308-558B: Fundamentals of Computer Vision
Shape from Shading

Mani Ghasemlou

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Shape from Shading

Shape from shading is a means by which *shape* is reconstructed from a single intensity image. It uses certain elements, such as patterns of lighting and shading, as a means of inferring the shape of the objects in view. Shape from shading has been shown to be reasonably successful in specific applications, such as astronomy.

The reflectance map

There is a specific fundamental equation that relates image intensity to the slope of a surface. To begin to understand this equation, we must first give the simplest example of a lit surface, the *Lambertian Surface equation*

$$L(\mathbf{P}) = \rho \mathbf{i}^\top \mathbf{n} \tag{1}$$

gives the radiance L at point $\mathbf{P}$, with surface albedo ρ, illumination direction $\mathbf{i}$, and surface normal $\mathbf{n}$.

To express this equation in terms of the illumination of the surface normal of a point for any albedo and illuminant direction, we have

$$R_{\rho,\mathbf{i}} = \rho \mathbf{i}^\top \mathbf{n} \tag{2}$$

This particular equation describes an example of a reflectance map.

The fundamental equation

In chapter 2 of [1], the *image irradiance equation* was defined as

$$E(\mathbf{p}) = L(\mathbf{P}) \frac{\pi}{4} \left(\frac{d}{f}\right)^2 \cos^4 \alpha \tag{3}$$

where $E(\mathbf{p})$ denoting the brightness on the image plane at coordinate $\mathbf{p}$. Given the above, two assumptions are made, along with two other assumptions for convenience:

1. The constant terms in (3) are ignore, with the assumption that the optical system has been calibrated accordingly.
2. All the points on the visible portions of the image are receiving direct light.
3. The visible surface is far away from the viewer.
4. The visible surface may be described as the function $Z = Z(X, Y)$.

These assumptions allow us to significantly simplify (3). Now, with (1) and (2) in mind, we can say

$$E(\mathbf{p}) = R_{\rho,\mathbf{i}}(\mathbf{n}) \tag{4}$$

This is the *fundamental equation of shape from shading.*

The above assumptions also simplify the problem of shape from shading. Assumption 3 and 4 allow us to adopt the weak-perspective camera model and to describe x and y by

$$x = f \frac{X}{Z_0} \tag{5}$$

and

$$y = f\frac{Y}{Z_0} \tag{6}$$

where Z_0 is the average distance of the surface $Z(X,Y)$ and the image plane. Plugging (5) and (6) into Z, we have

$$Z = Z(x,y) \tag{7}$$

Taking x and y partial derivatives of (an arbitrary) normal vector on the surface Z, we get $[1, 0, \frac{\delta Z}{\delta x}]^\top$ and $[0, 1, \frac{\delta Z}{\delta y}]^\top$. Armed with the knowledge that these two vectors are on the tangent plane of the surface at (x,y), we can describe $\mathbf{n}$ as their normalized vector product:

$$\mathbf{n} = \frac{1}{\sqrt{1+\frac{\delta Z}{\delta x}^2+\frac{\delta Z}{\delta y}^2}}\left[-\frac{\delta Z}{\delta x}, -\frac{\delta Z}{\delta y}, 1\right]^\top \tag{8}$$

Now, with (4) and (8), we can say:

$$E(\mathbf{p}) = \frac{\rho}{\sqrt{1+\frac{\delta Z}{\delta x}^2+\frac{\delta Z}{\delta y}^2}}\mathbf{i}^\top\left[-\frac{\delta Z}{\delta x}, -\frac{\delta Z}{\delta y}, 1\right] \tag{9}$$

This equation can be considered one of the basic means of solving *shape from shading*. It is often difficult to compute, however, particularly because of two reasons:

1. Computing the x and y partial derivatives for the entire surface is not trivial.
2. ρ, $\mathbf{i}$, are generally unknown.

Reason two leads us to the final section.

Estimating ρ and $\mathbf{i}$

Often, given an intensity image, the albedo (ρ) and illumination direction ($\mathbf{i}$) are unknown. Therefore, we must derive ways to estimate these two values.

Assumptions

Given a range of tilt and slant angles (with *tilt and slant* described by Fig.1), we have

$$\mathbf{n} = [\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta]^\top \tag{10}$$

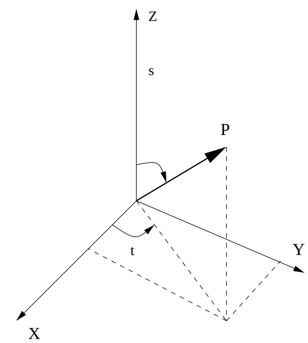

Figure 1: tilt (t) and slant (s)

where $\alpha \in [0, 2\pi]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$. Given this, we may describe the distribution of normal vectors ($\mathcal{P}$) as:

$$\mathcal{P}(\alpha,\beta) = \frac{\cos\beta}{2\pi} \tag{11}$$

Section 9.3.2 of [1] describes a three-step method for estimating ρ and $\mathbf{i}$:

1. Using 11, precompute the averages of the image brightness and its derivatives.
2. Evaluate the corresponding averages from the image brightness.
3. Enter the results into equations that can be solved for ρ and $\mathbf{i}$.

Given slant σ and tilt τ, we have:

$$\mathbf{i} = [\cos\tau\sin\sigma, \sin\tau\sin\sigma, \cos\sigma]^\top \tag{12}$$

Since the image brightness, as a function of α and β, can be described as:

$$E(\alpha,\beta) = \rho(\cos\alpha\sin\beta\cos\tau\sin\sigma + \sin\alpha\sin\beta\sin\tau\sin\sigma + \cos\beta\cos\sigma) \tag{13}$$

we can write the average $< E >$ as

$$< E > = \int_0^{2\pi} d\alpha \int_0^{\frac{\pi}{2}} d\beta \mathcal{P}(\alpha,\beta)E(\alpha,\beta) \tag{14}$$

Simplifying the above, we get

$$< E > = \frac{\pi}{4}\rho\cos\sigma \tag{15}$$

We will also need the average of the square of the image brightness, $< E^2 >$, which is given by

$$< E^2 >= \frac{1}{6}\rho^2 \left(1 + 3\cos^2\sigma\right) \tag{16}$$

Given (15) and (16), we can estimate ρ and σ:

$$\rho = \frac{\gamma}{\pi} \tag{17}$$

and

$$\cos\sigma = \frac{4 < E >}{\gamma} \tag{18}$$

where $\gamma = \sqrt{6\pi^2 < E^2 > -48 < E >^2}$. Finally, we can recover τ using:

$$\tan\tau = \frac{< \hat{E}_y >}{< \hat{E}_x >} \tag{19}$$

where $\hat{E}_y >$ and $\hat{E}_x >$ are the vertical and horizontal components of the direction of the image spatial gradient: $\left[\hat{E}_x, \hat{E}_y\right]^\top = \left(E_x^2 + E_y^2\right)\left[E_x, E_y\right]^\top$.

Given these estimated values, we can plug them into 12 to obtain a good value for $\mathbf{i}$. We also have a good estimate of τ. We can now plug these values into 9 to solve the *shape from shading problem.*

References

[1] E.Trucco, A.Verri, "Introductory Techniques for 3-D Computer vision". Prentice-Hall, New Jersey (1998).