**COMP 766: Assignment 2**
**Available: Monday, March 24th, 2014**
**Due Date: Tuesday, April 8th, 2014**

This assignment explores aspects of the heat equation and the geometric heat equation. Use Matlab for implementation and for visualization. Organize and present your results carefully and be sure to discuss them. Submit a single PDF document under mycourses; as you did for assignment 1. I do not necessarily need to see your implementations but I do want you to describe what you have done clearly.

# 1 Part I: (30%) THE HEAT EQUATION

Let $I(x)$ be a bounded function defined for $x \in \mathcal{R}^n$. Let $u(x,t)$ be a bounded solution to the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u,$$
$$u(x,0) = f(x).$$

Then it can be shown that the solution is given by convolution with a Gaussian kernel:

$$u(x,t) = \int_{\mathcal{R}^n} K(x-y,t) I(y) dy,$$

where

$$K(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}.$$

In other words, $u(x,t)$ is obtained by blurring $I(x)$ by increasingly diffuse Gaussians, parametrized by $t > 0$ with standard deviations $\sigma$ satisfying $2\sigma^2 = 4t$. This suggests two different strategies for Gaussian blurring, which I would like you to explore using Van Gogh's starry night painting, which you can download from here:

`http://uploads2.wikipaintings.org/images/vincent-van-gogh/the-starry-night-1889.jpg`

You can convert this to a greyscale image which you will treat as $I$.

1. Blur $I$ with Gaussians having standard deviations $\sigma = 2, 4, 16$ pixels, using the straightforward convolution routine in Matlab.

2. Now attempt to achieve the same result by directly implementing the Heat Equation, and applying it for the appropriate number of iterations. Use central differences to implement the partial derivatives in space $x$ and simple first-order differencing for time $t$.

3. Compare the results obtained using the two strategies above. A useful way to do this is to visualize the evolution of some well chosen isophotes (level curves of fixed intensity) overlayed on the blurred images. In theory these level curves should be the same in both cases.

## 2 Part II: (60%) THE GEOMETRIC HEAT EQUATION

The geometric heat equation is given by:

$$\frac{\partial u}{\partial t} = div(\frac{\nabla u}{|\nabla u|})|\nabla u|,$$

$$u(x, 0) = I(x).$$

If you think of this in terms of a level set formulation, all isophotes in $u$ are evolving (independently) by their curvature.

1. I would like you to prove that with $u(x, 0) = I(x)$, where $I(x)$ is a 2D intensity image the above equation reduces to $I_t = I_{rr}$, where $r$ is the local direction tangent to the isophote, i.e., the direction orthogonal to the direction $\nabla I$. In other words, comparing with the heat equation in Part I, $I_t = \Delta I = I_{rr} + I_{r'r'}$, with $r'$ the direction of $\nabla I$, the geometric heat equation blurs only along level curves. This interpretation is possible because the Laplacian used in Part I is rotationally invariant, so it can be expressed in terms of any two orthogonal directions locally.

2. Now implement the geometric heat equation by discretizing it, again using central differences for the partial derivatives in space and simple first-order differencing in time. Apply it to the starry night image and visualize the results. Here it would again be useful to look at the evolution of some well chosen isophotes and to discuss how these evolve differently than those for the heat equation in Part I. What can you observe about the shape of these evolving isophotes?