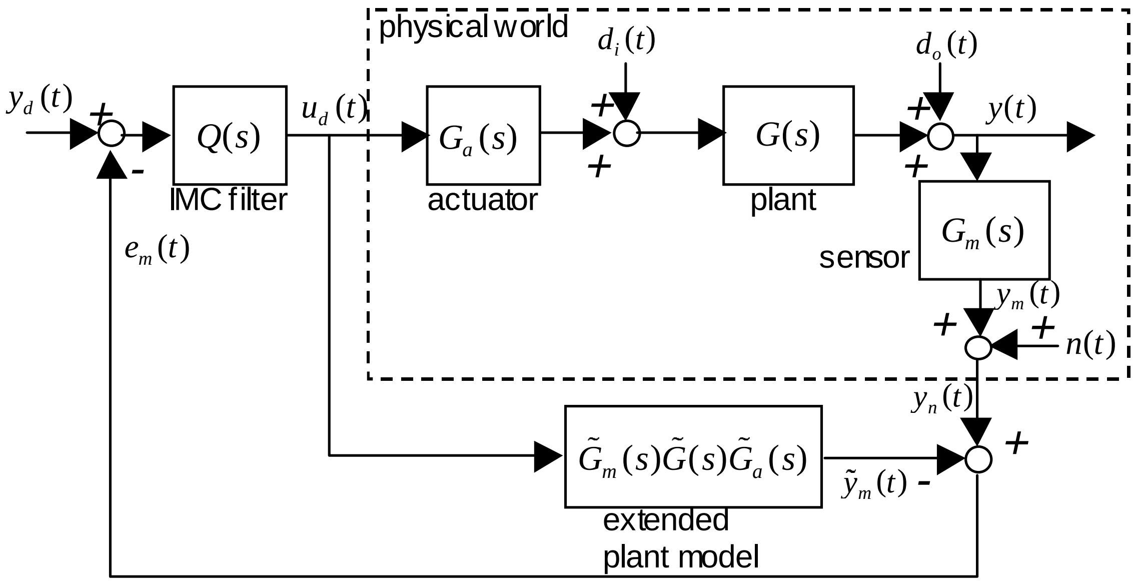
$$\begin{aligned}
\mathcal{F}_L[P(s), K(s)] &= P_{11}(s) + P_{12}(s)K(s)\left[I - P_{22}(s)K(s)\right]^{-1} P_{21}(s) \\
&= \begin{bmatrix} 0 & G & 1 & 0 \end{bmatrix} + GG_a K\left(1 + G_m GG_a K\right)^{-1}\begin{bmatrix} 1 & -G_m G & -G_m & -G_m \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{GG_a K}{1+G_m GG_a K} & G - \dfrac{GG_a K G_m G}{1+G_m GG_a K} & 1 - \dfrac{GG_a K G_m}{1+G_m GG_a K} & \dfrac{-GG_a K G_m}{1+G_m GG_a K} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{GG_a K}{1+G_m GG_a K} & \dfrac{G}{1+G_m GG_a K} & \dfrac{1}{1+G_m GG_a K} & \dfrac{-G_m GG_a K}{1+G_m GG_a K} \end{bmatrix}
\end{aligned}$$

Thus, the Laplace transform of the output signal $y(t)$ is given by:

$$\begin{aligned}
\hat{y}(s) &= \mathcal{F}_L[P(s), K(s)]\hat{w}(s) \\
&= \begin{bmatrix} \dfrac{GG_a K}{1+G_m GG_a K} & \dfrac{G}{1+G_m GG_a K} & \dfrac{1}{1+G_m GG_a K} & \dfrac{-G_m GG_a K}{1+G_m GG_a K} \end{bmatrix} \begin{bmatrix} \hat{y}_d \\ \hat{d}_i \\ \hat{d}_o \\ \hat{n} \end{bmatrix} \\
&= \frac{GG_a K}{1+G_m GG_a K}\hat{y}_d + \frac{G}{1+G_m GG_a K}\hat{d}_i + \frac{1}{1+G_m GG_a K}\hat{d}_o - \frac{-G_m GG_a K}{1+G_m GG_a K}\hat{n}
\end{aligned}$$

## 3.4 Internal model control (IMC) configuration

We have spent some time developing LTI models of different types of plants because all of the controller design techniques presented in this course are *model-based*. A different and interesting way to look at a feedback control system is to separate out explicitly the plant model $G(s)$ in the controller. This is the so-called internal model control (IMC) configuration of Morari and Zafiriou (1989) shown below. The key assumption here is that the plant, the actuators and the sensors are stable, so that all the poles of $G(s), G_a(s), G_m(s)$ are in the open left half-plane. Note that the controller is the feedback interconnection of the extended plant model and the stable IMC controller $Q(s)$.




• Figure 10: General IMC configuration

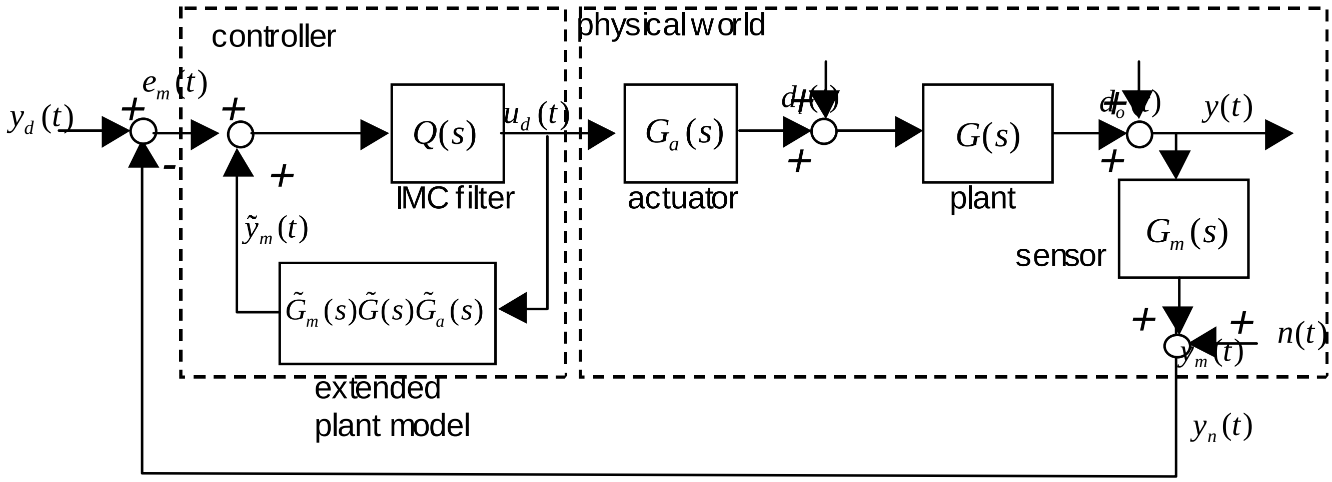
The basic idea behind IMC is the following. Suppose that the transfer matrix models for our plant, actuators and sensors are perfect, i.e., $\tilde{G}_m(s) = G_m(s), \tilde{G}(s) = G(s), \tilde{G}_a(s) = G_a(s)$. Also assume that there are no disturbance or noise in the process for now. Then, since the control signal $u_c(t)$ is applied to both the extended plant and its model, we have

$$\hat{e}_m(s) = \hat{y}_n(s) - \hat{\tilde{y}}_n(s) = \left[G_m(s)G(s)G_a(s) - \tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]\hat{u}_c(s) = 0 \qquad (1.118)$$

and therefore there is no internal feedback in the controller that would result from a deviation from the actual measured output of the plant $\tilde{y}_n(t)$ from the desired measured output $y_n(t)$. Hence both the extended plant and its model operate in open-loop. Now a basic requirement for the IMC filter $Q(s)$ is that it must be stable. Thus, the response of the plant will be the open-loop response of the stable cascade interconnection of $Q(s)$ and $G_m(s)G(s)G_a(s)$:

$$\hat{y}_n(s) = G_m(s)G(s)G_a(s)Q(s)\hat{y}_d(s)\,. \qquad (1.119)$$

From this equation, it is now clear what the ideal IMC controller should be, that is $Q(s) = \left[G_m(s)G(s)G_a(s)\right]^{-1}$ so that we get perfect tracking $\hat{y}_n(s) = \hat{y}_d(s)$. This is the concept of *perfect control* in the IMC literature. However, it is not possible to use such a controller (we don't get anything for free!) because it leads to infinite control signals as we now show. The classical controller $K(s)$ corresponding to this $Q(s)$ can be found by computing the transfer matrix $e_m \mapsto u_d$ in the equivalent block diagram shown below.




• Figure 11: Block diagram equivalent to IMC configuration

We can see that $K(s)$ is given by the feedback interconnection of $Q(s)$ and the extended plant model $\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)$:

$$\begin{aligned}\hat{u}_d(s) &= Q(s)\hat{e}_m(s) + Q(s)\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\hat{u}_d(s)\\ \hat{u}_d(s) &= \left[I - Q(s)\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]^{-1} Q(s)\hat{e}_m(s)\end{aligned} \quad . \qquad (1.120)$$

Thus,

$$K(s) = \left[I - Q(s)\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]^{-1} Q(s)$$

and for the "perfect" IMC filter, we obtain an infinite controller gain.

$$\begin{aligned}K(s) &= \left[I - \left[\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]^{-1}\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]^{-1}\left[\tilde{G}_m(s)\tilde{G}(s)\tilde{G}_a(s)\right]^{-1}\\ &= \infty\end{aligned} \qquad (1.121)$$

Even though perfect control is impossible, the concept is important as it provides an ideal scenario to approach in an actual controller design. So the controller design problem using IMC consists of finding a good stable IMC filter such that it approaches perfect control while satisfying all other constraints such as actuator and sensor saturation limits, etc.

Benefits of using an IMC configuration not only for design but also in the actual implementation include:

- It is intuitive,
- There is an internal representation in the controller of what the output signal of the plant should be. This information could be presented to the process operator as a target signal,

- Actuator nonlinearities such as saturation can be included in the internal model such that the control signal generated by the controller takes it into account,
- The IMC filter can be tuned on-line to improve closed-loop performance or robustness.
- Plant delays can be included in the internal model. The controller is then called a *Smith Predictor*.

IMC for unstable plants is the subject of on-going research.

# 4 Nominal stability and performance of LTI feedback control systems

Before we begin our study of controller design techniques, we must look at what the objectives of feedback control are, and quantify these objectives as specifications.

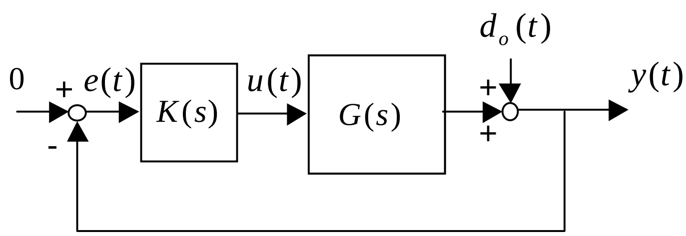
## 4.1 Sensitivity and complementary sensitivity functions

### 4.1.1 Sensitivity Function

We introduced the sensitivity function as the transfer matrix (or transfer function in the SISO case) from the output disturbance to the output of the plant,

$$S(s) := d_0 \mapsto y = [I + K(s)G(s)]^{-1}, \qquad (1.122)$$

for the standard block diagram:




The name sensitivity can be attributed in part to the fact that the transfer matrix $S(s)$ represent the level of sensitivity of the output to an output disturbance.

$$\hat{y}(s) = S(s)\hat{d}_0(s) \qquad (1.123)$$

If the disturbance signal has a Fourier transform, then (assuming $S(s)$ is stable and hence has a frequency response $S(j\omega)$), the Fourier transform of the output is given by

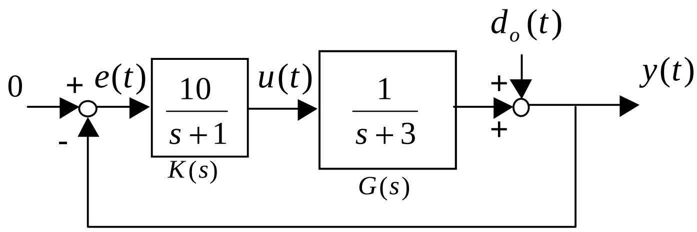
$$\hat{y}(j\omega) = S(j\omega)\hat{d}_0(j\omega) . \qquad (1.124)$$

Hence, the frequency response of the sensitivity $S(j\omega)$ amplifies or attenuates the output disturbance at different frequencies. Note that since the error is simply the negative of the output, we also have

$$\hat{e}(s) = -S(s)\hat{d}_0(s)\,. \tag{1.125}$$

*Example*

Suppose $d_0(t) = e^{-t}q(t)$ for the following SISO control system:




The Fourier transform of $d_0(t)$ is given by

$$\hat{d}_0(j\omega) = \frac{1}{1+j\omega} \tag{1.126}$$

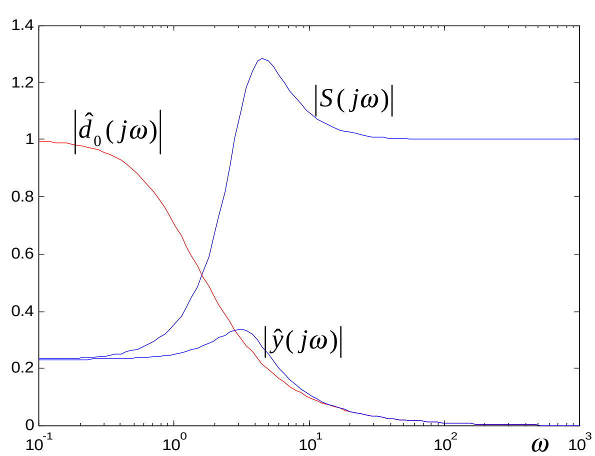
The sensitivity function is calculated as follows:

$$S(s) = \frac{1}{1+\frac{10}{(s+1)(s+3)}} = \frac{(s+1)(s+3)}{s^2+4s+13} = \frac{(s+1)(s+3)}{(s+2-j3)(s+2+j3)}, \tag{1.127}$$

This sensitivity function is stable because its complex conjugate poles lie in the open left half-plane. Its frequency response is

$$S(j\omega) = \frac{(j\omega+1)(j\omega+3)}{13-\omega^2+j4\omega}. \tag{1.128}$$

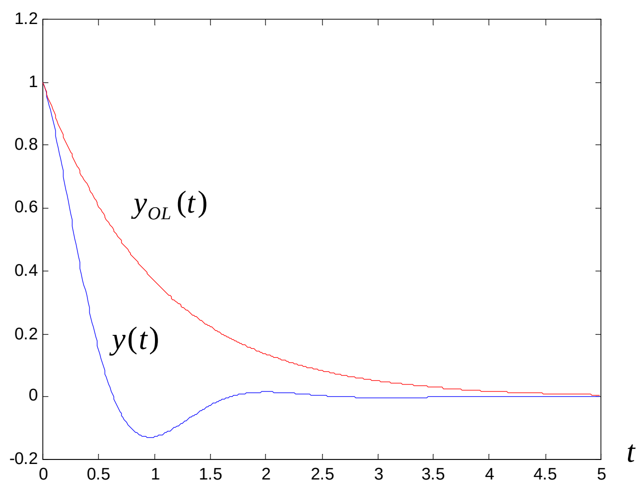
The magnitudes of $S(j\omega), \hat{d}_0(j\omega), \hat{y}(j\omega)$ are plotted below.



It can be seen that the controller reduces the effect of the output disturbance roughly by a factor of 5 in the frequency domain, as opposed to the open-loop case. That is, the sensitivity was made small in the bandwidth of the disturbance. In the time domain, the closed-loop plant response

$$y(t) = \mathcal{L}^{-1}\left\{S(s)\hat{d}_0(s)\right\} = e^{-2t}\left(\cos 3t + \frac{1}{3}\sin 3t\right)u(t) \tag{1.129}$$

to the disturbance is smaller in magnitude than the open-loop response $y_{OL}(t) = d_0(t) = e^{-t}q(t)$ as shown below.




However, if the disturbance signal had had energy around $\omega = 4\,\mathrm{rd/s}$, it would have been *amplified* around this frequency.

The main reason why $S(s)$ is called the sensitivity function is because it is equal to the sensitivity of the closed-loop transmission $T(s)$ to an infinitesimally small perturbation of the loop gain defined as $L(s) := G(s)K(s)$. In the SISO case, for an infinitesimally small relative change $\frac{dL(s)}{L(s)}$ in the loop gain, the corresponding relative change $\frac{dT(s)}{T(s)}$ in the transmission is given by

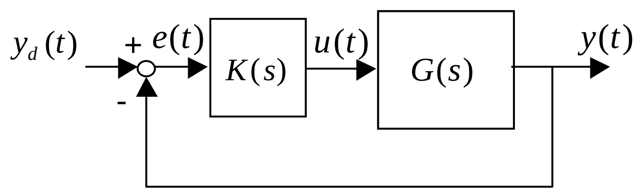
$$\begin{aligned}\frac{\frac{dT(s)}{T(s)}}{\frac{dL(s)}{L(s)}} &= \frac{L(s)}{T(s)}\frac{dT(s)}{dL(s)} = \left[1+L(s)\right]\frac{d}{dL(s)}\frac{L(s)}{1+L(s)} \\ &= \left[1+L(s)\right]\frac{(1+L(s))-L(s)}{\left[1+L(s)\right]^2} = \frac{1}{1+L(s)} = S(s)\end{aligned} \tag{1.130}$$

### 4.1.2 Complementary sensitivity function

The *complementary sensitivity function* is the closed-loop transfer matrix $T(s)$ that we called *transmission* earlier on. It is called the complementary sensitivity function because it "complements" the sensitivity function in the sense that

$$S(s)+T(s)=I\,. \tag{1.131}$$

We have seen that $T(s)$ is the closed-loop transfer matrix from the reference signal (desired output) to the plant output in tracking control problems for the standard unity feedback control system shown below.

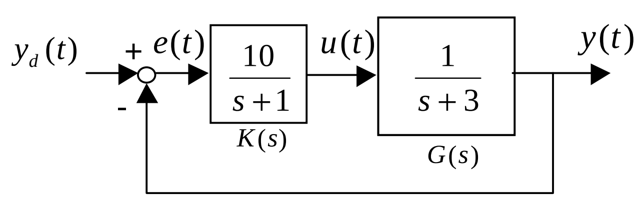



$$T(s) := y_d \mapsto y = [I+G(s)K(s)]^{-1} G(s)K(s)\,, \tag{1.132}$$

Usually, reference signals have the bulk of their energy or power at low frequencies (e.g., piecewise continuous signals) but not always (e.g., fast robot joint trajectories). The main objective for tracking is to make $T(j\omega)\approx I$ over the frequency band where the reference has most of its energy.

*Example*

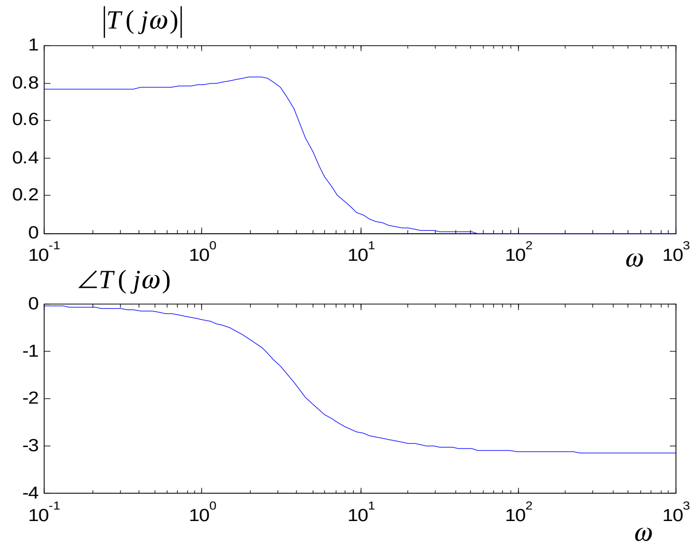
Consider the previous example set up as a tracking system.




The complementary sensitivity function can be calculated using (1.131):

$$T(s)=1-S(s)=1-\frac{s^2+4s+3}{s^2+4s+13}=\frac{10}{s^2+4s+13}\,. \tag{1.133}$$

Although the DC gain of $T(0)=10/13=0.77$ is not that close to 1, the magnitude of its frequency response is reasonably flat and the phase reasonably close to 0 up to $\omega = 1\,\mathrm{rd/s}$, as shown below.




Suppose that the reference signal is a causal rectangular pulse of 10-second duration:

$$y_d(t) = u(t) - u(t-10). \tag{1.134}$$

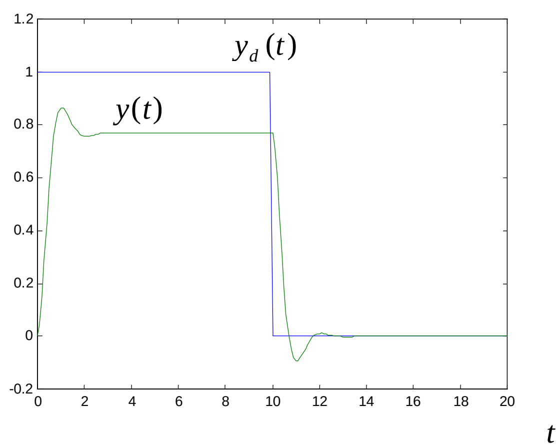
Its Laplace transform is

$$\hat{y}_d(s) = \frac{1 - e^{-10s}}{s}. \tag{1.135}$$

so that the Laplace transform of the plant output is

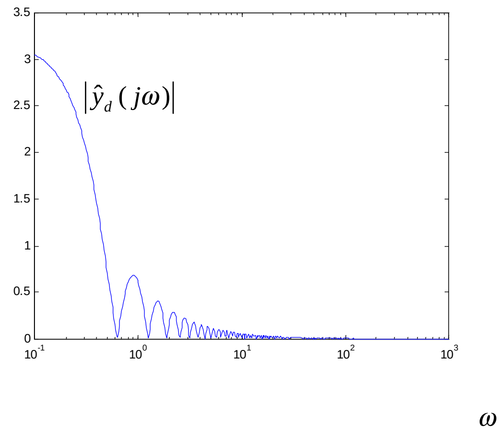
$$\hat{y}(s) = T(s)\hat{y}_d(s) = \frac{10(1 - e^{-10s})}{s(s^2 + 4s + 13)}. \tag{1.136}$$

The corresponding time-domain output signal is plotted in the figure below, together with the reference signal.



Apart from a settling value lower than 1 due to the DC gain, the response is not too bad. This must mean that in the frequency domain, the Fourier transform of the reference should have most of its energy in the "passband" of the transmission. It is indeed the case, as seen in the plot of $\left|\hat{y}_d(j\omega)\right|$.

$$\hat{y}_d(j\omega) = \frac{10e^{-j5\omega}}{\pi}\operatorname{sinc}\left(\frac{5\omega}{\pi}\right). \tag{1.137}$$



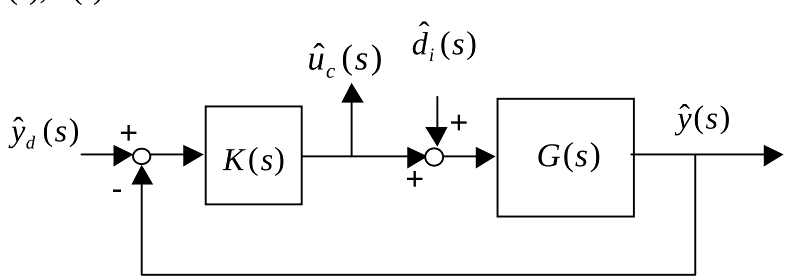

## 4.2 Nominal internal stability

The most fundamental property of a feedback control system is its *stability*. Obviously, an unstable feedback system is useless (unless the goal was to build an oscillator using positive feedback in the first place!) In this section, we give four equivalent theorems to check the stability of a unity feedback control system.

We shall use the following definition of *BIBO stability*, (or *stability* in short) for a unity feedback system. This is a requirement that any bounded input signal injected at any point in the control system results in bounded output signals measured at any point in the system.

*Definition of stability*

The unity feedback system below is said to be *stable* if, for all bounded inputs $y_d(t), d_i(t)$, the corresponding outputs $y(t), u(t)$ are also bounded.



The idea behind this definition is that it is not enough to look at a single input-output pair for BIBO stability. Note that some of these inputs and outputs may not appear in the original problem formulation, but they can be defined as "fictitious" inputs or outputs. If actuators and sensors are considered separately from the plant, more inputs and outputs must be added and the definition is easily extended.

Recall that an *open-loop* transfer function $G(s)$ is BIBO stable iff

(a) All of its poles are in the open left half-plane and

(b) It is *proper* (i.e., $\lim_{s\to\infty} G(s) < \infty$ or the order of the numerator is $\leq$ order of the denominator for G(s) rational).

The second condition is required because otherwise a bounded input $|u(t)| < M$ with arbitrarily fast variations would produce an unbounded output for an improper transfer function. Consider for example a pure differentiator $G(s) = s$ with the input $u(t) = \sin(t^2)$. Its output is $y(t) = 2t\cos(t^2)$ which is unbounded as $t \to +\infty$.

*Closed-Loop Stability Theorem I*

The closed-loop system of the above figure is stable if and only if the transfer functions $T(s)$, $G^{-1}(s)T(s)$ and $G(s)S(s)$ are stable (unstable pole-zero cancellations are allowed in these products).

*Proof*

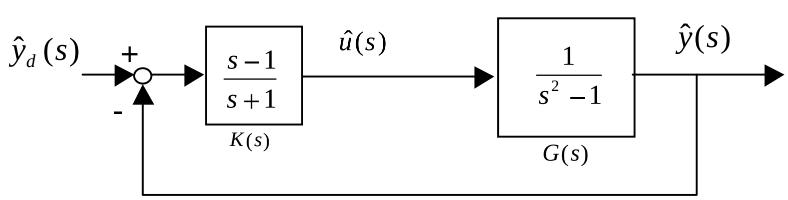
It is easy to find the relationship between the inputs and the outputs (do it as an exercise):

$$\begin{bmatrix} \hat{y}(s) \\ \hat{u}(s) \end{bmatrix} = \begin{bmatrix} T(s) & G(s)S(s) \\ G^{-1}(s)T(s) & -T(s) \end{bmatrix} \begin{bmatrix} \hat{y}_d(s) \\ \hat{d}_i(s) \end{bmatrix}. \qquad (1.138)$$

This matrix of transfer function (called a transfer matrix) is stable iff each individual transfer function entry of the matrix is stable. The theorem follows.

*Example*

Let's assess the stability of the SISO tracking control system shown below.




Note that the open-loop plant is unstable. We calculate the three transfer functions

$$\begin{aligned} T(s) &= \frac{K(s)G(s)}{1+K(s)G(s)} = \frac{1}{s^2+2s+2} \; (stable) \\ G^{-1}(s)T(s) &= \frac{K(s)}{1+K(s)G(s)} = \frac{s^2-1}{s^2+2s+2} \; (stable) \\ G(s)S(s) &= \frac{G(s)}{1+K(s)G(s)} = \frac{1}{(s^2-1)(s^2+2s+2)} \; (unstable) \end{aligned} \tag{1.139}$$

and conclude that this closed-loop system is unstable. The problem here is that the controller attempts to cancel out the plant's unstable pole which would lead to instability in an actual implementation.

Note that the poles of $S(s)$ and $T(s)$ are the same, and if any of these two transfer matrices is proper, then the other one is also proper. This is easy to see from the identity:

$$T(s) = I - S(s). \tag{1.140}$$

If $S(p_i) = \infty$, then $T(p_i) = -\infty$ and vice-versa.

Another equivalent closed-loop stability theorem can now be stated (without proof) for a unity feedback control system if we explicitly rule out pole-zero cancellation occurring in the closed right half-plane when forming the loop gain $G(s)K(s)$. One only needs to check the stability of either $S(s)$ or $T(s)$.

*Closed-Loop Stability Theorem II*

The unity feedback control system is stable if and only if

(a) either $S(s)$ or $T(s)$ is stable, and

(b) no pole-zero cancellation occurs in the closed right half-plane when forming the loop gain $G(s)K(s)$.

Now suppose that the SISO plant and controller transfer functions are written as

$$G(s) = \frac{n_G(s)}{d_G(s)}, \; K(s) = \frac{n_K(s)}{d_K(s)}, \tag{1.141}$$

where the plant numerator and denominator $n_G(s), d_G(s)$ are coprime polynomials, i.e., have no common factors, and likewise for the controller numerator and denominator. Define the *characteristic polynomial* $p(s) := n_G(s)n_K(s) + d_G(s)d_K(s)$ of the closed-loop system. The *closed-loop poles* of the system are defined to be the zeros of the characteristic polynomial $p(s)$. We have our third stability result for SISO systems (stated without proof, can be extended to MIMO systems):

*Closed-Loop Stability Theorem III*

The unity feedback control system is stable if and only if the closed-loop poles of the system are in the open left half-plane.

*Example*

For our previous example, we have

$$n_G(s) = 1,\ d_G(s) = (s+1)(s-1),\ n_K(s) = (s-1),\ d_K(s) = (s+1) \tag{1.142}$$

and

$$p(s) = (s-1) + (s+1)^2(s-1) \tag{1.143}$$

which clearly has a zero at $s = 1$. Therefore the control system is unstable.

Finally, we have a fourth equivalent theorem on closed-loop stability, which is really just a restatement of CLS Theorem II (taking $S(s)$ ).

*Closed-Loop Stability Theorem IV*

The unity feedback control system is stable if and only if

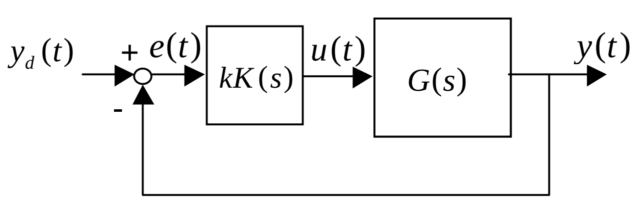
(a) The transfer function $\det[I + G(s)K(s)]$ has no closed RHP zeros,

(b) no pole-zero cancellation occurs in the closed right half-plane when forming the loop gain $G(s)K(s)$.

### 4.2.1 The Root Locus (this section is based on course notes by Prof. B. Francis at the University of Toronto)

The root locus is a classical tool for stability analysis of SISO feedback control systems. It is the locus in the *s*-plane described by the closed-loop poles (i.e., the zeros of the characteristic polynomial $p(s)$) as a real parameter $k$ varies from 0 to $+\infty$ in the loop gain. It is a method for studying the effect of a parameter on the locations of the closed-loop poles, in particular to find out for what value of the parameter they become unstable.

The parameter is usually the controller gain, but it could be a parameter of the plant transfer function. The usual setup is this:




Factor $G(s),\ K(s)$ as ratios of coprime polynomial: