Lecture 6:

Derivation of Fundamental Equation of Radiometric Image Formation.

$\rightarrow (d)$

Optical Parameters: lens type, focal length, field of view, angular apertures.

Photometric Parameters: have to do with models of light energy reaching the sensor:
- type, intensity, direction of illumination
- reflectance properties of viewed surfaces
- sensor to photoreceptor.

Geometric Parameters: determine the position on which a 3-D (world) point is projected

- type of projections
- position and orientation of camera
- persp. distortions
- discrete nature of photoreceptors (layout)
- quantization of intensity

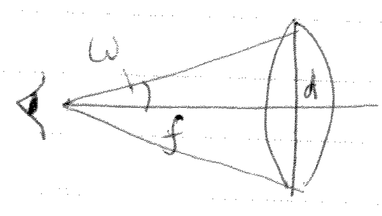
Fundamental Eq of Thin Lenses (Fig 2.4)

$$\boxed{\frac{1}{\hat{Z}} + \frac{1}{\hat{z}} = \frac{1}{f}}$$

- wider aperture to gather more light
- focus to get a sharp image

Field of View is half of the angle subtended by the lens diameter as seen from the focus

$$\omega = \tan^{-1}\left(\frac{d}{2f}\right)$$




$\hat{Z} = Z$, $\hat{z} = Z'$

sometimes *radiance* refers to the energy radiated from a surface (emitted or reflected), whereas *irradiance* refers to the energy incident on a surface.

***Surface Reflectance and Lambertian Model.*** A model of the way in which a surface reflects incident light is called a *surface reflectance model*. A well-known one is the *Lambertian model*, which assumes that each surface point appears equally bright from all viewing directions. This approximates well the behavior of rough, nonspecular surfaces, as well as various materials like matte paint and paper. If we represent the direction and amount of incident light by a vector $\mathbf{I}$, the scene radiance of an ideal Lambertian surface, $L$, is simply proportional to the dot product between $\mathbf{I}$ and the unit normal to the surface, $\mathbf{n}$:

SCENE RADIANCE

$$L = \rho \mathbf{I}^\top \mathbf{n} \tag{2.4}$$

with $\rho > 0$ a constant called the surface's *albedo*, which is typical of the surface's material. We also assume that $\mathbf{I}^\top \mathbf{n}$ is *positive*; that is, the surface faces the light source. This is a necessary condition for the ray of light to reach $\mathbf{P}$. If this condition is not met, the scene radiance should be set equal to 0.

surface and light source

We will use the Lambertian model in several parts of this book; for example, while analyzing image sequences (Chapter 8) and computing shape from shading (Chapter 9). Intuitively, the Lambertian model is based on the exact cancellation of two factors. Neglecting constant terms, the amount of light reaching *any* surface is always proportional to the cosine of the angle between the illuminant and the surface normal $\mathbf{n}$ (that is, the effective area of the surface as seen from the illuminant direction). According to the model, a Lambertian surface reflects light in a given direction $\mathbf{d}$ proportionally to the cosine of $\theta$, the angle between $\mathbf{d}$ and $\mathbf{n}$. But since the surface's area seen from the direction $\mathbf{d}$ is inversely proportional to $\cos\theta$, the two $\cos\theta$ factors cancel out and do not appear in (2.4).

***Linking Surface Radiance and Image Irradiance.*** Our next task is to link the amounts of light reflected by the surfaces, $L$, and registered by the imaging sensor, $E$.

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**Assumptions and Problem Statement**

Given a thin lens of diameter $d$ and focal length $f$, an object at distance $Z$ from the lens, and an image plane at distance $Z'$ from the lens, with $f$, $Z$, and $Z'$ as in (2.1), find the relation between image irradiance and scene radiance.

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In order to derive this fundamental relation, we need to recall the geometric notion of *solid angle*. The solid angle of a cone of directions is the area cut out by the cone on the unit sphere centered in the cone's vertex. Therefore, the solid angle $\delta\omega$ subtended by a small, planar patch of area $\delta A$ at distance $r$ from the origin (Figure 2.6) is

solid angle

$$\delta\omega = \frac{\delta A \cos\psi}{r^2} \tag{2.5}$$

1 define $\delta\omega$ and ask why $r^2$?

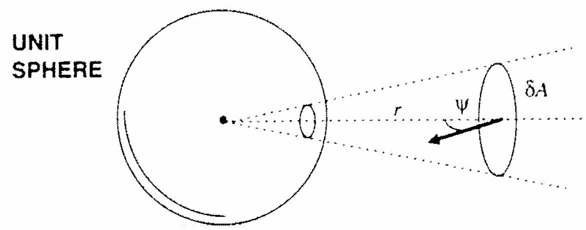


Figure 2.6 The definition of solid angle.

with $\psi$ the angle between the normal to $\delta A$ and the ray that points from the origin to $\delta A$. The factor $\cos\psi$ ensures the proper foreshortening of the area $\delta A$ as seen from the origin.

We now write the image irradiance at an image point, **p**, as the ratio between $\delta P$, the power of light over a small image patch, and $\delta I$, the area of the small image patch:

2) Define image irradiance

$$E = \frac{\delta P}{\delta I}. \tag{2.6}$$

If $\delta O$ is the area of a small surface patch around **P**, $L$ the scene radiance at **P** in the direction toward the lens, $\Delta\Omega$ the solid angle subtended by the lens, and $\theta$ the angle between the normal to the viewed surface at **P** and the principal ray (Figure 2.7), the power $\delta P$ is given by $\delta O L \Delta\Omega$ (the total power emitted in the direction of the lens) multiplied by $\cos\theta$ (the foreshortening of the area $\delta O$ as seen from the lens):

3) Explain all the terms in 2.7

$$\delta P = \delta O L \Delta\Omega \cos\theta. \tag{2.7}$$

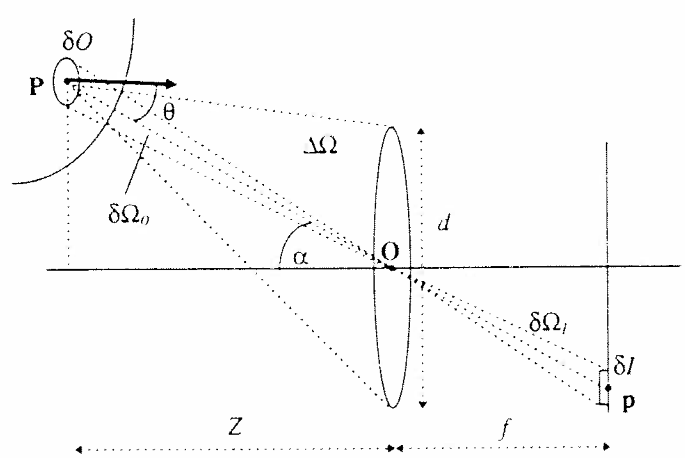


Figure 2.7 Radiometry of the image formation process.

Combining (2.6) and (2.7), we find

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$$E = L\Delta\Omega \cos\theta \frac{\delta O}{\delta I}. \tag{2.8}$$

We still need to evaluate $\Delta\Omega$ and $\delta O/\delta I$. For the solid angle $\Delta\Omega$ (Figure 2.7), (2.5) with $\delta A = \pi d^2/4$ (lens area), $\psi = \alpha$ (angle between the principal ray and the optical axis), and $r = Z/\cos\alpha$ (distance of $P$ from the lens center) becomes

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$$\Delta\Omega = \frac{\pi}{4} d^2 \frac{\cos^3\alpha}{Z^2}. \tag{2.9}$$

For the solid angle $\delta\Omega_I$, subtended by a small image patch of area $\delta I$ (see Figure 2.7), (2.5) with $\delta A = \delta I$, $\psi = \alpha$, and $r = f/\cos\alpha$ gives

$$\delta\Omega_I = \frac{\delta I \cos\alpha}{(f/\cos\alpha)^2}. \tag{2.10}$$

7 Similarly, for the solid angle $\delta\Omega_O$ subtended by the patch $\delta O$ on the object side, we have

$$\delta\Omega_O = \frac{\delta O \cos\theta}{(Z/\cos\alpha)^2}. \tag{2.11}$$

It is clear from Figure 2.7 that $\delta\Omega_I = \delta\Omega_O$; hence, their ratio is 1, so that dividing (2.11) by (2.10) we obtain

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$$\frac{\delta O}{\delta I} = \frac{\cos\alpha}{\cos\theta}\left(\frac{Z}{f}\right)^2 \tag{2.12}$$

Ignoring energy losses within the system, and plugging (2.9) and (2.12) into (2.8), we finally obtain the desired relation between $E$ and $L$.

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**The Fundamental Equation of Radiometric Image Formation**

Plug 8 and 6 in 5

image irradiance

$$E(\mathbf{p}) = L(\mathbf{P}) \frac{\pi}{4} \left(\frac{d}{f}\right)^2 \cos^4\alpha. \tag{2.13}$$

scene radiance

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Equation (2.13) says that *the illumination of the image at* **p** *decreases as the fourth power of the cosine of the angle formed by the principal ray through* **p** *with the optical axis*. In the case of small angular aperture, this effect can be neglected; therefore, the image irradiance can be regarded as uniformly proportional to the scene radiance over the *whole* image plane.

The nonuniform illumination predicted by (2.13) is hard to notice in ordinary images, because the major component of brightness changes is usually due to the spatial gradient of the image irradiance. You can try a simple experiment to verify the effect predicted by (2.13) by acquiring an image of a Lambertian surface illuminated by diffuse light (see Exercise 2.2).