Belief State as an Information State
ECSE-506

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Consider a particular ω and $y_{1:T}, u_{1:T}$:

$$\pi_{t+1}(x_{t+1}) = \mathbb{P}(X_{t+1} = x_{t+1} \mid Y_{1:t+1} = y_{1:t+1}, U_{1:t} = u_{1:t}) \tag{1}$$

$$= \frac{\mathbb{P}(X_{t+1} = x_{t+1}, Y_{t+1} = y_{t+1} \mid Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t})}{\sum_{\widetilde{x}_{t+1}} \mathbb{P}(X_{t+1} = \widetilde{x}_{t+1} \mid Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t})} \tag{2}$$

Rewriting the numerator with the law of total probability:

$$\sum_{\widetilde{x}_t} \mathbb{P}(Y_{t+1} = y_{t+1} \mid X_{t+1} = x_{t+1}, x_t = \widetilde{x}_t, Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) \tag{3}$$

$$\times \mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = \widetilde{x}_t, Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) \tag{4}$$

$$\times \mathbb{P}(X_t = \widetilde{x}_t \mid Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) \tag{5}$$

By independence upon the basic random variables, (3) and (4) simplify to:

$$\mathbb{P}(Y_{t+1} = y_{t+1} \mid X_{t+1} = x_{t+1}, x_t = \widetilde{x}_t, Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) = \mathbb{P}(Y_{t+1} = y_{t+1} \mid X_{t+1} = x_{t+1})$$

$$\mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = \widetilde{x}_t, Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) = \mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = \widetilde{x}_t, U_t = u_t)$$

As for (5), we apply Baye's rule again:

$$\mathbb{P}(X_t = \widetilde{x}_t \mid Y_{1:t} = y_{1:t}, U_{1:t} = u_{1:t}) = \frac{\mathbb{P}(X_t = \widetilde{x}_t, U_t = u_t \mid Y_{1:t} = y_{1:t}, U_{1:t-1} = u_{1:t-1})}{\sum_{\widehat{x}_t} \mathbb{P}(X_t = \widehat{x}_t, U_t = u_t \mid Y_{1:t} = y_{1:t}, U_{1:t-1} = u_{t-1}} \tag{6}$$

and write the numerator as

$$\mathbb{P}(U_t = u_t \mid X_t = \widetilde{x}_t, Y_{1:t} = y_{1:t}, U_{1:t-1} = u_{1:t-1}) \times \mathbb{P}(X_t = \widetilde{x}_t \mid W_{1:t} = y_{1:t}, U_{1:t-1} = u_{1:t-1}) \tag{7}$$

$$= \frac{\mathbf{1}_{u_t}(g_t(y_{1:t}, u_{1:t-1})\pi_t(\widetilde{x}_t))}{\mathbf{1}_{u_t}(g_t(y_{1:t}, u_{1:t-1})) \sum_{\widehat{x}_t} \pi_t(\widehat{x}_t)} \tag{8}$$

$$= \pi_t(\widetilde{x}_t) \tag{9}$$

Under these simplifications, the numerator of (2) becomes:

$$\sum_{\widetilde{x}_t} \mathbb{P}(Y_{t+1} = y_{t+1} \mid X_{t+1} = x_{t+1}) \times \mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = \widetilde{x}, U_t = u_t) \times \pi_t(\widetilde{x}_t) = A_t(y_{t+1}, x_{t+1}, u_t, \pi_t) \tag{10}$$

The belief state at $t+1$ is then of the form:

$$\pi_{t+1}(x_{t+1}) = \frac{A_t(y_{t+1}, x_{t+1}, U_t, \pi_t)}{\sum_{\widetilde{x}_{t+1}} A_t(y_{t+1}, \widetilde{x}_{t+1}, U_t, \pi_t)} = \varphi_t(\pi_t, y_{t+1}, u_t)(x_{t+1}) \tag{11}$$