Finding Sheets in Fibrous Structures

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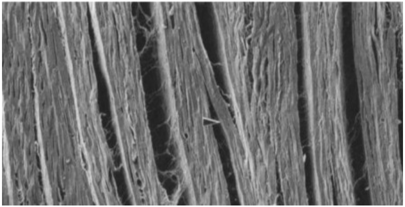
Sheets in the heart

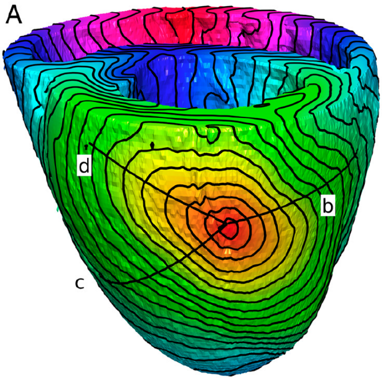
- Muscle fibers in the heart are arranged along laminar sheets.



Histological section showing sheet like organization of myocytes. LeGrice et al., Amer. J. Physiology, 1995.

Sheets in the heart

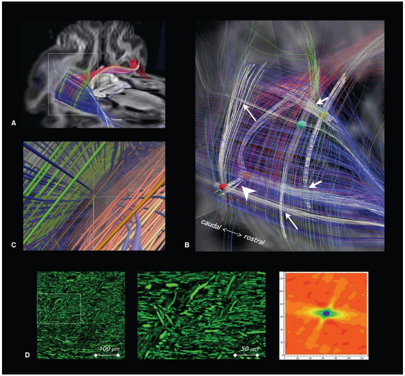
- Muscle fibers in the heart are arranged along laminar sheets.
- Sheet like organization also plays an important role in electrophysiology.



Young and Panfilov, PNAS, 2010

Sheets in the brain

- White matter tracts in the mammalian brain are hypothesized to lie along sheets, which intersect at right angles.



Sheet like organization of fiber tracts in the monkey brain. Van J. Wedeen et al. Science, 2012.

Sheets in the brain

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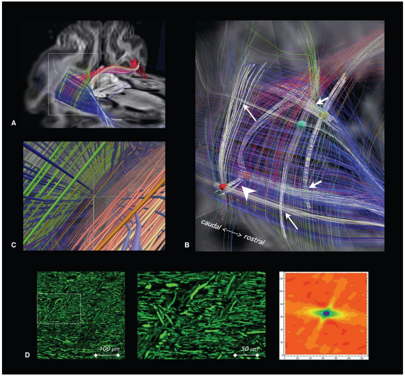
- This sheet like organization is localized within regions.



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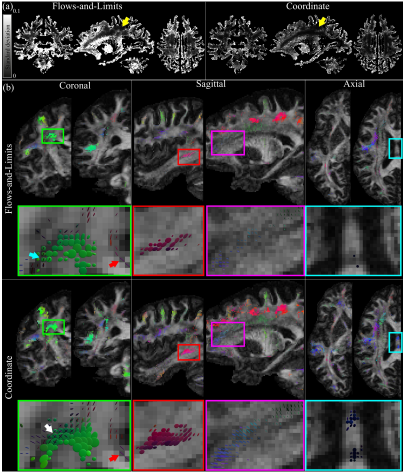
Sheet Probability Index

- Tax et al. have proposed a sheet probability index, as a measure of sheet likelihood in the brain from DTI.



Tax et al., Medical Image Analysis, 2017.

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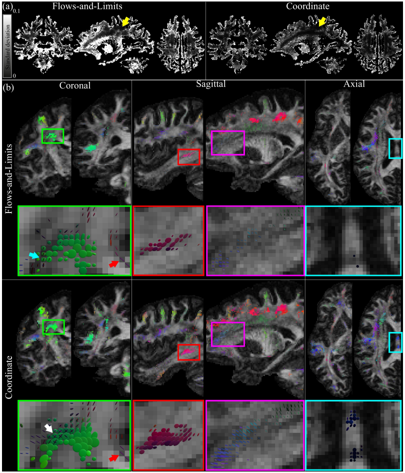
- Ankele et al., have used the eigenvectors of diffusion tensors directly.



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- Claims of such organization have been based on qualitative descriptions.
- Quantitative descriptions, such as the sheet probability index, use the second and third principle directions from DTI, which may be unreliable.

Holonomic Vector fields

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$$\langle \mathbf{n}, \operatorname{curl} \mathbf{n} \rangle = 0.$$

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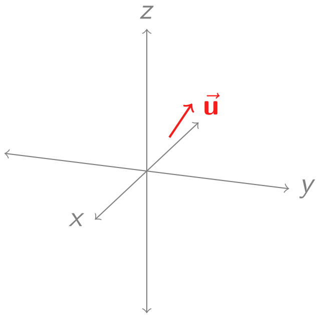
$$\rho = \langle \mathbf{u} \times \mathbf{v}, [\mathbf{u}, \mathbf{v}] \rangle.$$

Non-Holonomicity Energy

- Consider two orthogonal unit vector fields: $\mathbf{u}$ which is fixed,



vector field visualization

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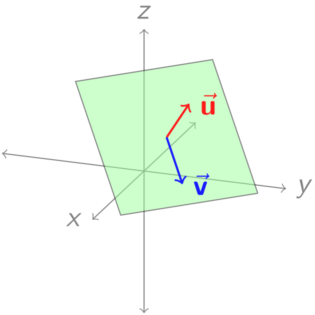

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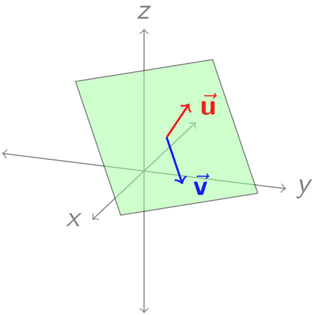

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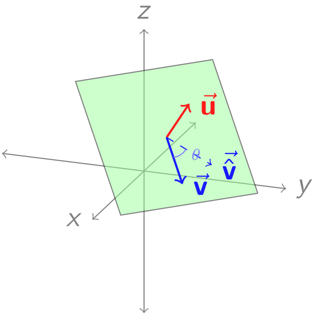

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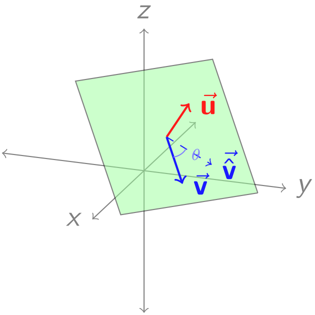
$$E(\mathbf{u}, \hat{\mathbf{v}}) = \rho^2(\mathbf{u}, \hat{\mathbf{v}}).$$



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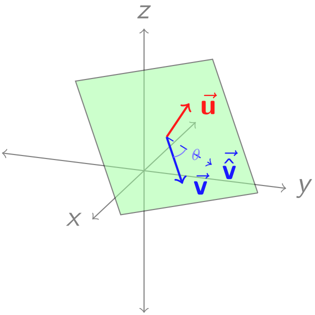

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Non-holonomicity with rotation

- Non-holonomicity of vector fields $\mathbf{u}$ and $\mathbf{v}$, say $\rho^{\mathbf{uv}}$ is given by

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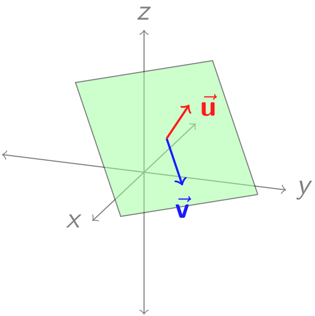

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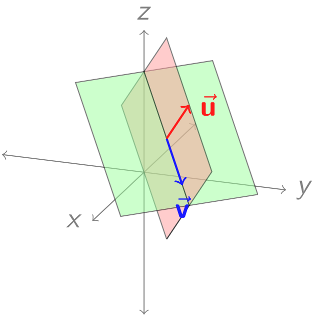

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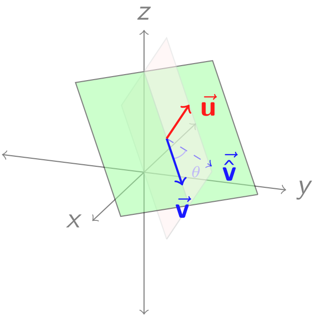

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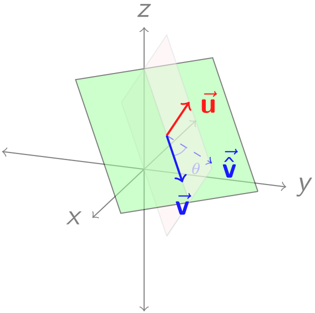

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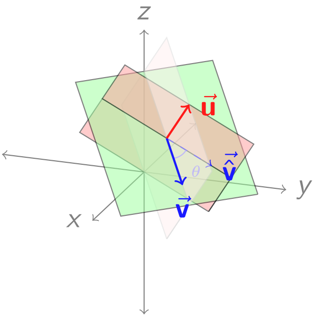


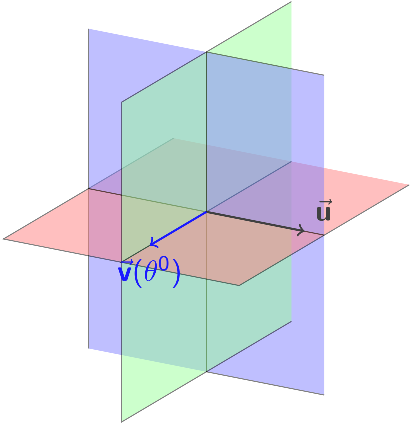
Configuration of vector fields

Minimization

- Given a fixed input field **u** we pose the estimation of field **v** as the following minimization problem

$$\mathbf{v}^* = arg\min_{\mathbf{v}} E(\mathbf{u}, \mathbf{v}, \theta)$$

$$\text{subject to } \langle \mathbf{u}, \mathbf{v} \rangle = 0.$$



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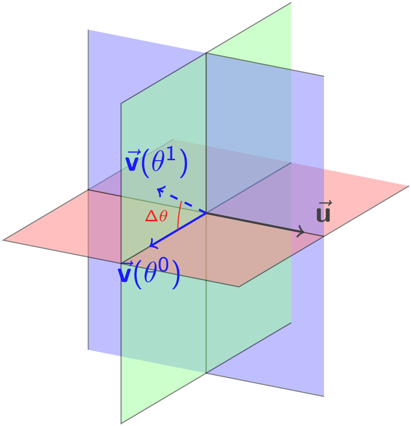
- Gradient descent update for parameter θ

$$\theta^{t+1} = \theta^t - \eta \frac{\partial E}{\partial \theta}.$$



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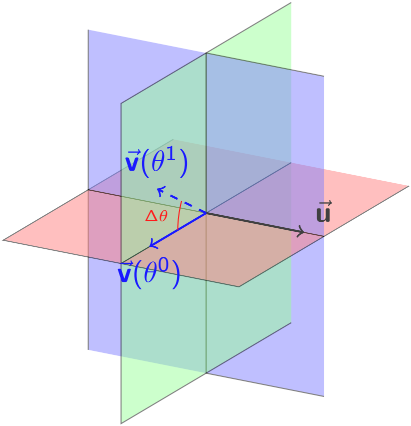

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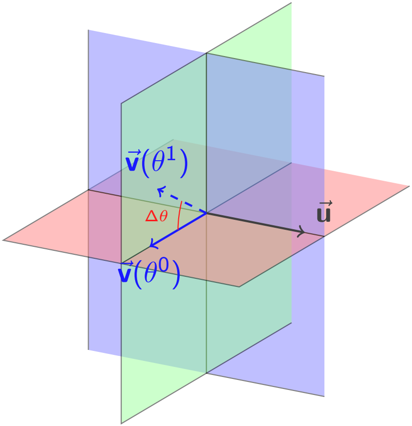
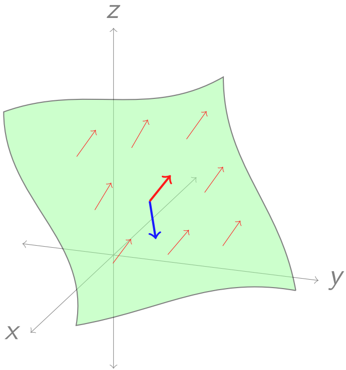
$$\mathbf{v}(\theta^1) = \mathbf{R}^{\mathbf{u}}_{\theta^1} \mathbf{v}$$

where, $\mathbf{R}^{\mathbf{u}}_{\theta^1} = \cos\theta^1 \mathbf{I} + sin \theta^1 [\mathbf{u}]_\times + (1 - \cos\theta^1)\mathbf{u}\mathbf{u}^T$



Non-local Energy

- The energy function $E = \rho^2$ is non-local as it depends on values of $\mathbf{v}$ at neighbouring points as well.



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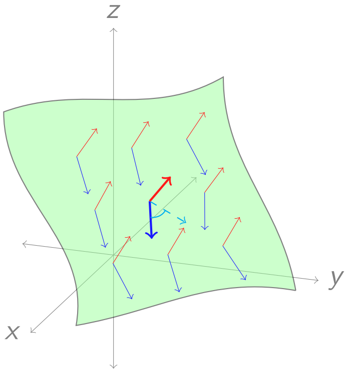

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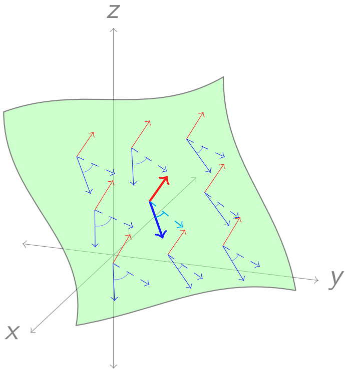
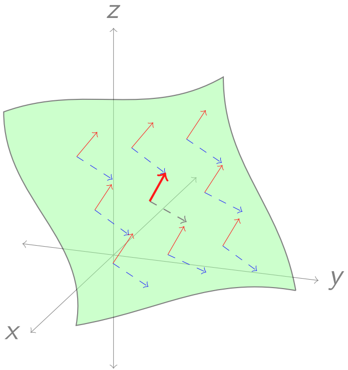


Non-local Energy

- The energy function $E = \rho^2$ is non-local as it depends on values of $\mathbf{v}$ at neighbouring points as well.
- The non-local holonomicity energy may or may not decrease after a gradient descent update!



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$$\frac{\partial E}{\partial \theta} = 2\Big(\rho_s(\theta) + \nabla_{\mathbf{u}}\theta\Big)\Big((\rho^{\mathbf{un}} - \rho^{\mathbf{uv}})\sin 2\theta + 2\alpha^{\mathbf{uv}}\cos 2\theta + \operatorname{div}\mathbf{u}\Big).$$

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There exists a positive η such that energy decreases if

$$4\alpha^{\mathbf{uv}}\left(2\alpha^{\mathbf{uv}} + \operatorname{div}\mathbf{u}\right) + \nabla_{\mathbf{u}}\mathcal{E}_0^{\rho} > 0.$$

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For an input field which doesn't fan in or out, we have

$$-8(\alpha^{\mathbf{uv}})^2 < \nabla_u\mathcal{E}_0^\rho.$$

Convergence of Gradient Descent

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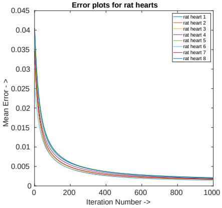
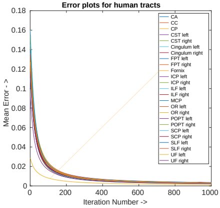
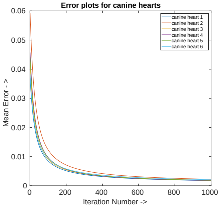
  - η should be chosen based on smoothness properties of the input vector field.

Convergence Rate and Runtime

- With $\eta = 0.1$ the error flattens out in about 500 iterations.

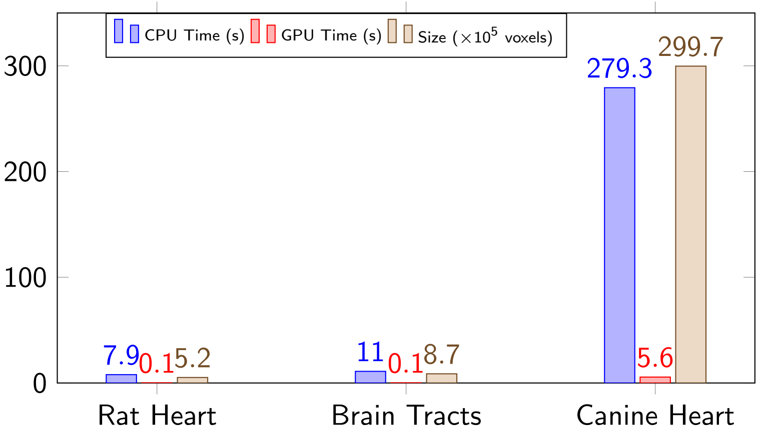






Convergence Rate and Runtime

- The algorithm converges to a reasonable minimum in a few seconds for small datasets and in a few minutes for larger datasets.



A synthetic example - 1

A short axis slice of circumferential streamlines.

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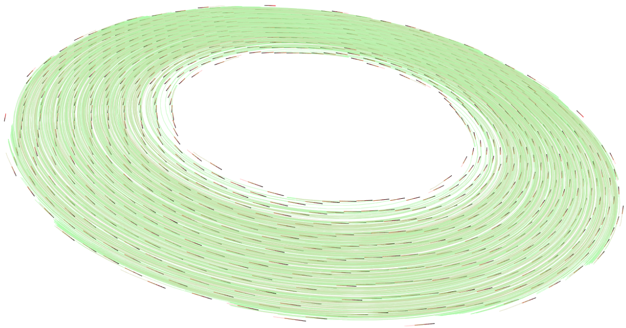
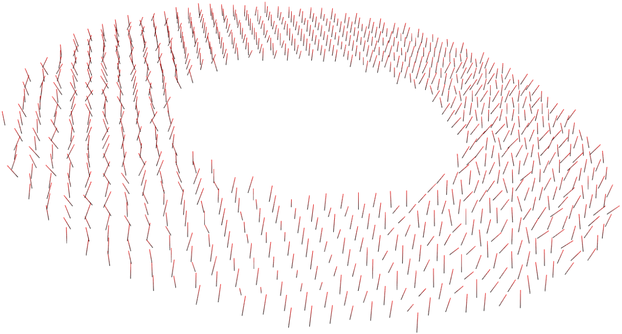


The tangent vector field to the circumferential streamlines.

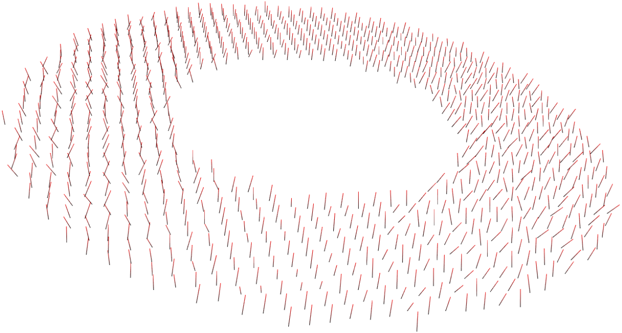
A synthetic example - 1



A randomly seeded vector field in the plane perpendicular to the circumferential vector field.

A synthetic example - 1

Minimizing non-holonomicity on the aforementioned vector fields



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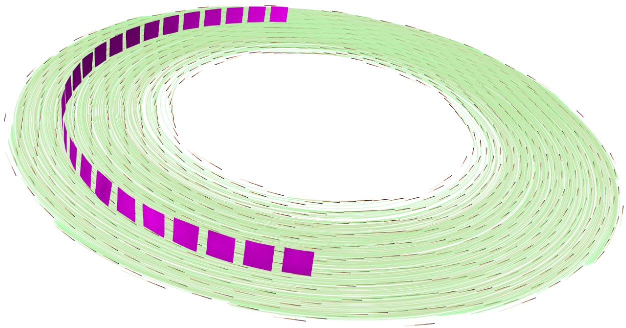
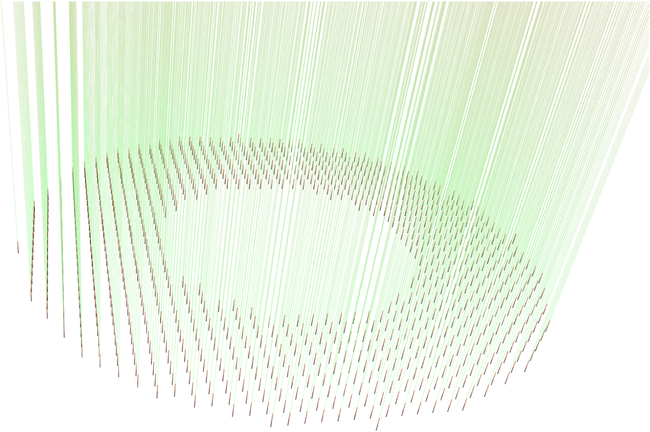


Sheets fit to the original and estimated vector fields.

A synthetic example - 2

A set of streamlines along the long axis.

A synthetic example - 2



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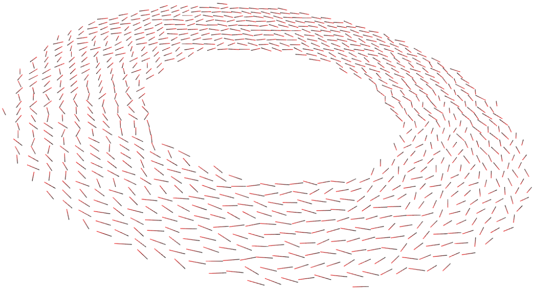

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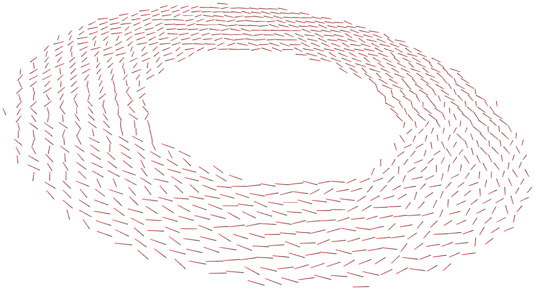
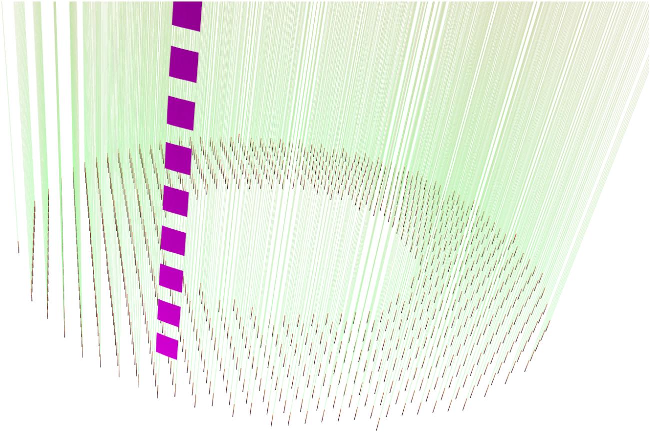
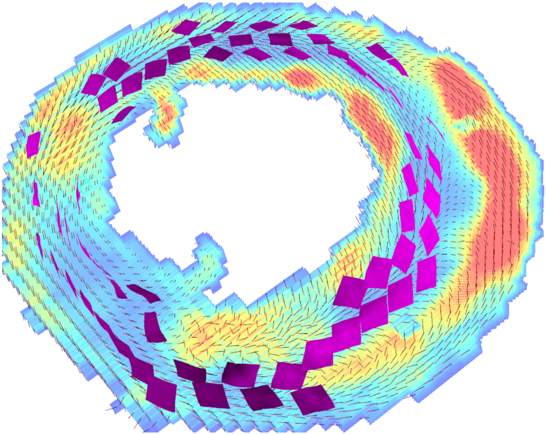


A synthetic example - 2



Sheets fit to the original and estimated vector fields.

An ex-vivo rat heart



Sheets fit to a short axis slice of DTI from a rat heart.

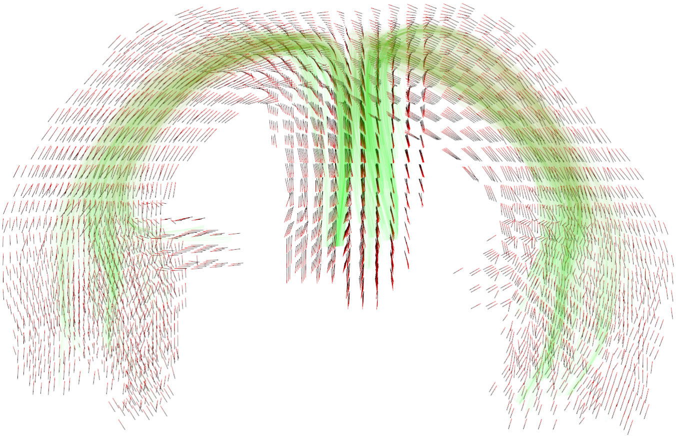
The Fornix Tract

Fornix tract streamlines from an HCP atlas.

The Fornix Tract

Fornix tract streamlines from an HCP atlas.

The Fornix Tract

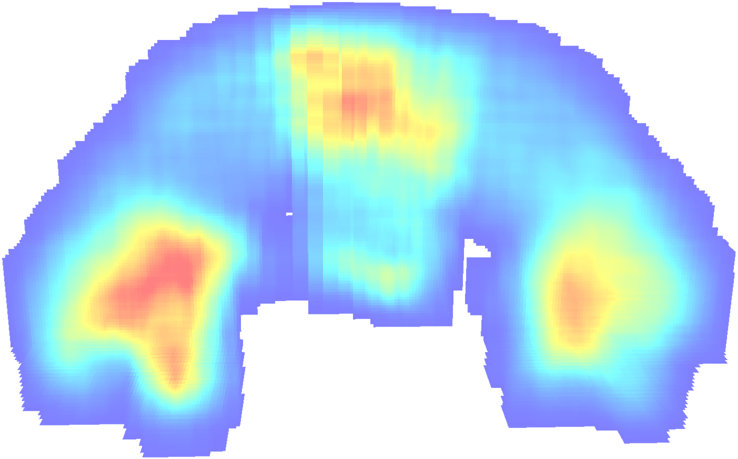


The vector field tangent to the Fornix tract streamlines.

The Fornix Tract

Sheets estimated on the vector field tangent to the Fornix tract streamlines.

The Fornix Tract



Volume rendering of error of fit (non-holonomicity).

Conclusion

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- Our efficient GPU based implementations using PyTorch can be shared.