Area and Length Minimizing Flows for Shape Segmentation

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- $\mathcal{C}(p) = (x(p), y(p))$ is a smooth closed curve.

- $\mathcal{T} = (x_p, y_p)/\sqrt{x_p^2 + y_p^2}$ is the unit tangent.

- $\mathcal{N} = (-y_p, x_p)/\sqrt{x_p^2 + y_p^2}$ is the unit normal.

- $\kappa\mathcal{N} = \frac{1}{|\frac{\partial\mathcal{C}}{\partial p}|}\frac{\partial}{\partial p}\left[\frac{\frac{\partial\mathcal{C}}{\partial p}}{|\frac{\partial\mathcal{C}}{\partial p}|}\right]$,

where κ is the Euclidean curvature.

- $\mathcal{C} = \mathcal{C}(p,t)$ is a smooth family of closed curves.
- $0 \leq p \leq 1$.
- $\mathcal{C}(0,t) = \mathcal{C}(1,t)$.
- $\mathcal{C}'(0,t) = \mathcal{C}'(1,t)$.

Length functional:

$$\begin{aligned} L(t) &= \int_0^1 \|\frac{\partial \mathcal{C}}{\partial p}\| \, dp \\ &= \int_0^1 (x_p^2 + y_p^2)^{1/2} dp. \end{aligned}$$

$$
\begin{aligned}
L'(t) &= \int_0^1 \frac{\partial}{\partial t}[(x_p^2 + y_p^2)^{1/2}]dp \\
&= \int_0^1 \frac{2(x_p x_{pt} + y_p y_{pt})}{2(x_p^2 + y_p^2)^{1/2}} dp \\
&= \int_0^1 \frac{\langle \frac{\partial \mathcal{C}}{\partial p}, \frac{\partial^2 \mathcal{C}}{\partial p \partial t} \rangle}{\| \frac{\partial \mathcal{C}}{\partial p} \|} dp.
\end{aligned}
$$

Integration by parts:

Let $u = \frac{\mathcal{C}_p}{||\mathcal{C}_p||}$, $dv = \frac{\partial^2 \mathcal{C}}{\partial p \partial t} dp$.

$$L'(t) = \langle \frac{\mathcal{C}_p}{||\mathcal{C}_p||}, \mathcal{C}_t \rangle \Big]_0^1 - \int_0^1 \langle \frac{\partial \mathcal{C}}{\partial t}, \frac{1}{||\frac{\partial \mathcal{C}}{\partial p}||} \frac{\partial}{\partial p} \left[\frac{\frac{\partial \mathcal{C}}{\partial p}}{||\frac{\partial \mathcal{C}}{\partial p}||} \right] ||\frac{\partial \mathcal{C}}{\partial p}|| \rangle \, dp.$$

$$L'(t) = -\int_0^{L(t)} \langle \frac{\partial \mathcal{C}}{\partial t}, \kappa \mathcal{N} \rangle \, ds.$$

Area functional:

$$A(t) = -\frac{1}{2}\int_0^L \langle \mathcal{C}, \mathcal{N} \rangle \, ds = -\frac{1}{2}\int_0^1 \langle \mathcal{C}, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \rangle \, dp.$$

Taking the first variation:

$$A'(t) = \underbrace{-\frac{1}{2}\int_0^1 \langle \mathcal{C}_t, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \rangle \, dp}_{I_1}$$
$$\underbrace{-\frac{1}{2}\int_0^1 \langle \mathcal{C}, \begin{pmatrix} -y_{pt} \\ x_{pt} \end{pmatrix} \rangle \, dp.}_{I_2}$$

Integration by parts for I_2:

Let $u = \mathcal{C}$, $dv = \begin{pmatrix} -y_{pt} \\ x_{pt} \end{pmatrix}$

$$\begin{aligned} I_2 &= -\frac{1}{2}\left(\left. \langle \mathcal{C}, \begin{pmatrix} -y_t \\ x_t \end{pmatrix} \rangle \right]_0^1 - \int_0^1 \langle \mathcal{C}_p, \begin{pmatrix} -y_t \\ x_t \end{pmatrix} \rangle dp \right) \\ &= -\frac{1}{2}\int_0^1 \langle \mathcal{C}_t, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \rangle \, dp = I_1 \end{aligned}$$

$$A'(t) = -\int_0^L \langle \mathcal{C}_t, \mathcal{N} \rangle ds.$$

$\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ is a family satisfying:

$$\boxed{\frac{\partial \mathcal{C}}{\partial t} = \Gamma \mathcal{N}\,.}$$

Let $\Psi : \mathbf{R}^2 \times [0,\tau) \to \mathbf{R}$ be a Lipschitz continuous function such that $\mathcal{C}(p,t)$ is its zero level set:

$$\mathcal{C}(p,t) = \{(x,y) \in \mathbf{R}^2 \,:\, \Psi(x,y,t) = 0\}\,.$$

Differentiating with respect to t:

$$\begin{aligned} 0 &= \Psi_x x_t + \Psi_y y_t + \Psi_t \\ \Psi_t &= -\langle \nabla \Psi, \mathcal{C}_t \rangle \end{aligned}$$

Differentiating with respect to p:

$$\Psi_x x_p + \Psi_y y_p = 0$$

$$\nabla \Psi \perp \mathcal{T} \Rightarrow \mathcal{N} = -\frac{\nabla \Psi}{\|\nabla \Psi\|}$$

Substituting for $\mathcal{C}_t$ and $\nabla\Psi$ in $\Psi_t = \langle \mathcal{C}_t, -\nabla\Psi \rangle$:

$$\Psi_t = \langle \Gamma\mathcal{N}, \|\nabla\Psi\|\, \mathcal{N} \rangle$$

$$\boxed{\Psi_t = \Gamma\ \|\nabla\Psi\|.}$$

Example of a level set flow:

$$\Psi_t = \left(\beta_0 + \beta_1 \mathrm{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right)\right) \|\nabla\Psi\| \,.$$

- $\Psi_t = \phi(x,y)\left(\beta_0 + \beta_1 \text{div}\left(\frac{\nabla\Psi}{\|\nabla\Psi\|}\right)\right)\|\nabla\Psi\|$.

Caselles *et al.* 93, Malladi *et al.* 94, Tek and Kimia 95

- $\Phi(x,y) : \mathbf{R}^2 \to \mathbf{R}^+$ has local minima at "edges":

$$\phi = \frac{1}{1 + \|\nabla G_\sigma * I\|^n}$$

- Euclidean metric: $ds^2 = dx^2 + dy^2$.
- Conformal metric: $ds_\phi^2 = \phi^2(dx^2 + dy^2)$.
- $\phi : \mathbf{R}^2 \to \mathbf{R}^+$ is differentiable

ϕ-Length functional:

$$L_\phi(t) = \int_0^1 \phi \, \|\frac{\partial \mathcal{C}}{\partial p}\| \; dp.$$

First variation and integration by parts:

$$L'_\phi(t) = -\int_0^{L_\phi(t)} \langle \frac{\partial \mathcal{C}}{\partial t}, \phi\kappa\mathcal{N} - \langle \nabla\phi, \mathcal{N}\rangle\rangle \, ds.$$

$$\begin{aligned} \mathcal{C}_t &= (\phi\kappa - \langle \nabla\phi, \mathcal{N}\rangle)\,\mathcal{N}. \\ \Rightarrow \Psi_t &= \left(\phi\kappa + \langle \nabla\phi, \frac{\nabla\Psi}{\|\nabla\Psi\|}\rangle\right) \|\nabla\Psi\| \,. \end{aligned}$$

ϕ-Area functional:

$$A_\phi(t) = -\frac{1}{2}\int_0^{L(t)} \phi\langle \mathcal{C}, \mathcal{N}\rangle \, ds = -\frac{1}{2}\int_0^1 \phi \left\langle \mathcal{C}, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \right\rangle dp.$$

Taking the first variation:

$$\begin{aligned} -2A'_\phi(t) &= \underbrace{\int_0^1 \phi_t \left\langle \mathcal{C}, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \right\rangle dp}_{I_1} \\ &+ \underbrace{\int_0^1 \phi \left\langle \mathcal{C}_t, \begin{pmatrix} -y_p \\ x_p \end{pmatrix} \right\rangle dp}_{I_2} \\ &+ \underbrace{\int_0^1 \phi \left\langle \mathcal{C}, \begin{pmatrix} -y_{pt} \\ x_{pt} \end{pmatrix} \right\rangle dp}_{I_3}, \end{aligned}$$

$$
\begin{aligned}
I_1 &= \int_0^L \langle \nabla\phi, \mathcal{C}_t \rangle \langle \mathcal{C}, \mathcal{N} \rangle \, ds, \\
I_2 &= \int_0^L \phi \langle \mathcal{C}_t, \mathcal{N} \rangle \, ds, \\
I_3 &= \int_0^1 \langle \phi \, \mathcal{C}, \begin{pmatrix} -y_{pt} \\ x_{pt} \end{pmatrix} \rangle \, dp.
\end{aligned}
$$

Use integration by parts on I_3 ...

$$
A'_\phi(t) = -\int_0^L \langle \mathcal{C}_t, \left(\phi + \frac{1}{2} \langle \mathcal{C}, \nabla\phi \rangle \right) \mathcal{N} \rangle ds.
$$

$$
\begin{aligned}
\mathcal{C}_t &= \left(\phi + \frac{1}{2} \langle \mathcal{C}, \nabla\phi \rangle \right) \mathcal{N} \\
\Rightarrow \Psi_t &= \left(\phi + \frac{1}{2} \langle \mathcal{C}, \nabla\phi \rangle \right) \|\nabla\Psi\| \\
\Psi_t &= \frac{1}{2} \text{div} \left(\begin{pmatrix} x \\ y \end{pmatrix} \phi \right) \|\nabla\Psi\| \, .
\end{aligned}
$$

- ϕ surface area minimizing flow:

$$\Psi_t = \left\{ \phi \ \text{div} \left(\frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \langle \nabla \phi, \frac{\nabla \Psi}{\|\nabla \Psi\|} \rangle \right\} \|\nabla \Psi\|$$

- ϕ volume minimizing flow:

$$\Psi_t = \frac{1}{3} \ \text{div} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \phi \right) \|\nabla \Psi\| \ .$$