Combined Questions

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1 Exam Questions

Q1: DYNAMIC PROGRAMMING *Given are tow sequences of decimal digits,* $x_1, ..., x_n$ *and* $y_1, ...y_n$ *and an* $n \times m$ *matrix L, where L[i,j] is the length of the longest common subsequence among* $x_1, ...x_i$ *and* $y_1, ...y_j$. *Give an* $O(m+n)$ *time algorithm that outputs* $i_i < i_2, ... < i_k$ *and* $j_1 < ... < j_k$, *where L[n,m] and* $x_{i_1} = y_{j_1}, ..., x_{i_k} = y_{j_k}$.

Q2: LOWER BOUNDS

i *What is meant by the statement that the lower bound for Wordle is at most 5? [Answer with a sentence and put the qualifiers in the correct order.]*

ii *Textbooks often mention that the lower bound for sorting is* $\lceil \log_2 n! \rceil$. *What is meant by that phrase?[Answer with a sentence and put the qualifiers in the correct order.]*

Q3: COMPLEXITY *Freely use the appropriate landau symbols. Assume that we are in a RAM model of comutation.*

i *Give a tight upper bound for the complexity of the standard dynamic programming algorithm for finding the traveling salesman tour of n cities.*

i *How fast can you compute* M^n *where M is a given* 3×3 *matrix holding nine integers?*

iii *How fast can you determine the* $2n+1$ *coefficients of the product polynomial* $(a_0 + a_1 x + ... + a_n x^n)(b_0 + b_1 x + ... + b_n x^n)$, *where the* a_i*'s and* b_i*'s are given integers? What is the name of this method?*

Q4: ALGORITHM DESIGN *Compute the array* $A[0], ...A[n]$, *where* $A[i]$ *is the number of zeros(ones) in the binary expansion of i. Your algorithm should take time* $O(n)$ *under the RAM model in which you can perform integer addition, integer division, integer subtraction, integer multiplication and the modulo operation in constant time.*

Q5: SOLUTIONS OF RECURRENCES *State solutions in* Θ *notation for the following recurrences,* *when $T_0 = 0$:*

$$
\begin{aligned}
T_n &= 3T_{\lfloor \frac{n}{4} \rfloor} + 1, n \geq 1 \\
T_n &= 4T_{\lfloor \frac{n}{5} \rfloor} + 1, n \geq 1 \\
T_n &= T_{\lfloor \frac{n}{3} \rfloor} + T_{\lfloor \frac{2n}{3} \rfloor} + n, n \geq 1 \\
T_n &= 3T_{\lfloor \frac{n}{3} \rfloor} + n, n \geq 1 \\
T_n &= 2T_{\lfloor \frac{n}{2} \rfloor} + n, n \geq 1
\end{aligned}
$$

- $\Theta(n^{\log_4 3})$
- $\Theta(n^{\log_5 4})$
- $\Theta(n \log n)$
- $\Theta(n \log n)$
- $\Theta(n \log n)$

Q6: TREES

- *Answer yes of no: does the preorder and postorder listing of a binary tree determine the shape of the tree?*
- *Same question for an ordered tree.*
- *If a ternary tree has 1000 nodes of degree 3 and 20(21) nodes of degree 2, then how many leaves are there?*
- *The level order number of a node is its position in the level order numbering. If the level order number of a node in a complete binary tree is n, then how far is it from the root?*

Q7: ALGORITHM ON TREES *A binary tree with root pointer t has cells with three fields,* `left[.]`, `right[.]` *and* `odd[.]`*(*`size[.]`*), where the latter is supposed to be one if the subtree of the node is of odd size(where the node itself is included in the count) and zero otherwise (where the latter contains size of the subtree). Write a simple linear time recursive algorithm (i.e., fill in) the* `odd` *field for all nodes.*

2 Divide And Conquer

Q1: *A generalization of the Fibonacci sequence looks like this:*

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3},$$

with $x_0 = 0$, $x_1 = 0$, $x_2 = 1$.

1. *Compute* x_n *in* $O(\log n)$ *time in a RAM model, by generalizing the method we saw in class.*
2. *Would you be able to find a mathematical expression for* x_n*?*
3. *Show that* $x_n = \Theta(3^n)$
4. *Still under RAM model, how can you compute* x_n *in time* $O(1)$ *when*

$$x_n = x_{n-1} - x_{n-2} + x_{n-3}, \quad x_0 = 0, x_1 = 0, x_2 = 1$$

Q2: *We have an equality oracle, which answers the question "Is* $x = y$*". Determine in* $O(n)$ *time (i.e. uses of the oracle) whether a set* $x_1, ..., x_n$ *contains a majority element, (an element that occurs* $> \frac{n}{2}$ *times) and if so return that element.*

Q3: *Let* A_n *be an* $n \times n$ *matrix of integers. To compute the determinant of* A_n*, there is a standard method that sums weighted* $(n-1) \times (n-1)$ *matrix determinants. Derive a RAM model recursion for the time complexity* T_n *of the divide-and-conquer approach. Show that* $T_n = \Theta(n!)$.

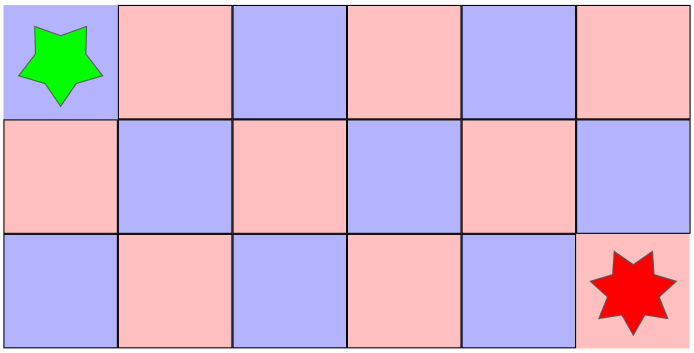
3 Dynamic Programming

Q1: *[Count] We are given a flight of n stairs. At each step and we can move up by 1 or 2 steps at a time. How many different ways are there to reach the top?*

Q2: *[Min Cost] Given a flight of n stairs where we move up by 1 or 2 steps at a time. Each move (of 1 or 2 steps) will cost us* c_j. *What is the minimum cost required to reach the top?*

Q3: *[Variation, Count] Can jump upto M steps and every next jump can be >= k where k is size of last jump.*

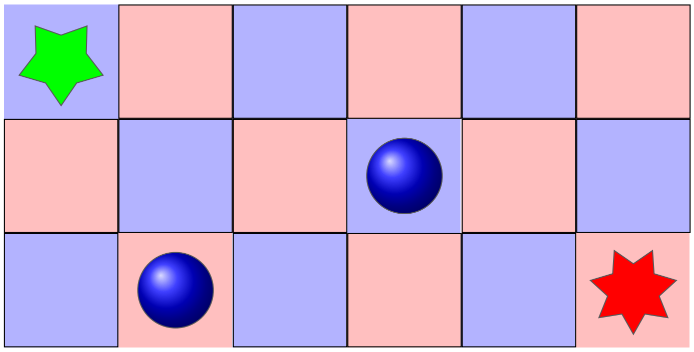
Q4: *[Count] We are given an* $m \times n$ *2D gird and an agent who starts at top left corner. The agent can either move one step to the left or one step to the right. Calculate the total number of ways to reach the bottom right corner?*



Question 4: Starting position: Green (5-pointed) star, Final destination: Red (7-pointed) Star

Q5: *[Count] In the previous question, suppose you are given another* $m \times n$ *array B where* $(i,j)^{th}$ *entry specifies weather grid location* (i,j) *is blocked. Count the number of ways to reach bottom right corner*

Q6: *[Min Cost] Given an* $m \times n$ *2D grid and an agent that can move one step to the right or one step down. Agent spend* c_{ij} *units to move through grid location* g_{ij}. *If the agent starts at top left corner, what is the minimum cost required for the agent to reach bottom right.*



Question 5: Starting position: Green (5-pointed) star, Final destination: Red (7-pointed) Star, Obstacles: Blue (round) Rock

1	5	1
7	2	3

Question 6: Sample 2×3 Grid with cost values c_{ij} overlaid.

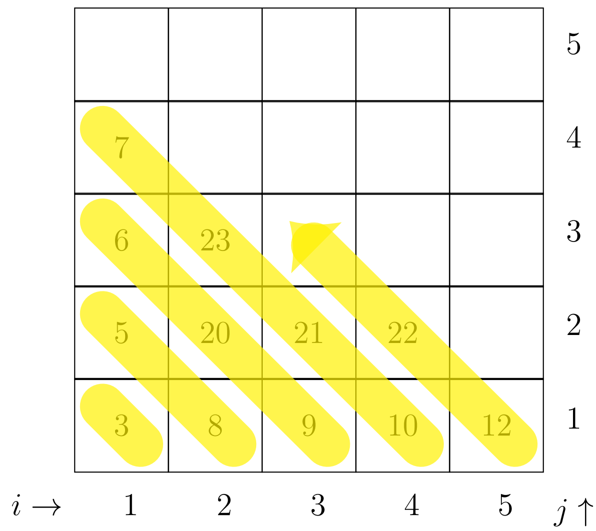
Q7: *[Count] Given a set S of K integer $s_1, s_2, ..., s_K$ and a target integer T, calculate the number of possible ways to add up to T using numbers from S? What if the order of chosen element doesn't matter?*

Q8: *[Min Cost] Given a set S of K integers $s_1, ...s_K$ and a target t, Compute minimum number of elements required to sum up to t (Each element s_i can be used multiple times).*

4 Data Structures

Q1: *Implement a* **`bitSet<N>`** *ADT which stores an N bit binary number. The data structure should support following operations:* **`all, any, none`** *which check if all, any or none of the bits are set to* 1*,* **`count`** *which return number of bits set to* 1*,* **`set(k)/reset(k)`** *which sets bit k to* 1/0 *and* **`flip`** *which toggles the values of all bits. All operations should be $O(1)$*

Q2: DATA STRUCTURES *A beap (biparental heap) stores keys in a matrix by filling diagonals. The key at position (i, j) is smaller than all keys to the right and above it. The last diagonal is filled bottom to top. Write algorithms for $INSERT$ and $DELETE\text{-}MIN$ that take $O(\sqrt{n})$ time.*



Question 2.1: Beap example

5 Basic Trees

Q1: *Give a non-recursive stack-based algorithm that produces at once pre, post and in-order sequence of a binary tree.*

Q2: *Given a binary tree in which the root has no right child, write an algorithm that outputs the ordered tree it represents where every node contains a linked list of its children*

Q3: BST *What is the chance that a random binary search tree on n nodes is a chain?*

Q4: BST *Given a binary search tree, write an $O(n)$ algorithm for outputting a standard binary heap for the keys (smallest on top) without using any key comparisons.*

Q5: TREAP *In a treap what is the chance that the element with smallest key is a leaf?*

Q6: QUAD TREE *In a quadtree on n nodes, how many regions are there?*

Q7: RED-BLACK TREE *In a red-black tree, if we flip the colors of all internal nodes, do we sill have a red-black tree.*

Q8: RED-BLACK TREE *Given a red-balck tree with a color field, write an algorithm that computes ranks of all nodes.*

Q9: RED-BLACK TREES *Given a tree with a 'red/black' color field, write an algorithm to determine if it is a valid red-balck tree.*

Q10: RED-BLACK TREE *Write an algorithm for joining two red-black trees t_1 and t_2 where t_1 only has negative keys and t_2 has only positive keys. Assume that the rank of the root of t_1 is one more than the rank of the root of t_2. (You can use Lazy Delete)*

Q11: ALGORITHM DESIGN *How would you find the 7 smallest of* n *items using no more than* $n + 6\log_2 n$ *comparisons?*

Q12: OPTIMAL VALUE *We are given sticks of length* $l_1, ...l_k$ *(not necessarily integer-valued). Find the longest length* L *such that we can make* m *sticks of length* L *by stick-breaking.*

Q13: SUFFIX TRIE *Find a suffix trie for a binary text file of length* n *that has size* $\Omega(n^2)$

Q14: DNA TRIE *Assume that we have DNA strings of* n *people, and that symbols A,C,T and G occur with about equal frequency and independently of each other. How deep will most leaves in the trie for data be?*

Q15: RABIN-KARP *Suggest possible hash functions* h *for Rabin- Karp pattern matching algorithm*

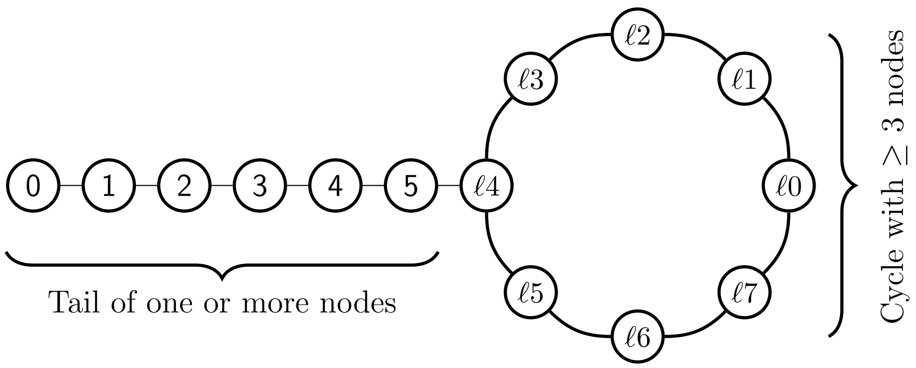
Q16: BINARY SEARCH *To determine if an integer* x *is in a sorted array of integers, binary search could be used. If bit complexity is your main concern, is a binary trie preferable or worse?*

6 Graphs

Q1: FOREST *A graph $G = (V, E)$ has N components that are all trees. How are $|V|$, $|E|$ and $|N|$ related?*

Q2: ARTICULATION NODE *An* articulation node *v in a graph $G = (V, E)$ is such that its removal disconnects the graph. How can you determine that a node v is an articulation node?*

Q3: LOLLIPOP *Write an algorithm that determines in time $O(|E| + |V|)$ if a given graph $G = (V, E)$ is a lollipop*

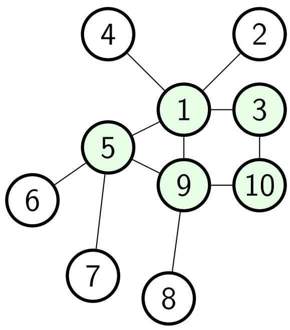


Question 3.1: Lollipop graph

Q4: CYCLE *Write an algorithm to determine if a directed graph $G = (V, E)$ has a cycle.*

Q5: CORE GRAPH *We are given a connected graph $G = (V, E)$. A clean up removes all nodes of degree 1 (also called leaves). We repeat "clean-up" until no nodes of degree 1 are left. What remains is called the* core *of the graph. Write an efficient algorithm for finding the core*

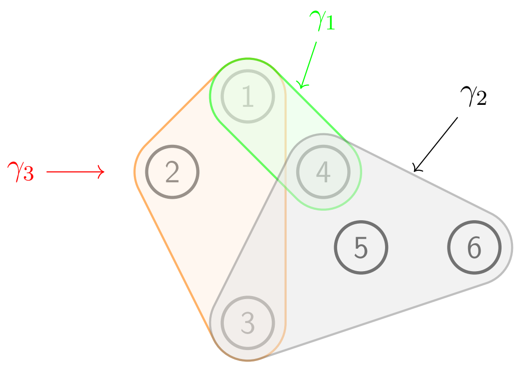
Q6: MST *Let all edges of a connected graph $G = (V, E)$ have a length 1,2 or 3. Find the minimum spanning tree (MST) and the shortest path tree for node 1 in worst-case time*



Question 5.1: Core Graph in green

$O(|V| + |E|)$.

Q7: SOCIAL NETWORKS *People organize themselves in groups of friends or by area of interest. This creates an intricate graph, also called a* hypergraph, *and denoted by* $G = (V, E)$. *Here* V *is a set of nodes (people), and* E *is a collection of groups. A group* $\gamma \in E$ *is an unordered set from* V. *It is understood that* $|\gamma| \geq 2$. *If all groups* γ *have* $|\gamma| = 2$, *then we obtain the standard graph. At the data structure level, groups* γ *are given as linked lists, and nodes* u *have linked lists of groups they belong to. Propose a DFS inspired traversal algorithm that takes time* $O\left(\sum_{\gamma \in E} |\gamma| + |V|\right)$.



Question 7.1: Hyper Graph with vertices $V = \{1, 2, 3, 4, 5, 6\}$ and edges $E\{\gamma_1, \gamma_2, \gamma_3\}$