A Spatio-temporal Generalization of Canny's Edge Detector

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Abstract

Optical flow is traditionally found by working on two input images. On the other hand, most edge detectors locate edges only on static images. As we have more and more computational power, it is natural to consider optical flow extraction and edge detection in the spatio-temporal domain to utilize as much temporal information as possible. In this paper, image edges are assumed to be stochastic in time and Canny's edge detector is generalized to the spatial-temporal domain. Some properties of this generalized edge detector are presented.

1 Introduction

People have proposed various algorithms in edge detection. Some of them employ high spatial frequency enhancement and thresholding; some of them fit the image with small planar surfaces or facets; some of them attempt to enhance edges by linear filtering; some of them use statistical classification approaches; regularization theory and mathematical morphology were also used. All of the above algorithms operate on a single static image, or a snapshot, and try to extract 'edges' according to their definitions of edges. Another important low-level vision task is motion detection. There are basically two motion detection schemes in computer vision, namely, the intensity-based gradient scheme and the token-matching scheme. These algorithms mainly operate on two images sampled at different but close time epochs. The intensity-based gradient scheme recovers a dense optical flow and avoids the correspondence problem. On the other hand, the token-matching scheme provides more accurate but sparser optical velocity data over the input images. A number of biological motion models have been developed, e.g., Reichardt's correlation models[12], the directionally tuned linear filter models[1], and Adelson's energy models[1]. They are basically built on the spatio-temporal domain, not just on two image snapshots only. They are local (use local image information only), and therefore suitable for explaining short-range motion detection. Some experimental evidence also supports uniformity (similar operations in different image locations) in motion detection.

The oldest edge detector is virtually a simplest local spatial filter of a minimum size. In retrospect, the more computational power we had or the more accurate estimation of the edge location we wanted to achieve, the more complex was the edge detector proposed. As we possess more computational power, it is natural to consider edge detection and velocity flow extraction in the spatio-temporal domain. This trend actually leads us to develop computational algorithms compatible with the above spatio-temporal biology models. A subclass of simple cells (S type) are known to respond preferentially to edges of light when the edge has the appropriate location, orientation, polarity or the direction of motion[11]. This also motivates our investigation in combining edge detection and velocity estimation which could be useful in understanding the cooperative behavior of the simple cells. To make the most use of SIMD parallel machines and special purpose image processors, the proposed low-level vision algorithms should be local and uniform, which is compatible with the biology studies as well. Recently, some local, uniform, and spatio-temporal algorithms have been proposed to obtain the image flow. For instance, Heeger used the Gabor filtering to extract the velocity flow of a local image pattern possessing a flat power spectrum[6]. In a different approach, employing the d'Alembertian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{u^2} \cdot \frac{\partial^2}{\partial t^2}$, Buxton and Buxton were able to detect the velocity of the zero crossings in a sequence of images[3]. In our research, we consider an image edge as a locally straight line with an abrupt intensity change and intend to extract the location and the velocity of the edge. As a first step, we generalize Canny's edge detector to the spatio-temporal domain and discuss its properties in this paper. As a continuation, in a companion paper, [7], we present a spatio-temporal data-fusion framework which is able to provide continuous edge location and local (with aperture problems) velocity outputs perpendicular to the edge orientation.

2 Generalizing Canny's Detector

Canny[4] assumed that edge detection is performed by convolving a step edge (with Gaussian noise) with a spatially antisymmetric function $f_{Canny}(x)$ and marking edges at the maxima in the output of the convolution. He formulated three performance criteria, namely, (1) good detection, (2) good localization, and (3) one response to a single edge. The general optimal solution on the interval $[0, W]$ turns out to be $f_{Canny}(x) = a_1 \cdot e^{\alpha x} \sin(\omega x) + a_2 \cdot e^{\alpha x} \cos(\omega x) + a_3 \cdot e^{-\alpha x} \sin(\omega x) + a_4 \cdot e^{-\alpha x} \cos(\omega x) + C$. Deriche[5] pushed the boundary conditions to the infinite extent as $f_{Canny}(0) = 0, f_{Canny}(+\infty) = 0, f'_{Canny}(0) = S, f'_{Canny}(+\infty) = 0$, where S is a negative real number and obtained the solution $f_D(x) = -c \cdot e^{-\alpha|x|} \sin(\omega x)$, with α, ω, and c positive reals. Actually, the Lagrange multipliers in Canny's original derivations were not solved analytically, otherwise we could have

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determined the optimal relationship between α and ω. In the case that $\alpha \gg \omega$, the optimal filter can be approximated by

$$f_{D1}(x) = -c \cdot xe^{-\alpha|x|}$$

f_{D1} performs better than the Gaussian function that Canny used as an approximation[5]. Also, f_{D1} is very simple and depends on only one parameter α. Decreasing α will lower the edge localization, but yield better signal-to-noise ratio and vice versa. Boie et al.[2] formalized the edge detection problem from a matched filter perspective and showed that, despite common belief, good detection and good localization need not to be in opposition to each other. The reason is that the optimal criteria they chose are different from Canny's. However, the implementation results from both perspectives are comparable[2].

Consider an edge in a sequence of sampled images. If the edge is a physical edge belonging to some part of an object in the world space, a contour edge or an internal edge, it only disappears when that part of the object is occluded by other objects or when it gets out of the scene that is imaged. If the edge is a boundary of illumination differences, it would persist like physical edges unless there is an abrupt change in the lighting condition. Moreover, even if the edge exists in the images, its velocity and shape might change as time goes by. If the accelerations of the world-space objects and the light sources are finite and there are no light flashes, the velocities of image edges would change continuously in most cases. The edge detection problem that used to be considered was on a snap-shot static image, so it is plausible to use deterministic functions to model them. For the edge detection in the spatio-temporal domain, the use of stochastic functions to model the uncertain temporal behavior of edges appears necessary. We will consider the optimal detection for one dimensional edges only.

Definition 1 *In the two-dimensional spatio-temporal domain, let one-dimensional edges be modelled by $A(t) \cdot I_0(x - v_0 t)$ around $t = 0$, where v_0 is a fixed edge velocity and $I_0(x)$ is the deterministic spatial description of the moving edge. $A(t)$ is a stochastic function with a value 1 when $t \in [-t_m, t_p]$, $t_m > 0$ and $t_n > 0$, and 0 otherwise.*

Because we do not know when the edge might disappear and when its velocity might change, t_m and t_p are considered as random variables with some probability distribution. The random interval $[-t_m, t_p]$ would be the period of time when the velocity v_0 and the shape $I_0(x)$ of the one-dimensional edge assumed fixed. In $[-t_m, t_p]$, moving edges not only have to exist (temporal persistence), but also maintain the same velocity. This is a stricter constraint than the speed coherence, path coherence, and consistent edge motion constraints described by Kahn[9].

Let the variance of the additive uncorrelated white Gaussian noise be n_0^2, then the input signal $I(x,t)$ can be represented by $A(t) \cdot I_0(x - v_0 t) + n(x,t)$. Assume that the moving *step* edge is centered at $x = 0$ when $t = 0$. There should be a local maximum in the response at $x_0 = v_0 t_0$ in the plane $t = t_0$, in the absence of noise. The expected mean-squared output of the filter $E[O_s^2(0,0)]$ to the edge signal would be $a_0^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+v_0 t} f(x,t)dx \int_{-\infty}^{+v_0 t'} f(x',t')dx' R_{AA}(-t,-t')dt'$ dt, where $R_{AA}(t_1,t_2)$ is the autocorrelation of a stochastic process $A(t)$, i.e., $R_{AA}(t_1,t_2) = E[A(t_1)A(t_2)]$. The expected mean-squared response of the filter $E[O_n^2(0,0)]$ to the noise would be $n_0^2[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^2(x,t)dxdt]$. Hence, the output signal-to-noise-ratio is $SNR = |\frac{O_s}{O_n}| = \frac{a_0}{n_0} \cdot \Sigma$, where $\Sigma(f, v_0)$ is

$$\frac{[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+v_0 t} f(x,t)dx \int_{-\infty}^{+v_0 t'} f(x',t')dx' R_{AA}(-t,-t')dt'dt]^{\frac{1}{2}}}{[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^2(x,t)dxdt]^{\frac{1}{2}}}$$

This is in contrast to [2] where $|\frac{O_s}{O_n}|^2$ is maximized.

We will mark edges at local maxima of the filter output on a particular plane, say t_0. Suppose there is a local maximum in the total response at the point $x = x_1$ at t_0, then $\frac{\partial O_s}{\partial x}(x_1,t_0) + \frac{\partial O_n}{\partial x}(x_1,t_0) = 0$. The Taylor's expansion of $\frac{\partial O_s}{\partial x}(x_1,t_0)$ about the track of the moving edge, $x = v_0 t$, gives $\frac{\partial O_s}{\partial x}(x_1,t_0) = \frac{\partial O_s}{\partial x}(v_0 t_0, t_0) + (x_1 - v_0 t_0)\frac{\partial^2 O_s}{\partial x^2}(v_0 t_0, t_0) + o((x_1 - v_0 t_0)^2)$. The response of the filter to the edge signal should have a local maximum at $x = v_0 t_0$ at any t_0, i.e., $\frac{\partial O_s}{\partial x}(v_0 t_0, t_0) = 0$. Taking expected values, we obtain $E[(\frac{\partial O_s}{\partial x_0})^2(x_1,t_0)] \approx E[(x_1 - v_0 t_0)^2] \cdot E[(\frac{\partial^2 O_s}{\partial x_0^2})^2(v_0 t_0, t_0)]$. Defining $f_x(x,t) = \frac{\partial f(x,t)}{\partial x}$ and simplifying, $E[(x_1 - v_0 t_0)^2] \approx$

$$\frac{n_0^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_x^2(x,t)dxdt}{a_0^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_x(x,t)|_{x=v_0 t} R_{AA}(t_0-t, t_0-t') f_{x'}(x',t')|_{x'=v_0 t'} dtdt'}$$

Let the spatio-temporal edge localization be the reciprocal of $\sqrt{E[(x_1 - v_0 t_0)^2]}$ at $t_0 = 0$, i.e., localization $= \frac{a_0}{n_0} \cdot \Lambda$, where

$$\Lambda(f, v_0) = \frac{[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_x(x,t)|_{x=v_0 t} R_{AA}(-t,-t') f_{x'}(x',t')|_{x'=v_0 t'} dtdt']^{\frac{1}{2}}}{[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_x^2(x,t)dxdt]^{\frac{1}{2}}}$$

and,

$$\frac{\partial O_s}{\partial x}(v_0 t_0, t_0) = \int_{-\infty}^{+\infty} A(t_0 - t) f(v_0 t, t)dt = 0, \quad \forall \quad A(t), \ t_0 \qquad (1)$$

Just as Canny did in one dimension, we seek to maximize $\Pi(f, v_0) = \Sigma(f, v_0) \cdot \Lambda(f, v_0)$ subject to (1). Therefore, we have shown[1]

Theorem 1 *Let the two-dimensional spatio-temporal edge signal be $a_0 A(t) \cdot u_{-1}(x - v_0 t)$ and the variance of the additive uncorrelated white Gaussian noise be n_0^2. The optimal filter $f(x,t)$ jointly maximizing the edge localization and the signal-to-noise-ratio maximizes $\Pi(f, v_0) = \Sigma(f, v_0) \cdot \Lambda(f, v_0)$ subject to (1).*

Corollary 1.1 *If $f^{w_x, w_t}(x,t) = f(\frac{x}{w_x}, \frac{t}{w_t})$, where w_x, w_t are positive constants, then for all stochastic processes $A(t)$, $\Pi(f^{w_x, w_t}, v_0) = \Pi(f^{1, w_t}, \frac{v_0}{w_x})$.*

Unlike Canny's edge detector, the optimal step edge detector in the spatio-temporal domain is not spatially scalable. Examining these scaling effects on Π suggests us to use a more suitable coordinate system. Let us define the new coordinate system (s,t) as follows:

[1]For all of the missing proofs and the details in this paper, please see [8].

$$s = x - v_0 t, \quad t = t$$

Assume that the optimal filter in this new coordinate system is called $h(s,t)$, i.e., $h(s,t) = f(s + v_0 t, t)$. For a moving step edge centered at $x = v_0 t$, $h(s,t)$ can be assumed to be antisymmetric with respect to s. Therefore, $h(0,t) = 0$ and the boundary condition (1) is automatically satisfied. The scalability of $h(s,t)$ along the spatio-temporal axis s suggests us to investigate the general product-form solutions of $h(s,t)$, that is, $h(s,t) = k(s)g(t)$. Therefore, we have $\Pi(f, v_0) = \frac{1}{2} \cdot \Pi'(h)$ and $\Pi'(h) = \Pi'_s(k) \cdot \Pi'_t(g)$.

$$\Pi'_s(k) = \frac{\mid \int_{-\infty}^{0} k(s)ds \mid}{[\int_{-\infty}^{0} k^2(s)ds]^{\frac{1}{2}}} \cdot \frac{\mid k_s(0) \mid}{[\int_{-\infty}^{0} k_s^2(s)ds]^{\frac{1}{2}}} \qquad (2)$$

$$\Pi'_t(g) = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(t) R_{AA}(-t,-t') g(t') dt dt'}{\int_{-\infty}^{+\infty} g^2(t) dt} \qquad (3)$$

To limit the number of peaks in the response so that there will be a low probability of declaring more than one edge, we like to make the distance between peaks in the noise response at particular time epoch approximate the width of the response of the operator to a single step. By Rice's theorem[10] and following Canny's arguments[4], the distance between adjacent maxima in the noisy response of $f(x,t)$ at the time epoch $t = 0$ would be

$$\frac{x_{max}}{2\pi} = \left(-\frac{R_{f_x f_x}(0,0)}{\frac{\partial^2 R_{f_x f_x}(0,0)}{\partial x^2}}\right)^{\frac{1}{2}} = \left(\frac{\int_{-\infty}^{+\infty} k_s^2(s)ds}{\int_{-\infty}^{+\infty} k_{ss}^2(s)ds}\right)^{\frac{1}{2}}$$

We set the distance to be some fraction of the operator spatial width W_x, i.e., $x_{max} \propto W_x$. Hence,

$$\left(\frac{\int_{-\infty}^{+\infty} k_s^2(s)ds}{\int_{-\infty}^{+\infty} k_{ss}^2(s)ds}\right)^{\frac{1}{2}} \propto W_x \qquad (4)$$

The equations (2) and (4) comprise the optimization problem Canny solved for the one dimensional case[4].

Theorem 2 *If $h(s,t) = f(s + v_0 t, t)$ is the optimal spatio-temporal filter in detecting a moving step edge with a velocity v_0 and $h(s,t)$ can be written as a product form, namely $k(s)g(t)$, then $k(s)$ is the one-dimensional version of Canny's edge detector $f_{Canny}(s)$ and $g(t)$ maximizes $\Pi'_t(g)$ in (3).*

To determine $g(t)$, we need to maximize $\Pi'_t(g)$ in which $R_{AA}(t,t')$, i.e., the stochastic behavior of the step edge, has to be assumed or experimentally determined in advance. Referring to (3), maximizing $\Pi'_t(g)$ is equivalent to maximizing $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(t) R_{AA}(-t,-t') g(t') dt dt'$ subject to $\int_{-\infty}^{+\infty} g^2(t) dt = c$, where c is a constant. That is, we seek some $g(t)$ from a space of admissible functions that maximizes $\Psi(g) = \int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{+\infty} g(t') R_{AA}(-t,-t') dt' dt - \lambda(\int_{-\infty}^{+\infty} g^2(t) dt - c)$, where λ is the Lagrange multiplier. By the variational approach, one necessary condition is that

$$\int_{-\infty}^{+\infty} g(t') R_{AA}(-t,-t') dt' = \lambda g(t) \qquad (5)$$

$\Pi'_t(g)$ in (3) is known as the Rayleigh quotient. Because $R_{AA}(t,t')$ is symmetric in the sense that $R_{AA}(t,t') = R_{AA}(t',t)$, the linear operator in (5) is Hermitian and hence all of the eigenvalues are real and the eigenfunctions are orthogonal to each other. Furthermore, when $g(t)$ is the eigenfunction, say $g_0(t)$, corresponding to the largest eigenvalue λ_0, in the eigenvalue problem (5), the Rayleigh quotient $\Pi'_t(g)$ is maximized at λ_0.

In the definition for moving edges, if $t_m = t_p = \epsilon_t$, where ϵ_t is a very small positive number, then the edge only exists around $t = 0$ or we are dealing with a single image only. For $-\epsilon_t \le t \le \epsilon_t$, $R_{AA}(-t,-t') = E[A(-t)A(-t')] = E[A(-t')]$ and then $\int_{-\epsilon_t}^{+\epsilon_t} g(t') dt' = \lambda g(t)$, i.e., $g(t)$ is a nonzero constant within the small interval $[-\epsilon_t, \epsilon_t]$ and the corresponding largest eigenvalue (Rayleigh quotient) is $2\epsilon_t$. For $t < -\epsilon_t$ or $t > \epsilon_t$, $R_{AA}(-t,-t') = E[A(-t)A(-t')] = 0$ and then $g(t) = 0$. Detecting the edges on a snapshot can be considered as the case that ϵ_t is a small fixed number and the edge velocity is zero. Hence, $h(s,0) \propto k(s) = k(x)$ which reduces to Canny's edge detector in one dimension.

Theorem 3 *Canny's edge detector f_{Canny} is an optimal spatio-temporal edge detector on the condition that (1) $A(t) = 1$ when $-\epsilon_t \le t \le \epsilon_t$; $A(t) = 0$ elsewhere, where ϵ_t is a very small number, and (2) the edge velocity is zero when $-\epsilon_t \le t \le \epsilon_t$.*

Generally speaking, there is a period of time between now and the future that edges would not disappear or change the velocities. Similarly, this 'waiting time' also exists between now and the past that edges appeared or had different velocities from now. Researchers in the field of the queuing theory and reliability engineering have been using the exponential function extensively to model waiting times between the job arrivals and the life span of the products. Thanks to the neatness and the simplicity of the exponential function, it become easy to work with some of the problems in these two fields. If we courageously adopt the exponential function as the probability density functions of the waiting times[2], then we have the following definition:

Definition 2 *Both t_m and t_p follow the exponential function $P_e(x) = \tau e^{-\tau x}, \tau > 0, x > 0$.*

$A(t)$ can only be either 0 or 1, so $R_{AA}(t,t') = E[A(t)A(t')] = 1 \cdot \text{Prob}[A(t) = A(t') = 1]$ where Prob[E] is the probability that the event E is true. Under the above definition, if t and t' are both positive, $R_{AA}(t,t') = \text{Prob}[A(t_{max}) = 1] = \int_{t_{max}}^{+\infty} \tau e^{-\tau x} dx = e^{-\tau t_{max}}$, where t_{max} is the larger number between t and t'. Similarly, if t and t' are both negative, $R_{AA}(t,t') = e^{\tau t_{min}}$ where t_{min} is the smaller negative number between t and t'. If $t < 0 < t'$, by assuming independence of two random variables t_m and t_p, $R_{AA}(t,t') = \text{Prob}[A(t) = 1] \cdot \text{Prob}[A(t') = 1] = [\int_{-\infty}^{t} \tau e^{\tau x} dx] \cdot [\int_{t'}^{+\infty} \tau e^{-\tau x} dx] = e^{\tau(t-t')}$. Similarly, if $t' < 0 < t$, $R_{AA}(t,t') = e^{\tau(t'-t)}$. If different probability distributions are assumed for t_m and t_p, we can still calculate $R_{AA}(t,t')$.

$\Pi'_t(g(t)) = \Pi'_t(g(-t))$ (since $R_{AA}(t,t') = R_{AA}(-t,-t')$) and the eigenfunction $g_0(t)$ corresponding to the largest eigenvalue λ_0 is unique for the Hermitian linear operator, so $g_0(t) = g_0(-t)$. We only have to solve $g(t)$ for the case

[2] We notice later that in an one-dimensional multi-edge model proposed in [13], the number of the edge points in a fixed interval is assumed to have a Poisson distribution, which is equivalent to assuming that the interval between the adjacent edge points has the exponential distribution. This is another example where the exponential function is used in the stochastic model.

that $t > 0$. Substituting known $R_{AA}(t, t')$ into (5) and simplifying, we obtain $\frac{d^2g(t)}{dt^2} + \tau\frac{dg(t)}{dt} + \frac{\tau}{\lambda}e^{-\tau t}g(t) = 0$, for $t > 0$. Let $x = e^{-\tau t}$ and $\lambda_i = \lambda^{-1}$, then

$$x^2\frac{d^2g(x)}{dx^2} + \frac{\lambda_i x}{\tau}g(x) = 0$$

where $x \in (0, 1)$, known as the Sturm-Liouville equation. It is actually the transformed Bessel's equation of order 1. The solution is $g(x) = x^{\frac{1}{2}}\{C_1J_1(2\sqrt{\frac{\lambda_i}{\tau}}x^{\frac{1}{2}}) + C_2Y_1(2\sqrt{\frac{\lambda_i}{\tau}}x^{\frac{1}{2}})\}$. The set of λ_i's that provide nontrivial solutions to the integral equation (5) can be shown to consist of the roots of the equation $J_1'(2\sqrt{\frac{\lambda_i}{\tau}}) = 0$. The smallest z that makes $J_1'(z) = 0$ is about 1.84118, so the smallest λ_i is about 0.847486τ. Therefore, the largest eigenvalue in (5) is about $\lambda_0 = 1.17996\tau^{-1}$. The corresponding eigenfunction $g_0(t)$ that maximizes (3) is $g_0(t) = ce^{-\frac{1}{2}\tau t}J_1(1.84118e^{-\frac{1}{2}\tau t})$. Notice that $\Pi_t'(e^{-\tau t}) = 1.16667\tau^{-1}$, so the approximation $e^{-\tau t}$ is in error by about 1.1 % in terms of Π_t'.

It can be shown that $ce^{-\frac{1}{2}\tau t}J_1(2.40483e^{-\frac{1}{2}\tau t})$ is the optimal noncausal filter and the approximation $e^{-\tau t}$ is in error by 3.6 %.

Theorem 4 *If $h(s, t) = f(s + v_0t, t)$ is the optimal spatio-temporal filter in detecting a moving step edge with a velocity v_0 and $h(s, t)$ can be written as a product form, namely $k(s)g(t)$, then $k(s) = f_D(s) \propto e^{-\alpha|s|}\sin(\omega s)$ and $g(t) = g_0(t) \propto e^{-\frac{1}{2}\tau|t|}J_1(1.84118e^{-\frac{1}{2}\tau|t|})$. Therefore, the optimal spatio-temporal filter can be written as $-ce^{-\alpha|x-v_0t|}\sin(\omega(x-v_0t))e^{-\frac{1}{2}\tau|t|}J_1(1.84118e^{-\frac{1}{2}\tau|t|})$. If the optimal spatio-temporal <u>causal</u> filter in the product from is sought, then it can be written as $-ce^{-\alpha|x-v_0t|}\sin(\omega(x - v_0t))e^{-\frac{1}{2}\tau t}J_1(2.40483e^{-\frac{1}{2}\tau t})$, for $t > 0$.*

Corollary 4.1 *If we choose the function $f_{D1}(s)$ for $k(s)$, and $e^{-\tau|t|}$ (noncausal) or $e^{-\tau t}$ (causal) for $g_0(t)$, then the approximated optimal filter would be $f(x, t) = -c \cdot (x - v_0t)e^{-\alpha|x-v_0t|}e^{-\tau|t|}$. The approximated causal one is $f(x, t) = -c \cdot (x - v_0t)e^{-\alpha|x-v_0t|}e^{-\tau t}$, for $t > 0$.*

3 Some Properties

The following theorems are derived by using the approximated functions $-c \cdot (x - v_0t)e^{-\alpha|x-v_0t|}e^{-\tau|t|}$ as the optimal edge detector in the spatio-temporal domain.

Theorem 5 *The absolute value of the Fourier transform $\mathcal{F}\{f\}(u, w)$ of the optimal spatio-temporal edge detector $f(x, t)$ is maximized at the line $uv_0 + w = 0$ in the frequency domain, particularly maximized at $(\frac{\alpha}{\sqrt{3}}, -\frac{\alpha v_0}{\sqrt{3}})$ and $(-\frac{\alpha}{\sqrt{3}}, \frac{\alpha v_0}{\sqrt{3}})$.*

Corollary 5.1 *The optimal spatio-temporal edge detector is a bandpass filter. The expected convolutional output with a stochastic step moving edge would be maximized when the edge velocity is the same as the velocity the edge detector is tuned to.*

Theorem 6 *Let the optimal noncausal filter be $f(x, t) = -c(x-v_0t)e^{-\alpha|x-v_0t|}e^{-\tau|t|}$ and the nonideal step edge be $A(t) \cdot a_0u_{-1}(x + d - v_1t)$. Let $D_v = | v_1 - v_0 |$ and $D_d = | d |$. The expected output $E[O_s(x, t)]$, at $x = 0$ and $t = 0$ would be*

$$(-4ca_0)\frac{-\alpha^3D_v^3e^{-\frac{2\tau D_d}{D_v}} + [(\alpha^3\tau D_d + 3\alpha^2\tau)D_v^2 - 4\alpha\tau^3D_d - 4\tau^3]e^{-\alpha D_d}}{\alpha^2(\alpha^4D_v^4 - 8\alpha^2\tau^2D_v^2 + 16\tau^4)}$$

for $\forall D_d$ and $\forall D_v$ except when $D_v = \frac{2\tau}{\alpha}$. When $D_v = \frac{2\tau}{\alpha}$, $E[O_s(0, 0)] = \frac{ca_0(3+\alpha D_d+\alpha^2D_d^2)e^{-\alpha D_d}}{4\alpha^2\tau}$.

When $D_d = 0$, $E[O_s(0, 0)]$ becomes $\frac{4ca_0(\alpha D_v+\tau)}{\alpha^2(\alpha D_v+2\tau)^2}$. When $D_v = 0$, it becomes $\frac{ca_0(1+\alpha D_d)e^{-\alpha D_d}}{\alpha^2\tau}$. Finally, when $D_d = D_v = 0$, it becomes $\frac{ca_0}{\alpha^2\tau}$ which is the expected output on the ideal edges satisfying the original assumption. We can work out the sensitivity analysis for the causal filters similarly[8]. Notice that the expected noncausal or causal outputs are not monotonically decreasing with respect to D_d or D_v for some α's and τ's. However, they are monotonically decreasing when $D_v = 0$ or $D_d = 0$.

Theorem 7 *The optimal causal edge detector is realizable recursively in the temporal domain.*

The optimal spatio-temporal edge detectors are optimal only when they are exactly tuned to the edge velocity. An optimal edge detector would not produce a maximized output at the edge position large enough to be claimed edges in the case that the real edge velocity is very different from the assumed velocity. Therefore, several edge detectors tuned to different velocities and employed at the same time appear necessary to detect the edges moving with different unknown velocities.

To derive optimal three-dimensional spatio-temporal filters for *two-dimensional* image edges, we need one more stochastic function, say $B()$, independent of $A()$, to model the spatial direction along the image edge lines. Let n be an axis perpendicular to the edge lines and $n^\perp$ axis perpendicular to the n axis. Just like $A(t)$, $B(n^\perp)$ has a value 1 when $n^\perp \in [-n_m^\perp, n_p^\perp]$, $n_m^\perp > 0$ and $n_p^\perp > 0$, and 0 otherwise. Also, let us assume the random variables $n_m^\perp$ and $n_p^\perp$ follow the exponential function $P_e(n^\perp) = \tau_{n^\perp}e^{-\tau_{n^\perp}\cdot n^\perp}, \tau_{n^\perp} > 0, n^\perp > 0$.

Theorem 8 *The optimal causal filter $f(n, n^\perp, t)$ is $-c\cdot(n-v_0t)e^{-\alpha|n-v_0t|}e^{-\tau_{n^\perp}\cdot|n^\perp|}e^{-\tau_t t}$, for $t > 0$ and $f(n, n^\perp, t) = 0$, for $t < 0$. The noncausal one is $f(n, n^\perp, t) = -c \cdot (n - v_0t)e^{-\alpha|n-v_0t|}e^{-\tau_{n^\perp}\cdot|n^\perp|}e^{-\tau_t|t|}$.*

Theorem 9 *The absolute value of the Fourier transform $\mathcal{F}\{f\}(u, v, w)$ of the optimal edge detector $f(n, n^\perp, t)$ is maximized at a line, which is the intersection of two planes, $uv_0 + w = 0$ and $v = 0$, in the frequency domain, particularly maximized at $(\frac{\alpha}{\sqrt{3}}, 0, -\frac{\alpha v_0}{\sqrt{3}})$ and $(-\frac{\alpha}{\sqrt{3}}, 0, \frac{\alpha v_0}{\sqrt{3}})$.*

Because the optimal filters are tuned to specific edge orientation and edge velocity, they are like the oriented receptive fields of Hubel and Wiesel or the linear impulse response of Ross and Burr[1], and it is natural and interesting to compare them with the Gabor filters. Both of them are band-pass filters, highly suited to parallel processing, and the convolutional outputs are maximized only when they are tuned to the right spatial orientations and the right normal velocities. Gabor filters minimize the joint localization of the spatio-temporal domain and the spatio-temporal frequency domain. However, the optimal edge detectors optimize Canny's criteria. They are optimal in different sense, so they have different shapes in the frequency domain. Noise robustness is enhanced in the optimal edge detectors and the Gabor filters because the noise can be attenuated through the temporal axis as well as the two-dimensional spatial domain. Multiple Gabor filters operating simultaneously but tuned to different spatial orientations and different image flows are used in some algorithms[6] to extract the image flow. In a companion paper[7], we show how to achieve edge detection and velocity estimation in the same time by the optimal edge detectors. In order for the Gabor filters to separate more velocities, their spatio-temporal supports must also increase. The following is a similar theorem for the optimal edge detectors.

Theorem 10 *In order for the optimal edge detectors,* $f(n, n^\perp, t) = -c\cdot(n - v_0 t)e^{-\alpha|n-v_0 t|}e^{-\tau_{n^\perp}\cdot|n^\perp|}e^{-\tau_t|t|}$*, to have higher resolution in both velocity and edge orientation,* τ_n *and* $\tau_{n^\perp}$ *should be smaller. Increasing* α *increases the resolution.*

4 Conclusion

We model the moving step edges as a product of a deterministic function in space and a stochastic function in time which captures the edge shapes and the temporal uncertainties respectively. Under Canny's original optimality criteria, a set of optimal edge detectors are derived. They are in a product form, i.e. a product of a spatial function and a temporal function. The spatial function happens to be Canny's edge detector in one dimension and the temporal function can be well approximated by the exponential function. Generalizing Canny's edge detector to the temporal domain is not only theoretically interesting, but also practically useful and insightful. Our generalization of Canny's edge detectors provides better immunity to noise and can serve as one of the tools in understanding the temporal behavior of moving edges. We use them in a data-fusion framework in [7] to detect moving edges and their normal velocities simultaneously. For completeness, we derive some properties of the optimal edge detectors and compare them with Gabor filters.

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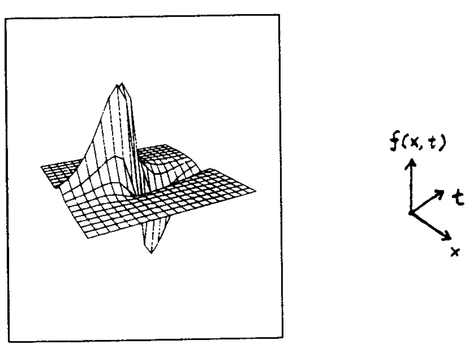

Figure 1: An example of the optimal noncausal filters $-(x - v_0 t)e^{-\alpha|x-v_0 t|}e^{-\tau|t|}$ with $\alpha = 1$, $\tau = 2$, and $v_0 = -2$.