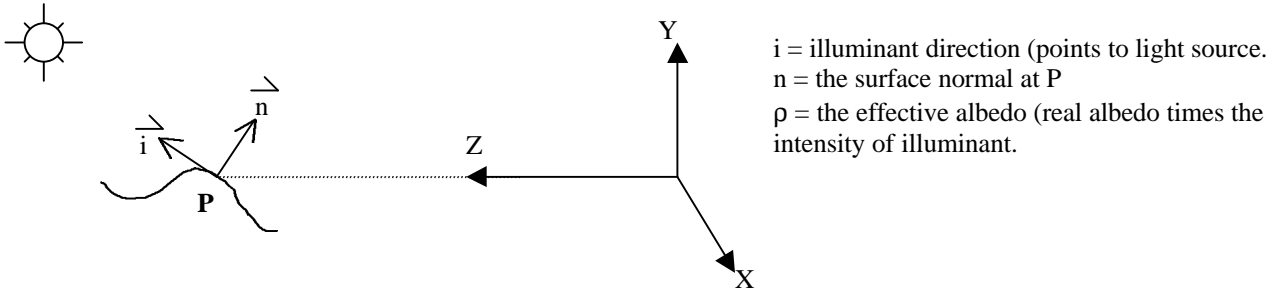


Shape From Shading

SFS uses the pattern of lights and shades in an image to infer the shape of the surfaces in view.

Lambertian Surfaces – uniformly illuminated Lambertian surface appears equally bright from all viewpoints.



Recall, Scene Radiance is: $L(P) = \rho < i, n >$ (Under the Lambertian assumption ρ is a constant)

Now assume the “Weak Perspective” model with: $x = (f/Z)X$ and $y = (f/Z)Y$

Let $R_{\rho,i}(n) = \rho < i, n >$ be the reflectance map of the surface

Image Irradiance Equation: (denoting with $p = [x, y]^T$ the image of P) $E(p) = L(P) (\Pi/4) (d/f)^2 \cos^4 \alpha$

$E(p)$ – the brightness measured on the image plane at p.

Assume: 1) neglect the constant term $(\Pi/4)$, and assume system has been calibrated to offset the $\cos^4 \alpha$
 2) All visible points of the surface receive direct illumination.

Therefore: $E(p) = R_{\rho,i}(n)$ – **FUNDAMENTAL EQUATION OF SHAPE FROM SHADING**

Further assume 3) The visible surface is far away from the viewer
 4) and the visible surface can be described as $Z = Z(X, Y)$

This enables us to adopt the weak-perspective camera model and obtain:

$$x = f(X/Z_0) \quad y = f(Y/Z_0)$$

Z_0 – the average distance of the surface from the image plane.

$Z = Z(x, y)$ Z is a function of the (x, y) coordinates on the image plane

A point P is now described by $(x, y, Z(x, y))$

Now compute the surface slopes by taking the x and y partial derivatives of $[x, y, Z(x, y)]^T$

$$\text{Let } \vec{V}_1 = [1, 0, \partial Z / \partial x]^T$$

$$\text{and } \vec{V}_2 = [0, 1, \partial Z / \partial y]^T$$

$$\vec{V}_1 \times \vec{V}_2 = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}$$

where $p = \partial Z / \partial x$
 and $q = \partial Z / \partial y$

$$= +\vec{i}(-p) - \vec{j}(q) + \vec{k}(1) =$$

$$\frac{\begin{bmatrix} -p \\ -q \\ 1 \end{bmatrix}}{\sqrt{1 + p^2 + q^2}}$$

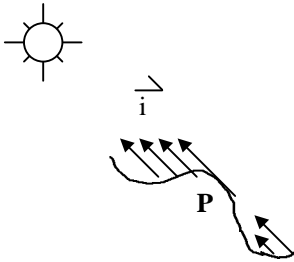
\vec{n} unit vector in the normal direction

Now plug \vec{n} into $\rho \langle \vec{i}, \vec{n} \rangle = E(p)$ to obtain:

$$E(p) = \frac{\rho}{\sqrt{1+p^2+q^2}} \langle \vec{i}, [-p \ -q \ 1] \rangle$$

Starting point of many shape from shading techniques

Problem: for certain illuminants, some image locations violate the assumption that the entire surface receives direct illumination, as in the diagram below:



Therefore if the image brightness would become negative, simply set it to 0.

$$E(x,y) = \max\{0, \frac{\rho}{\sqrt{1+p^2+q^2}} \vec{i}^T [-p, -q, 1] \}$$

The number of unknowns here may seem to suggest that this equation does not provide enough constraints to reconstruct p and q at all pixels.

Summary of assumptions:

- 1) The acquisition system is calibrated so that the image Irradiance, $E(p)$ equals the scene radiance, $L(P)$, with $p=[x,y]^T$ image of the 3D point $P=[X,Y,Z]^T$
- 2) All the visible surface points receive direct illumination
- 3) The surface is imaged under weak perspective
- 4) The optical axis is the Z axis of the camera, and the surface can be parameterized as $Z=Z(x,y)$

Now, our problem is to determine albedo and illuminant direction from a single image of the surface making further assumptions, namely:

- 5) The surface imaged is Lambertian, and
- 6) The direction of the surface normals are distributed uniformly in 3D space

Method for Recovering Albedo and Illuminant Direction

- 1 Compute averages of image brightness, $\langle E \rangle$, and its square, $\langle E^2 \rangle$.
- 2 Compute the spatial image gradient $[E_x, E_y]^T$. Compute the average of both components $\langle E_y \rangle$ and $\langle E_x \rangle$
- 3 Estimate ρ , $\cos \sigma$, and $\tan \tau$ via the equations:

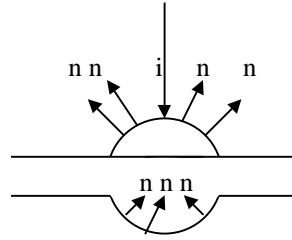
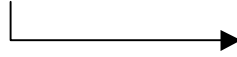
$$\rho = \Upsilon / \Pi \quad \cos \sigma = 4 \langle E \rangle / \Upsilon \quad \text{and} \quad \tan \tau = \langle \hat{E}_y \rangle / \langle \hat{E}_x \rangle$$
 where $\Upsilon = (6\Pi^2 \langle E^2 \rangle - 48 \langle E \rangle^2)^{1/2}$

Even under Lambertian assumptions, $E(p) = \frac{\rho}{\sqrt{1+p^2+q^2}} \langle \vec{i}, [-p \ -q \ 1] \rangle$ requires a nonlinear

partial differential equation in the presence of uncertain boundary conditions which could lead to an ill posed condition.

$$R(p,q) = E = \frac{\rho}{\sqrt{1+p^2+q^2}} \quad \langle \vec{i}, [-p, -q, 1] \rangle \text{ is an ill-posed question}$$

Ill-posed question: 1) Solution may not exist
2) Solution may not be unique



3) May not vary continuously with input data

one way to circumvent conditions 1 and 2 is to allow for some small deviation between image brightness and the reflectance map, and enforce a smoothness constraint.

ie, minimize the functional: $\mathcal{E} = \int \partial x \partial y \{ (E(x,y) - R(p,q))^2 + \lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2) \}$

\swarrow closeness to data term \searrow smoothness term

- smoothness constraint given by the sum of spatial derivatives of p and q
- λ is always positive and controls the relative influence of the two terms in the minimization process
- large λ equates to a very smooth solution, not close to the data
- small λ equates to a more irregular solution, closer to the data

For a functional \mathcal{E} which depends on two functions, p and q of two real variables x and y and on their first order spatial derivatives, we have the **Euler-Lagrange** equations:

$$\frac{\partial \mathcal{E}}{\partial p} - \frac{\partial}{\partial x} \frac{\partial \mathcal{E}}{\partial p_x} - \frac{\partial}{\partial y} \frac{\partial \mathcal{E}}{\partial p_y} = 0$$

and

$$\frac{\partial \mathcal{E}}{\partial q} - \frac{\partial}{\partial x} \frac{\partial \mathcal{E}}{\partial q_x} - \frac{\partial}{\partial y} \frac{\partial \mathcal{E}}{\partial q_y} = 0$$

Note: since $R(p,q)$ depends on p and q, but NOT p_x, p_y, q_x, q_y
 $E(x,y)$ does not depend on p or q, NOR p_x, p_y, q_x, q_y

Thus, the **Euler-Lagrange** Equations become:

$$-2(E - R) \frac{\partial R}{\partial p} - 2\lambda p_{xx} - 2\lambda p_{yy} = 0$$

and

$$-2(E - R) \frac{\partial R}{\partial q} - 2\lambda q_{xx} - 2\lambda q_{yy} = 0$$

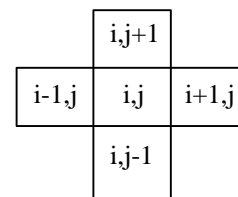
or in simplified form:

$\Delta p = -1/\lambda (E - R) \partial R / \partial p$	$\Delta q = -1/\lambda (E - R) \partial R / \partial q$
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p,q unknown

Δp and Δq denoting the Laplacian of p and q (ie $\Delta p = p_{xx} + p_{yy}$)

Solving this is easier in the discrete case as opposed to the Continuous case, thus denote $p_{i,j}$ and $q_{i,j}$ as the average of 4 neighbors of p and q at location i,j.

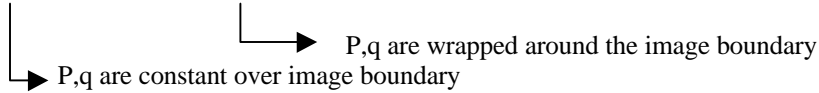


by letting, $\bar{p}_{i,j} = p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} / 4$ **and** $\bar{q}_{i,j} = q_{i+1,j} + q_{i-1,j} + q_{i,j+1} + q_{i,j-1} / 4$

we can create an iterative scheme, starting at an initial configuration for $p_{i,j}$ and $q_{i,j}$

$$\begin{array}{l} \bar{p}_{i,j}^{k+1} = p_{i,j}^k + 1/4\lambda (E - R) \partial R / \partial p |^k \\ \bar{q}_{i,j}^{k+1} = q_{i,j}^k + 1/4\lambda (E - R) \partial R / \partial q |^k \end{array} \quad \text{for } k = 0 \text{ to } N$$

In many cases, the boundary conditions are unknown, thus an attempt must be made to impose natural boundary conditions or cyclic boundary conditions



Conclusion: Algorithm for Shape From Shading

Input: image of unknown surface Z

Reflectance Map of surface

ρ - Surface Albedo

i - Illuminant (intensity and direction)

- surface slopes are assumed to wrap around the image boundaries
- until a suitable stopping criterion is met, iterate the following

1.) Update p and q via $\bar{p}_{i,j}^{k+1} = p_{i,j}^k + 1/4\lambda (E - R) \partial R / \partial p |^k$
and
 $\bar{q}_{i,j}^{k+1} = q_{i,j}^k + 1/4\lambda (E - R) \partial R / \partial q |^k$

- 2.) Compute FFT of p and q, estimate Z and p' and q' .
 Set $p = p'$ and $q = q'$

Output: is an estimate of Z, p and q.