Infinite Horizon LQG Graphon Mean Field Games: Explicit Nash Values and Local Minima

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Abstract

In this study, we generalize the analysis of infinite horizon linear quadratic Gaussian (LQG) Mean Field Games within the framework of Graphon Mean Field Games (GMFG) introduced in Caines and Huang (2018) for finite horizons. Graphon Mean Field Games (GMFGs) are non-uniform generalizations of Mean Field Games where the non-uniformity of agents is characterized by the nodes on which they are located in a network. Under mild assumptions on the structure of the network and parameters of the game, we obtain for almost every node, an explicit analytical expression for the Nash values (i.e. the cost at equilibrium). With additional assumptions, we provide sufficient conditions for nodes to have locally minimal Nash values. We illustrate the results for the uniform attachment network.

Keywords: Graphons, Mean Field Games, Nash Values, Infinite Horizon

1. Introduction

Graphon Mean Field Games, introduced in Caines and Huang (2018), Caines and Huang (2019) and developed in Caines and Huang (2021), are a generalization of Mean Field Games which were introduced in Huang et al. (2003, 2004, 2006, 2007); Lasry and Lions (2006a,b) to the case where populations of agents are located at the nodes of large undirected graphs. These large networks are studied via a graph limit theory, where adjacency matrices, $(g^n_{i,j})_{i,j=1:n}, n \geq 1$, are shown to converge, in the so-called cut metric to bounded measurable function limits g, called graphons, where

$$g : [0,1] \times [0,1] \to [-1,1] \quad (\alpha,\beta) \mapsto g(\alpha,\beta).$$

See Lovász and Szegedy (2006); Borgs et al. (2008, 2012); Lovász (2012).

Recent work on Mean Field Games and Control with network-coupled populations of agents includes, for example, Delarue (2017); Caines and Huang (2018); Gao and Caines (2019); Parise and Ozdaglar (2019); Caines and Huang (2021); Foguen-Tchuendom et al. (2021); Gao et al. (2021b,a); Lacker and Soret (2022); Carmona et al. (2022); Aurell et al. (2022a,b).

The current paper focuses on establishing explicit analytical results on the Nash values and nodes with locally minimal Nash values based on assumptions on the initial conditions and graphon properties. When nodes may be construed to have physical locations, the search for nodes with minimal Nash values may be formulated within the recently developed theory of (Riemannian space) embedded graph limits (see Caines (2022)). The extension of the class of graphon functions to measures permitted by this theory will not be employed in this paper, however the embedded limit network theory facilitates any assumption of multidimensional arguments and, when required, the differentiablility with respect to location parameters of the limit functions g; this case occurs in Proposition 8, Section 6, below. We emphasize, however, that whichever background formulation of graph limits is adopted, the graph limit function g appearing in this work is always assumed to be a bounded measurable function.

The contributions over the earlier versions of this work in Foguen-Tchuendom et al. (2022b,a), are as follows: (i) the infinite rank graphon case is covered when proving existence and uniqueness of the graphon mean field; (ii) the consideration of initial means which depend linearly upon the nodes, and (iii) the inclusion of numerical illustrations indicating that the relationship between centralities of the system graph and Nash value local minima previously detected in Foguen-Tchuendom et al. (2022a) depends on the homogeneity

of the initial means. This paper merges and improves upon the results first published in Foguen-Tchuendom et al. (2022a,b), and both papers build upon mean field games with cost localities studied in Huang et al. (2010). The differences between Huang et al. (2010) and the current paper are that in Huang et al. (2010) each node is assumed to be associated with an individual agent and graphons are not employed.

The paper is organized as follows. In Section 2, after some preliminaries, we introduce the infinite horizon GMFGs. In Section 3, we characterize the solvability of the infinite horizon GMFGs by the solvability of an auxiliary system of infinite horizon forward-backward ordinary differential equations (FBODEs). In Section 4, we show that the auxiliary system of infinite horizon FBODEs has a unique solution. In Section 5, we obtain explicit closed form Nash values. Section 6 we provide sufficient conditions for nodes to be local minima of the Nash values. In Section 7 two numerical illustrations when the graphon is the uniform attachment graphon. Section 8 concludes and indicates open questions.

2. Infinite Horizon Large Games On Networks

2.1. Preliminaries

In this subsection, we introduce infinite horizon GMFGs as limit versions of infinite horizon games with a large but finite number, N, of agents denoted, $\{\mathcal{A}_i, 1 \leq i \leq N < \infty\}$, which are distributed over a finite network of n nodes with edges represented by the adjacency matrix $(g^n_{i,j})_{i,j=1:n}$ whose entries take values in [0,1] and describe the strength of the connection between two given nodes in the network.

Let $\{w^i_t,\ t \geq 0,\ i = 1, ..., N\}$ be a collection of independent Brownian motions defined on a probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ satisfying the usual conditions such that $\{\mathcal{F}_t,\ t \geq 0\}$ is right-continuous and $\mathcal{F}_0$ contains all the null events. The state of the agents evolves according to a set of N controlled linear stochastic differential equations (SDEs) over the infinite horizon. For each agent $\mathcal{A}_i$, its state denoted $x^i(\cdot) \in R$ is a solution to the SDE,

$$dx^i_t = \left(ax^i_t + bu^i_t\right)dt + \sigma dw^i_t,\ \ \forall t \geq 0, \tag{1}$$

where $u^i(\cdot) \in R$ denotes the agent's $\mathcal{A}_i$ control input.

We assume that at each node $l \in \{1, ..., n\}$ there is a cluster of agents denoted C_l such that the total number of agents satisfies $N = \sum_{l=1}^n |C_l|$. We assume that the initial state of an agent $\mathcal{A}_i$ is $x^i_0 \sim \mathcal{N}(m^l, \nu^2)$, if $\mathcal{A}_i$ lies in cluster C_l, $l \in \{1, .., n\}$, and that the real coefficients a, b, m^l with $l \in \{1..., n\}$, $\nu > 0$, $\sigma \geq 0$ are known.

For each agent $\mathcal{A}_i$ in cluster C_k, the coupling terms, called the nodal network mean fields, governing its interaction with other agents over the network for all $t \geq 0,\ i \in \{1, \cdots, N\},\ k \in \{1, \cdots, N\}$, are

$$z^{k,n}_t = \frac{1}{n}\sum_{l=1}^n g^n_{k,l}\frac{1}{|C_l|}\sum_{j\in C_l} x^j_t. \tag{2}$$

The family of nodal network mean fields, $(z^{k,n}_t)_{t\in[0,\infty),k\in\{1...n\}}$ relies on the sectional information $g^n_{k,\bullet}$ available to an agent $\mathcal{A}_i$ in cluster C_k. This sectional information represents the understanding of the network interactions from the point of view of agents in cluster $C_k,\ k \in \{1, ..., n\}$. From the point of view of any agent $\mathcal{A}_i$ in any cluster C_k, all individuals residing in cluster C_k have symmetric interactions and their average states generates an overall impact for that cluster. These averages are called the cluster mean fields and are averaged again over the whole network via the nodal network mean fields.

Constrained by these dynamics and network interactions, each agent $\mathcal{A}_i$ chooses its control in order to minimize its quadratic cost functional

$$J^N(u^i, u^{-i}) := \mathbb{E}\int_0^\infty e^{-\rho t}[r(u^i_t)^2 + (x^i_t - z^{k,n}_t)^2]dt, \tag{3}$$

where $1 \leq i \leq N$, $\rho > 0$, $r > 0$, and u^{-i} denotes the controls of all agents except agent $\mathcal{A}_i$. We assume that all agents chose their controls from the space

$$\mathbb{A} := \{u : \Omega \times [0, \infty) \mapsto \mathbb{R} \mid u \text{ is } \mathbb{F} - \text{progressively measurable and } \mathbb{E}\int_0^\infty e^{-\rho t}|u(t)|^2 dt < \infty\}.$$

2.2. Infinite Horizon LQG GMFGs

For these infinite horizon games on networks, we are interested in studying the Nash equilibrium when both the number of agents and number of nodes are very large, so we recall its definition.

Definition 1 (Nash Equilibrium)**.** *A collection of controls, $(u^{i*})_{i=1}^N \in \mathbb{A}^N$, is called a Nash equilibrium, if and only if, any unilateral deviation from $u^{i*} \in \mathbb{A}$ to any other control $u^i \in \mathbb{A}$ does not yield a lower cost, i.e.*

$$J^N_i(u^{i*}, u^{-i*}) \leq J^N_i(u^i, u^{-i*}),\ \forall i = 1, ..., N,\ \forall u^i \in \mathbb{A}. \tag{4}$$

Finding a Nash equilibrium in games on networks with finite number of agents and nodes gets increasingly complex as both the cluster size and the network size increases. When the network supporting the interaction between the agents is uniform (i.e. fully symmetric),

Mean Field Games (MFGs) are satisfactory to deal with this complexity. In the case of non-uniform dense networks whose limits are characterized by graphons, GMFGs generalize MFGs to investigate these games in the double limit, $n \to \infty$ and $\min_{l=1:n} |C_l| \to \infty$ (observe that it implies that the number of agents, denoted by $N = \sum_{l=1}^{n} |C_l|$, goes to infinity). GMFGs are asymptotic versions of the sequence of these games on networks.

At all node $\alpha \in [0,1]$ on the graphon, there is a representative agent, denoted $\mathcal{A}_\alpha$, whose state's evolution solves the SDE, for all $t \geq 0$

$$dx_t^\alpha = (ax_t^\alpha + bu_t^\alpha)\,dt + \sigma dw_t^\alpha,\ x_0^\alpha \sim \mathcal{N}(m^\alpha, \nu^2). \tag{5}$$

Agent $\mathcal{A}_\alpha$ aims at minimizing the quadratic cost,

$$J(u^\alpha, z^\alpha) := \mathbb{E}\int_0^\infty e^{-\rho t}[r(u_t^\alpha)^2 + (x_t^\alpha - z_t^\alpha)^2]dt, \tag{6}$$

where $r, \rho > 0$ and, the nodal graphon mean field z_t^α is,

$$z_t^\alpha := \int_0^1 g(\alpha,\beta)\mathbb{E}[x_t^\beta]d\beta,\ \forall t \in [0,\infty),\ \forall \alpha \in [0,1]. \tag{7}$$

Nash equilibria for the above population and network limit Linear Quadratic Gaussian Graphon Mean Field Game (LQG GMFG) are found as follows:

1. Fix a two-parameter deterministic flow of graphon mean fields $\{z_t^\alpha,\ t \in [0,\infty),\ \alpha \in [0,1]\}$.
2. Find optimal controls, denoted by $u^{\alpha,o} := (u_t^{\alpha,o})_{t\in[0,\infty)} \in \mathbb{A}$, such that

$$\begin{aligned} J(u^{\alpha,o}, z^\alpha) &= \min_{u^\alpha \in \mathbb{A}} J(u^\alpha, z^\alpha) \\ &= \min_{u^\alpha \in \mathbb{A}} \mathbb{E}\int_0^\infty e^{-\rho t}[r(u_t^\alpha)^2 + (x_t^\alpha - z_t^\alpha)^2]dt \end{aligned} \tag{8}$$

subject to the dynamics, for all $t \geq 0$ and $\alpha \in [0,1]$

$$dx_t^\alpha = (ax_t^\alpha + bu_t^\alpha)\,dt + \sigma dw_t^\alpha,\ x_0^\alpha \sim \mathcal{N}(m^\alpha, \nu^2). \tag{9}$$

3. Show that the optimal state trajectories $\{x_t^{\alpha,o},\ t \in [0,\infty),\ \alpha \in [0,1]\}$, satisfy the consistency conditions, for all $\alpha \in [0,1],\ t \geq 0$,

$$z_t^\alpha = \int_0^1 g(\alpha,\beta)\mathbb{E}[x_t^{\beta,o}]d\beta. \tag{10}$$

3. Solvability of (8)-(9)-(10)

3.1. Infinite Horizon FBODEs

The solvability of this problem requires the solvability of control problems (8)-(9) and the consistency conditions (10) sequentially. To solve the control problems (8)-(9) we follow the approach exposed in Huang et al. (2007) and introduce the algebraic Riccati equation

$$\rho\pi = 2a\pi - \frac{b^2}{r}\pi^2 + 1, \quad r > 0,\ \rho > 0, \tag{11}$$

whose unique strictly positive solution is

$$\pi = \sqrt{\frac{r^2(\rho - 2a)^2}{4b^4} + \frac{r}{b^2}} - \frac{(\rho - 2a)\,r}{2b^2} > 0. \tag{12}$$

Let $L^2[0,1]$ the space of square integrable functions endowed with the inner product $\langle x, y\rangle = \int_0^1 x(\beta)y(\beta)d\beta$ and norm $\|x\|_2 := \sqrt{\langle x, x\rangle}$. Let $C_b\left([0,\infty); L^2[0,1]\right)$ be the space of bounded and continuous functions over the interval $[0,\infty)$, with values in $L^2[0,1]$ and endowed with the norm $\|x\|_{2,\infty} := \sup_{t\in[0,\infty)} \sqrt{\langle x, x\rangle}$.

Proposition 1. *Assume that there exist a function $s \in C_b\left([0,\infty); L^2[0,1]\right)$ solution to the offset Ordinary Differential Equation (ODE),*

$$\frac{ds_t^\alpha}{dt} = \left(-a + \frac{b^2}{r}\pi + \rho\right)s_t^\alpha + z_t^\alpha. \tag{13}$$

Then, there exist optimal control processes for the infinite horizon optimal control problems above, namely, for all $\alpha \in [0,1]$,

$$u_t^{\alpha,o} = -\frac{b}{r}(\pi x_t^{\alpha,o} + s_t^\alpha),\ \forall t \geq 0, \tag{14}$$

where the optimal states $(x_t^{\alpha,o})_{t\in[0,T]}$ are given by

$$\begin{aligned} dx_t^{\alpha,o} &= \left[\left(a - \frac{b^2}{r}\pi\right)x_t^{\alpha,o} - \frac{b^2}{r}s_t^\alpha\right]dt + \sigma dw_t^\alpha, \\ x_0^{\alpha,o} &\sim \mathcal{N}(m^\alpha, \nu^2). \end{aligned}$$

Proof 1. *The proof is a standard application of LQG tracking control theory. See Huang et al. (2007).*

Proposition 2. *Assume that there exist a function $q \in C_b\left([0,\infty); L^2[0,1]\right)$ solution to the ODE,*

$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2. \tag{15}$$

Then, the optimal costs are given, for all $\alpha \in [0,1]$, by

$$\begin{aligned} J(u^\alpha, z) &= \pi\mathbb{E}[(x_0^{\alpha,o})^2] + 2s_0^\alpha\mathbb{E}[x_0^{\alpha,o}] + q_0^\alpha \\ &= \pi(\nu^2 + (m^\alpha)^2) + 2s_0^\alpha m^\alpha + q_0^\alpha. \end{aligned} \tag{16}$$

Proof 2. *The proof is also standard for linear quadratic Gaussian tracking problems. See Huang et al. (2007).*

The first proposition takes care of the solvability of control problems (8)-(9), assuming we can show the solvability of the Backwards ODEs (13) and (15), and the second proposition gives us a formula for computing the optimal costs. The next proposition covers the solvability of the consistency conditions (10).

Proposition 3. *Assume that there is a solution to* (13) *in* $C_b\big([0,\infty); L^2[0,1]\big)$. *Then the consistency conditions* (10) *are satisfied, if and only if, there exist a function* $z \in C_b\big([0,\infty); L^2[0,1]\big)$ *solution to the ODE,*

$$dz_t^\alpha = \Big[\Big(a - \frac{b^2}{r}\pi\Big) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta\Big] dt, \quad (17)$$
$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

Proof 3. *The consistency conditions* (10) *describes a fixed point problem on the optimal states. From Proposition 1, we have access to the SDEs satisfied by the optimal states. Thanks to the linear nature of these SDEs, by taking expectations, it is straightforward to show that the existence of the fixed point is equivalent to the existence of solutions to ODEs* (17). *The proof is complete.*

Compiling the previous three propositions, we deduce that our infinite horizon LQG GMFG is solvable, whenever there exist processes, $\{z, s, q\} \subset C_b\big([0,\infty); L^2[0,1]\big)$, solutions to the Forward-Backward ODEs (FBODEs) below.

$$\frac{dz_t^\alpha}{dt} = \Big(a - \frac{b^2}{r}\pi\Big) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta, \quad (18)$$
$$\frac{ds_t^\alpha}{dt} = \Big(-a + \frac{b^2}{r}\pi + \rho\Big) s_t^\alpha + z_t^\alpha, \quad (19)$$
$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2, \quad (20)$$
$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

3.2. Steady-State Conditions

A difficulty with the FBODEs (18)-(19)-(20), is the absence of steady-state information $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$ required for their solvability. We circumvent this obstacle by setting $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$ as a solution to the equation

$$0 = \frac{dz_\infty^\alpha}{dt} = \frac{ds_\infty^\alpha}{dt} = \frac{dq_\infty^\alpha}{dt}, \quad \forall \alpha \in [0,1], \quad (21)$$

which defines the family of algebraic equations

$$0 = \Big(a - \frac{b^2}{r}\pi\Big) z_\infty^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_\infty^\beta d\beta, \quad (22)$$
$$0 = \Big(-a + \frac{b^2}{r}\pi + \rho\Big) s_\infty^\alpha + z_\infty^\alpha, \quad (23)$$
$$0 = -\sigma^2\pi + \frac{b^2}{r}(s_\infty^\alpha)^2 + \rho q_\infty^\alpha - (z_\infty^\alpha)^2. \quad (24)$$

From (22)-(23), we derive the equation for s_∞ below

$$0 = \Big(a - \frac{b^2}{r}\pi\Big)\Big[\Big(a - \frac{b^2}{r}\pi\Big) - \rho\Big] s_\infty^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_\infty^\beta d\beta,$$

which is equivalent (with discrepancies on at most a set of measure zero) to

$$\Big[\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi - \rho\Big) I - \frac{b^2}{r} g\Big] \circ s_\infty = 0 \quad (25)$$

where $(g \circ s_\infty)(\cdot) := \int_0^1 g(\cdot,\beta) s_\infty(\beta) d\beta$, and I denotes the identity operator from $L^2[0,1]$ to $L^2[0,1]$. Observe that the operator involved in (25),

$$\Big[\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi - \rho\Big) I - \frac{b^2}{r} g\Big]$$

has a bounded inverse if the quantity

$$\frac{r}{b^2}\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi - \rho\Big)$$

is nonzero and not an eigenvalue of graphon operator g.

Remark 1. *Since it is assumed that* $|g(x,y)| \le 1$, *for all* $x, y \in [0,1]$, *the operator norm of* g *satisfies that*

$$\|g\|_{\text{op}} := \sup_{v \in L^2[0,1], v \neq 0} \frac{\|gv\|}{\|v\|} \le \|g\|_2 \le 1$$

(see e.g. (Gao et al., 2021b, Lemma 7)) which implies that the absolute values of all the eigenvalues of g *are less than or equal to* 1. *When* $a = 0$, *it follows from* (11), *that* $\pi(\frac{b^2}{r}\pi + \rho)I - g = I - g$. *Therefore it has a bounded inverse when* 1 *is not an eigenvalue of* g.

We introduce our first set of technical assumptions.

Assumption (A0): Assume the real valued quantity,

$$\tilde{\tau} := \Big(\frac{b^2}{r}\Big)^{-1}\Big(a - \frac{b^2}{r}\pi\Big)\Big(a - \frac{b^2}{r}\pi - \rho\Big) \quad \text{is nonzero.}$$

Assumption (A1): Assume that the spectrum of g does not contain $\tilde{\tau}$.

Under **Assumptions (A0)-(A1)**, the functional equation (25) admits the (unique) solution in $L^2([0,1])$

$$z_\infty^\alpha = 0 = s_\infty^\alpha, \quad a.e.\ \alpha \in [0,1], \quad (26)$$

and an application of (24) yields

$$q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}, \quad a.e.\ \alpha \in [0,1]. \tag{27}$$

Equipped with the steady state information $(z_\infty^\alpha, s_\infty^\alpha, q_\infty^\alpha)$, we proceed to the study of, $\{z, s, q\} \subset C_b\left([0,\infty); L^2[0,1]\right)$, solution to the FBODEs,

$$\frac{dz_t^\alpha}{dt} = \left(a - \frac{b^2}{r}\pi\right) z_t^\alpha - \frac{b^2}{r}\int_0^1 g(\alpha,\beta) s_t^\beta d\beta, \tag{28}$$

$$\frac{ds_t^\alpha}{dt} = \left(-a + \frac{b^2}{r}\pi + \rho\right) s_t^\alpha + z_t^\alpha, \tag{29}$$

$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2, \tag{30}$$

$$z_0^\alpha = \int_0^1 g(\alpha,\beta) m^\beta d\beta.$$

with the infinite horizon conditions

$$z_\infty^\alpha = 0 = s_\infty^\alpha, \quad q_\infty^\alpha = \frac{\sigma^2\pi}{\rho}, \quad a.e.\ \alpha \in [0,1]. \tag{31}$$

4. Existence and Uniqueness for (28)-(29)

We focus on the FBODEs (28)-(29). Any solution z, s to FBODEs (28)-(29) must satisfy the integral equations

$$s_t^\alpha = -\int_t^\infty \exp\left(\left(-a + \frac{b^2}{r}\pi + \rho\right)(t-s)\right) z_s^\alpha ds \tag{32}$$

$$z_t^\alpha = \exp\left(\left(a - \frac{b^2}{r}\pi\right) t\right) \int_0^1 g(\alpha,\beta) m^\beta d\beta \tag{33}$$

$$+ \frac{b^2}{r}\int_0^t \int_s^\infty \int_0^1 \exp\left(\left(a - \frac{b^2}{r}\pi\right)(t-s)\right) \tag{34}$$

$$\times \exp\left(\left(-a + \frac{b^2}{r}\pi + \rho\right)(s-\tau)\right) \tag{35}$$

$$\times g(\alpha,\beta) z_\tau^\beta d\beta d\tau ds. \tag{36}$$

We introduce the notation, $\gamma_1 := -\left(a - \frac{b^2}{r}\pi\right)$ and $\gamma_2 := \left(-a + \frac{b^2}{r}\pi + \rho\right) > \frac{\rho}{2} > 0$, and $\Phi(\cdot)$ defined for all $y \in C_b\left([0,\infty); L^2[0,1]\right)$ as

$$\Phi(y)_t^\alpha := \exp(-\gamma_1 t) \int_0^1 g(\alpha,\beta) m^\beta d\beta \tag{37}$$

$$+ \frac{b^2}{r}\int_0^t \int_s^\infty \int_0^1 \exp(-\gamma_1(t-s)) \tag{38}$$

$$\times \exp(\gamma_2(s-\tau))\, g(\alpha,\beta) y_\tau^\beta d\beta d\tau ds. \tag{39}$$

Assumption (A2): Assume that $\gamma_1 > 0$ and $\frac{b^2}{r\gamma_1\gamma_2} < 1$.

Proposition 4. *Let (A1) and (A2) hold, then $\Phi(y)$ is bounded and uniformly continuous for all $y \in C_b\left([0,\infty); L^2[0,1]\right)$, i.e. $\Phi(y) \in C_b\left([0,\infty); L^2[0,1]\right)$. Moreover, the map $\Phi(\cdot)$ admits a unique fixed point on $C_b\left([0,\infty); L^2[0,1]\right)$.*

Proof 4. *To show that $\Phi(y) \in C_b\left([0,\infty); L^2[0,1]\right)$, we show firstly that $\Phi(y)$ is bounded for all $t \in [0,\infty)$*

$$\|\Phi(y)_t\|_2 \le \exp(-\gamma_1 t)\, \| \int_0^1 g(\alpha,\beta) m^\beta d\beta \|_2 \tag{40}$$

$$+ \frac{b^2}{r}\int_0^t \int_s^\infty \exp(-\gamma_1(t-s)) \tag{41}$$

$$\times \exp(\gamma_2(s-\tau))$$

$$\times \| \int_0^1 g(\alpha,\beta) y_\tau^\beta d\beta \|_2 d\tau ds,$$

$$\le \exp(-\gamma_1 t)\, \|m\|_2 \tag{42}$$

$$+ \frac{b^2}{r}\int_0^t \int_s^\infty \exp(-\gamma_1(t-s)) \tag{43}$$

$$\times \exp(\gamma_2(s-\tau))\, \|y_\tau\|_2 d\tau ds \tag{44}$$

$$\le \exp(-\gamma_1 t)\, \|m\|_2 + \frac{b^2}{r}\frac{1}{\gamma_1\gamma_2}\|y\|_{2,\infty} < \infty, \tag{45}$$

and secondly that it is uniformly continuous, $0 \le s \le t$

$$\Phi(y)_t^\alpha - \Phi(y)_s^\alpha \tag{46}$$

$$= (\exp(-\gamma_1 t) - \exp(-\gamma_1 s)) \int_0^t g(\alpha,\beta) m^\beta d\beta$$

$$+ \frac{b^2}{r}\int_0^s \int_v^\infty \int_0^1 \left[\exp(-\gamma_1(t-v)) - \exp(-\gamma_1(s-v))\right]$$

$$\times \exp(\gamma_2(v-\tau))\, g(\alpha,\beta) y_\tau^\beta d\beta d\tau dv \tag{47}$$

$$+ \frac{b^2}{r}\int_s^t \int_v^\infty \int_0^1 \exp(-\gamma_1(t-v))$$

$$\times \exp(\gamma_2(v-\tau))\, g(\alpha,\beta) y_\tau^\beta d\beta d\tau dv \tag{48}$$

$$=: I_0 + I_1 + I_2, \tag{49}$$

where the terms I_0, I_1, I_2 are defined respectively by the three previously added terms. Uniform continuity is a consequence of the fact that for all $0 \le s \le t$

$$|\Phi(y)_t^\alpha - \Phi(y)_s^\alpha| \le |I_0| + |I_1| + |I_2|$$

$$\le \left[\gamma_1 |\langle 1, m\rangle| + \frac{2b^2}{r\gamma_2}\|y\|_{2,\infty}\right] |t-s|. \tag{50}$$

Indeed, it holds for all $0 \le s \le t$ that

$$|I_0| \le |\exp(-\gamma_1 t) - \exp(-\gamma_1 s)| \left| \int_0^t g(\alpha,\beta) m^\beta d\beta \right| \quad (51)$$

$$\le \exp(-\gamma_1 s) |\gamma_1 t - \gamma_1 s| \left| \int_0^t g(\alpha,\beta) m^\beta d\beta \right| \quad (52)$$

$$\le \exp(-\gamma_1 s) |\gamma_1 t - \gamma_1 s| |\langle 1, m\rangle| \quad (53)$$

$$\le \gamma_1 |\langle 1, m\rangle| |t-s|, \quad (54)$$

$$|I_1| \le \frac{b^2}{r} \|y\|_{2,\infty} \times \int_0^s \int_v^\infty |\exp(-\gamma_1(t-v)) - \exp(-\gamma_1(s-v))| \times \exp(\gamma_2(v-\tau))\, d\tau dv \quad (55)$$

$$\le \frac{b^2}{r} \|y\|_{2,\infty} \int_0^s \int_v^\infty \gamma_1 |t-s| \exp(-\gamma_1(s-v)) \times \exp(\gamma_2(v-\tau))\, d\tau dv \quad (56)$$

$$\le \frac{b^2}{r} \|y\|_{2,\infty} \frac{1}{\gamma_2} |t-s| \int_0^s \gamma_1 \exp(-\gamma_1(s-v))\, dv \quad (57)$$

$$\le \frac{b^2}{r} \frac{\|y\|_{2,\infty}}{\gamma_2} |t-s|, \quad (58)$$

$$|I_2| \le \frac{b^2}{r} \|y\|_{2,\infty} \times \int_s^t \int_v^\infty \exp(-\gamma_1(t-v)) \exp(\gamma_2(v-\tau))\, d\tau dv \quad (59)$$

$$\le \frac{b^2}{r} \|y\|_{2,\infty} \int_s^t \frac{1}{\gamma_2} \exp(-\gamma_1(t-v))\, dv \quad (60)$$

$$= \frac{b^2}{r} \frac{1}{\gamma_1\gamma_2} \|y\|_{2,\infty} [1 - \exp(-\gamma_1(t-s))] \quad (61)$$

$$\le \frac{b^2}{r} \frac{1}{\gamma_1\gamma_2} \|y\|_{2,\infty} \gamma_1 |t-s| z \le \frac{b^2}{r} \frac{1}{\gamma_2} \|y\|_{2,\infty} |t-s|. \quad (62)$$

Next, because $C_b\left([0,\infty); L^2[0,1]\right)$ is a Banach space when endowed with the norm $\|\cdot\|_{2,\infty}$, it is enough to show that Φ is a contraction. For any pair $y^1, y^2 \in C_b\left([0,\infty); L^2[0,1]\right)$ and for all $t \ge 0$ we have

$$\|\Phi(y^1)_t - \Phi(y^2)_t\|_2 = \quad (63)$$

$$\| \int_0^t \int_s^\infty \int_0^1 \frac{b^2}{r} \exp(-\gamma_1(t-s)) \exp(\gamma_2(s-\tau)) \times g(\alpha,\beta)(y^1_\tau - y^2_\tau)(\beta) d\beta d\tau ds \|_2 \quad (64)$$

$$\le \frac{b^2}{r} \|y^1 - y^2\|_{2,\infty} \times \int_0^t \int_s^\infty \exp(-\gamma_1(t-s)) \exp(\gamma_2(s-\tau))\, d\tau ds \quad (65)$$

$$= \frac{b^2}{r} \|y^1 - y^2\|_{2,\infty} \frac{1}{\gamma_2} \int_0^t \exp(\gamma_2(s-\tau))\, ds \quad (66)$$

$$\le \frac{b^2}{r} \|y^1 - y^2\|_{2,\infty} \frac{1}{\gamma_2} \frac{1}{\gamma_1} [1 - \exp(-\gamma_1 t)], \quad (67)$$

and therefore,

$$\|\Phi(y^1) - \Phi(y^2)\|_{2,\infty} \le \left[\frac{b^2}{r\gamma_1\gamma_2} \right] \|y^1 - y^2\|_{2,\infty}, \quad (68)$$

and it is a contraction. The proof is complete.

5. Explicit Nash Values

Graphons as linear operators are compact with discrete spectrum (Lovász (2012)). Let the spectral representation of the graphon g be given by

$$g(\alpha,\beta) = \sum_{\ell=1}^\infty \lambda_\ell f_\ell(\alpha) f_\ell(\beta),$$

where f_ℓ is the orthonormal eigenfunction associated with the non-zero eigenvalue λ_ℓ of g for all $\ell \ge 1$.

5.1. Calculating GMFG Equilibrium.

Since $s \in C_b\left([0,\infty); L^2[0,1]\right)$, following an analysis similar to that in (Gao et al., 2023, Lemma 2), we can demonstrate that the solution for (28) is a classical solution in the space $C_b\left([0,\infty); L^2[0,1]\right)$, and furthermore the differentiation dz_t/dt (in the classical sense) also lies in $L^2[0,1]$ for all $t \in [0,\infty)$. The same is true for s and ds_t/dt, that is, s is a classical solution to (29) and ds_t/dt lies in $L^2[0,1]$ for all $t \in [0,\infty)$. Then for all $t \ge 0$, it follows that (see e.g. (Rudin, 1987, Ch.4))

$$\frac{dz_t}{dt} = \sum_{\ell=1}^\infty \langle f_\ell, \frac{dz_t}{dt} \rangle f_\ell = \sum_{\ell=1}^\infty \frac{d}{dt} \langle f_\ell, z_t \rangle f_\ell \quad \in L^2[0,1]$$

$$\frac{ds_t}{dt} = \sum_{\ell=1}^\infty \langle f_\ell, \frac{ds_t}{dt} \rangle f_\ell = \sum_{\ell=1}^\infty \frac{d}{dt} \langle f_\ell, s_t \rangle f_\ell \quad \in L^2[0,1].$$

Some technical assumptions are now introduced for the explicit resolution of the FBODEs (28)-(29)-(30) together with steady-state information (31).

Assumption (A3): Assume the nonzero eigenvalues $\{\lambda_\ell,\ \ell \geq 1\}$ of the graphon g satisfy the following bound

$$\lambda_\ell < 1 + \frac{r}{b^2} a(a-\rho), \quad \forall \ell \geq 1. \tag{69}$$

Remark 2. *Assumptions (A3) is introduced to ensure that the equations* (28) *and* (29) *(to be introduced) have explicit solutions over the infinite time horizon* $[0,\infty)$*. Assumption (A3) is to ensure a crucial second order ODE* (77) *(to be introduced) has an exponentially stable solution. We note that when* $a = 0$ *Assumption (A3) holds if g does not have* 1 *as eigenvalue.*

Proposition 5. *Let Assumptions (A1)-(A2)-(A3) be in force. Then, the process* $\{z_t^\alpha, s_t^\alpha\ \alpha \in [0,1],\ t \in [0,\infty)\}$*, solution to* (28)-(29)*, is explicitly given, for all* $t \geq 0$*, a.s.* $\alpha \in [0,1]$*, by*

$$z_t^\alpha = \sum_{l=1}^{\infty} f_\ell(\alpha) z_t^\ell, \quad s_t^\alpha = \sum_{l=1}^{\infty} f_\ell(\alpha) s_t^\ell, \tag{70}$$

where $z^\ell, s^\ell,\ \forall \ell \geq 1$ *are given explicitly by*

$$z_t^\ell = \lambda_\ell \langle m, f_\ell \rangle \exp\left[\left(\frac{\rho}{2} - \theta(\lambda_\ell)\right) t\right], \tag{71}$$

$$s_t^\ell = -\frac{z_t^\ell}{(\theta(\lambda_\ell) + \theta(0))} \tag{72}$$

and $\theta(\cdot)$ *is a function defined by*

$$\theta(\tau) := \sqrt{\frac{(\rho-2a)^2}{4} + (1-\tau)\frac{b^2}{r}}, \quad \tau \in \mathbb{R}. \tag{73}$$

Proof 5. *Consider the graphon spectral decomposition*

$$g(\alpha,\beta) = \sum_{\ell=1}^{\infty} \lambda_\ell f_\ell(\alpha) f_\ell(\beta), \quad \forall \alpha,\beta \in [0,1], \tag{74}$$

where $\{f_\ell,\ \ell \geq 1\} \subset L^2[0,1]$ *is the orthonormal eigenfunction of g, and* λ_ℓ *is the eigenvalue associated with* f_ℓ*. By definition,* $g f_\ell = \lambda_\ell f_\ell$*. Following the spectral reformulation of two point boundary value problems developed in Gao et al. (2021b), we define the eigen processes*

$$z_t^\ell = \langle z_t, f_\ell \rangle, \quad s_t^\ell = \langle s_t, f_\ell \rangle, \quad t \in [0,\infty),\ \ell \geq 1.$$

These processes satisfy the FBODEs,

$$\frac{dz_t^\ell}{dt} = \left(a - \frac{b^2\pi}{r}\right) z_t^\ell - \lambda_\ell \frac{b^2}{r} s_t^\ell, \quad z_0^\ell = \lambda_\ell \langle m, f_\ell \rangle,$$

$$\frac{ds_t^\ell}{dt} = z_t^\ell + \left(-a + \frac{b^2\pi}{r} + \rho\right) s_t^\ell, \quad s_\infty^\ell = 0,$$

for which we seek an explicit solution that is compatible with the infinite horizon condition $z_\infty^\ell = 0$*, for all* $\ell \geq 1$*. (We notice that* $z^\ell = 0$ *and* $s^\ell = 0$ *when* $\lambda_\ell = 0$ *following the equations above).*

From the ODE for s^ℓ*, it is straightforward to compute that,*

$$s_t^\ell = -\int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds. \tag{75}$$

Next, we substitute s^ℓ *back into the ODE for* z^ℓ *and obtain the ODE*

$$\frac{dz_t^\ell}{dt} = \left(a - \frac{b^2\pi}{r}\right) z_t^\ell + \lambda_\ell \frac{b^2}{r} \int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds. \tag{76}$$

By differentiating this ODE once and making appropriate substitutions, we get the second order ODE for z^ℓ

$$\frac{d^2 z_t^\ell}{dt} - \rho \frac{dz_t^\ell}{dt} + \left[\lambda_\ell \frac{b^2}{r} - \left(a - \frac{b^2\pi}{r}\right)^2 + \rho\left(a - \frac{b^2\pi}{r}\right)\right] z_t^\ell = 0, \tag{77}$$

whose characteristic equation

$$\xi_\ell^2 - \rho\xi_\ell + \left[-a^2 + \rho a + \frac{b^2}{r}(\lambda_\ell - 1)\right] = 0, \tag{78}$$

admits as solution

$$\xi_\ell = \left(\frac{\rho}{2} - \sqrt{\frac{(\rho-2a)^2}{4} + \frac{b^2}{r}(1-\lambda_\ell)}\right) = \frac{\rho}{2} - \theta(\lambda_\ell), \tag{79}$$

where $\theta(\lambda_\ell) = \sqrt{\frac{(\rho-2a)^2}{4} + (1-\lambda_\ell)\frac{b^2}{r}}$ *is real whenever* $\lambda_\ell < 1 + \frac{r(\rho-2a)^2}{4b^2}$*, which follows from Assumption (A3). It also follows from Assumption (A3) that* $\xi_\ell < 0$*.*

Therefore, the explicit solution for z^ℓ *is*

$$z_t^\ell = \lambda_\ell \langle m, f_\ell \rangle \exp(\xi_\ell t),\ \forall t \geq 0, \tag{80}$$

and, because $\xi_\ell < 0$ *for all* $l \in \{1,\ldots,L\}$*, the infinite horizon condition* $z_\infty^\ell = 0$ *is satisfied, and the explicit solution for* s^ℓ*,* $\forall t \in [0,\infty)$ *is*

$$\begin{aligned} s_t^\ell &= -\int_t^\infty \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right)(t-s)\right) z_s^\ell ds \\ &= -\lambda_\ell \langle m, f_\ell \rangle \exp\left(\left(-a + \frac{b^2\pi}{r} + \rho\right) t\right) \\ &\quad \times \int_t^\infty \exp\left(\left(\xi_\ell + a - \frac{b^2\pi}{r} - \rho\right) s\right) ds. \end{aligned} \tag{81}$$

Observe that, since

$$\xi_\ell < 0, \quad \textit{and} \quad \left(a - \frac{b^2\pi}{r} - \rho\right) < 0, \tag{82}$$

the integral term is explicitly solvable such that

$$s_t^\ell = \lambda_\ell \langle m, f_\ell \rangle \exp(\xi_\ell t)\left(\xi_\ell + a - \frac{b^2\pi}{r} - \rho\right)^{-1}. \tag{83}$$

Also, for all $\ell \geq 1$, we have that

$$\begin{aligned}\xi_\ell + a - \frac{b^2\pi}{r} - \rho &= \frac{\rho}{2} - \theta(\lambda_\ell) + a - \frac{b^2\pi}{r} - \rho \\ &= -\theta(\lambda_\ell) + a - \frac{\rho}{2} - \frac{b^2\pi}{r} \\ &= -\theta(\lambda_\ell) - \left(\frac{(\rho - 2a)^2}{4} + \frac{b^2}{r}\right)^{\frac{1}{2}} = -\theta(\lambda_\ell) - \theta(0).\end{aligned}$$

Therefore, the explicit solution for s^ℓ is

$$s_t^\ell = -\frac{z_t^\ell}{\theta(\lambda_\ell) + \theta(0)}, \quad \forall \ell \geq 1. \tag{84}$$

We deduce from (80) *that $z_\infty^\ell = 0$ for all ℓ, which implies that $s_\infty^\ell = 0$. Based on* (74) *and the definition of the eigen processes, we can now reconstruct the solution $\{z_t^\alpha, s_t^\alpha\ \alpha \in [0,1],\ t \in [0,\infty)\}$ as below*

$$z_t^\alpha = \sum_{l=1}^{\infty} f_\ell(\alpha) z_t^\ell, \quad s_t^\alpha = -\sum_{l=1}^{\infty} f_\ell(\alpha) \frac{z_t^\ell}{\theta(\lambda_\ell) + \theta(0)}$$

where for all $t \geq 0$ and for almost all $\alpha \in [0,1]$. The proof is complete.

Remark 3 (Properties of the $\theta(\cdot)$ function)**.** *The $\theta(\cdot)$ function, which we recall here, has several interesting properties $\theta(\tau) := \sqrt{\frac{(\rho-2a)^2}{4} + (1-\tau)\frac{b^2}{r}}$. A first property is that for $\lambda_\ell \neq \lambda_k$ the following hold:*

$$\frac{1}{\theta(\lambda_\ell) + \theta(\lambda_k)} = \frac{\theta(\lambda_\ell) - \theta(\lambda_k)}{\theta(\lambda_\ell)^2 - \theta(\lambda_k)^2} = -\left(\frac{r}{b^2}\right)\frac{\theta(\lambda_\ell) - \theta(\lambda_k)}{(\lambda_\ell - \lambda_k)}. \tag{85}$$

A second property is that $\theta'(\tau) = \frac{-b^2}{2r\theta(\tau)}$. Finally, since $\theta(\tau)$ is monotonically decreasing with respect to τ, the equality above implies that $\theta'(\tau)$ is monotonically increasing with respect to τ.

5.2. Calculating Nash Values.

In this subsection, building on the two previous propositions, we compute explicitly the Nash values for almost every $\alpha \in [0,1]$. This computation will require the explicit computation of the solution $\{q_t^\alpha, \alpha \in [0,1], t \in [0,T]\}$ to ODE (30). We introduce the assumption,

Assumption (A4): Assume that only a finite number L of the eigenvalues of g are nonzero and that these are twice differentiable with respect to their arguments.

Proposition 6. *Let Assumptions (A1), (A2), (A3) and (A4) be in force. Then, the cost at equilibrium is explicitly given, for almost every $\alpha \in [0,1]$, by*

$$\begin{aligned}J(u^\alpha, z) &= \pi v^2 + \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho} - 2m^\alpha \sum_{l=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell \langle m, f_\ell \rangle \\ &+ \frac{1}{\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\lambda_\ell \langle m, f_\ell \rangle\right)^2 - \frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell \langle m, f_\ell \rangle\right)^2 \\ &- \sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha) f_\ell(\alpha) \langle m, f_k \rangle \langle m, f_\ell \rangle \left(\frac{\rho}{2} - \theta(\lambda_k)\right) \\ &\times \left(\frac{1}{\theta(\lambda_\ell) + \theta(\lambda_k)}\right)\left[\frac{2}{\rho}\lambda_k\lambda_\ell - \frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right],\end{aligned}$$

where we define,

$$\bar{\lambda}_\ell := \frac{\lambda_\ell}{\theta(\lambda_\ell) + \theta(0)}, \quad \ell \in \{1, ..., L\}. \tag{86}$$

Proof 6. *Given the explicit processes $\{z_t^\alpha, s_t^\alpha\ \alpha \in [0,1],\ t \in [0,\infty)\}$, calculated for almost every $\alpha \in [0,1]$, we proceed to calculate explicitly the process $\{q_t^\alpha,\ \alpha \in [0,1],\ t \in [0,\infty)\}$, for almost every $\alpha \in [0,1]$.*

It is straightforward to verify that,

$$q_t^\alpha = -\exp(\rho t)\int_t^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds, \tag{87}$$

with $\Theta(\alpha, t)$, defined on $\alpha \in [0,1],\ t \in [0,\infty)$, by

$$\Theta(\alpha, t) = -\sigma^2\pi - (z_t^\alpha)^2 + \frac{b^2}{r}(s_t^\alpha)^2,$$

is a solution to the offset ODE,

$$\frac{dq_t^\alpha}{dt} = -\sigma^2\pi + \frac{b^2}{r}(s_t^\alpha)^2 + \rho q_t^\alpha - (z_t^\alpha)^2, \tag{88}$$

such that by applying L'Hopital's Rule for its infinite time horizon limit, we have

$$\begin{aligned}\lim_{t\to\infty} q_t^\alpha &= \lim_{t\to\infty}\left(-\exp(\rho t)\int_t^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds\right) \\ &= \lim_{t\to\infty}\left(\frac{\int_t^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds}{-\exp(-\rho t)}\right) \\ &= \lim_{t\to\infty}\left(\frac{-\Theta(\alpha, t)\exp(-\rho t)}{\rho\exp(-\rho t)}\right) \\ &= \lim_{t\to\infty}\frac{\Theta(\alpha, t)}{-\rho} = \frac{-\sigma^2\pi}{-\rho} = \frac{\sigma^2\pi}{\rho} = q_\infty^\alpha.\end{aligned}$$

Next, recall that the optimal cost is given, for all $\alpha \in [0,1]$, by

$$J(u^\alpha, z) = \pi(\nu^2 + (m^\alpha)^2) + 2s_0^\alpha m^\alpha + q_0^\alpha.$$

To calculate the Nash values explicitly, for a.e. $\alpha \in [0,1]$, it is sufficient to calculate the quantities s_0^α, q_0^α.

For almost every $\alpha \in [0,1]$,

$$\begin{aligned} s_0^\alpha &= -\sum_{l=1}^{L} (\theta(\lambda_\ell) + \theta(0))^{-1} f_\ell(\alpha)\lambda_\ell \langle m, f_\ell\rangle \\ &= -\sum_{l=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell \langle m, f_\ell\rangle \,. \end{aligned} \tag{89}$$

And, for almost every $\alpha \in [0,1]$,

$$q_0^\alpha = -\int_0^\infty \Theta(\alpha, s)\exp(-\rho s)\, ds, \tag{90}$$

where $\Theta(\alpha, t)$, $\forall \alpha \in [0,1]$, $t \in [0,\infty)$, is defined by:

$$\begin{aligned} \Theta(\alpha, t) = -\sigma^2\pi &- \left(\sum_{l=1}^{L} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle \exp(\xi_\ell t)\right)^2 \\ &+ \frac{b^2}{r}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle \exp(\xi_\ell t)\right)^2. \end{aligned}$$

Integrating by parts yields

$$q_0^\alpha = -\frac{\Theta(\alpha, 0)}{\rho} - \frac{1}{\rho}\int_0^\infty \exp(-\rho s)\frac{d\Theta(\alpha, s)}{ds} ds$$

where it holds that

$$\begin{aligned} -\frac{\Theta(\alpha, 0)}{\rho} &= \frac{\sigma^2\pi}{\rho} + \frac{1}{\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\right)^2 \\ &\quad - \frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle\right)^2, \end{aligned}$$

and

$$\begin{aligned} &-\frac{1}{\rho}\int_0^\infty \exp(-\rho s)\frac{d\Theta(\alpha, s)}{ds} ds \\ &= \sum_{k=1}^{L}\sum_{\ell=1}^{L} \xi_k \left(\int_0^\infty e^{(\xi_k+\xi_\ell-\rho)s} ds\right)\left[\frac{2}{\rho}\lambda_k\lambda_\ell - \frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right] \\ &\qquad \times f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle. \end{aligned}$$

We compute explicitly the integral above and obtain,

$$\begin{aligned} &-\frac{1}{\rho}\int_0^\infty \exp(-\rho s)\frac{d\Theta(\alpha, s)}{ds} ds \\ &= \sum_{k=1}^{L}\sum_{\ell=1}^{L} \xi_k (\xi_k + \xi_\ell - \rho)^{-1}\left[\frac{2}{\rho}\lambda_k\lambda_\ell - \frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right] \\ &\qquad \times f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle. \end{aligned}$$

By the equality $(\xi_k + \xi_\ell - \rho) = -(\theta(\lambda_\ell) + \theta(\lambda_k))$, we deduce that,

$$\begin{aligned} q_0^\alpha = \frac{\sigma^2\pi}{\rho} &+ \frac{1}{\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\right)^2 \\ &- \frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle\right)^2 \\ &- \sum_{k=1}^{L}\sum_{\ell=1}^{L}\left(\frac{\rho}{2} - \theta(\lambda_k)\right)(\theta(\lambda_\ell) + \theta(\lambda_k))^{-1} \\ &\qquad \times\left[\frac{2}{\rho}\lambda_k\lambda_\ell - \frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right] \\ &\qquad \times f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle. \end{aligned}$$

By substituting the calculated terms appropriately in

$$J(u^\alpha, z) = \pi(\nu^2 + (m^\alpha)^2) + 2s_0^\alpha m^\alpha + q_0^\alpha \tag{91}$$

we obtain the desired result. The proof is complete.

The next proposition introduces simplifications.

Proposition 7. *Assume (A1), (A2), (A3) and (A4) hold. Then, the Nash values are explicitly given below: for almost every $\alpha \in [0,1]$,*

$$\begin{aligned} J(u^\alpha, z) = \pi\nu^2 &+ \pi(m^\alpha)^2 + \frac{\sigma^2\pi}{\rho} \\ &- \frac{2r}{b^2}m^\alpha\sum_{\ell=1}^{L} f_\ell(\alpha)(\theta(0) - \theta(\lambda_\ell))\langle m, f_\ell\rangle \\ &- \sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha) f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle\left(\frac{\rho}{\theta(\lambda_\ell) + \theta(\lambda_k)} - 2\right) \\ &\qquad \times\frac{1}{\rho}\Big[\lambda_\ell\lambda_k - \frac{r}{b^2}(\theta(0) - \theta(\lambda_\ell))(\theta(0) - \theta(\lambda_k))\Big], \end{aligned}$$

where $\theta(\tau) := \sqrt{\frac{(\rho-2a)^2}{4} + (1-\tau)\frac{b^2}{r}}$, $\tau \in \mathbb{R}$.

Proof 7. *We observe that*

$$\sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha)f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle\left(\frac{\rho}{2}-\theta(\lambda_k)\right) \times\left(\frac{1}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right)\left[\frac{2}{\rho}\lambda_k\lambda_\ell-\frac{2b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]$$
$$=\sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha)f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle\left(\frac{\rho-\theta(\lambda_k)-\theta(\lambda_\ell)}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right) \times\left[\frac{1}{\rho}\lambda_k\lambda_\ell-\frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]$$
$$=\sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha)f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)}\right) \times\left[\frac{1}{\rho}\lambda_k\lambda_\ell-\frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right] -\frac{1}{\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\lambda_\ell\langle m, f_\ell\rangle\right)^2+\frac{b^2}{r\rho}\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle\right)^2.$$

Taking the cost form in Prop. (6)*, then last three terms there can be further simplified, which leads to the following result*

$$J(u^\alpha, z)=\pi\nu^2+\pi(m^\alpha)^2+\frac{\sigma^2\pi}{\rho}-2m^\alpha\sum_{l=1}^{L} f_\ell(\alpha)\bar{\lambda}_\ell\langle m, f_\ell\rangle -\sum_{k=1}^{L}\sum_{\ell=1}^{L} f_k(\alpha)f_\ell(\alpha)\langle m, f_k\rangle\langle m, f_\ell\rangle \times\left(\frac{\rho}{\theta(\lambda_\ell)+\theta(\lambda_k)}-2\right)\left[\frac{1}{\rho}\lambda_k\lambda_\ell-\frac{b^2}{\rho r}\bar{\lambda}_k\bar{\lambda}_\ell\right]. \tag{92}$$

An application of the property (85) *yields*

$$\bar{\lambda}_\ell := \frac{\lambda_\ell}{\theta(\lambda_\ell)+\theta(0)} = -\frac{r}{b^2}(\theta(\lambda_\ell)-\theta(0)),\ \ell\in\{1,\dots,L\}.$$

Replacing $\bar{\lambda}_k$ *and* $\bar{\lambda}_\ell$ *in* (92) *yields the desired result. The proof is complete.*

6. Nash Values Local Minima

Assumption (A5): Assume that the initial means are linear. That is

$$m^\alpha = \bar{m}+k\alpha \quad \forall\ \alpha\in[0,1]. \tag{93}$$

Assumption (A6): Assume that all the nonzeros eigenvalues are the same. That is,

$$\lambda_\ell=\lambda, \quad \forall \ell\in\{1,...,L\}. \tag{94}$$

Proposition 8. *Let (A1) to (A6) be in force. Assume that there is* $\alpha^*\in(0,1)$ *such that*

$$0 = 2k\pi\,(k\alpha^*+\bar{m})+B_\theta\sum_{\ell=1}^{L} f_\ell(\alpha^*)\langle m, f_\ell\rangle, \tag{95}$$

and for all $\ell\in\{1,...,L\}$,

$$\partial_\alpha f_\ell(\alpha^*)=0, \tag{96}$$

$$\partial^2_{\alpha\alpha} f_\ell(\alpha^*)\langle m, f_\ell\rangle\left\{(A_\theta+B_\theta\alpha^*)+2C_\theta\sum_{\ell=1}^{L} f_\ell(\alpha^*)\langle m, f_\ell\rangle\right\} \geq -\frac{2k^2\pi}{L}, \tag{97}$$

where

$$A_\theta := -2\bar{m}\frac{r}{b^2}\,(\theta(0)-\theta(\lambda)) \tag{98}$$

$$B_\theta := -2k\frac{r}{b^2}\,(\theta(0)-\theta(\lambda)) \tag{99}$$

$$C_\theta := -\frac{1}{\rho}\left(\frac{\rho}{2\theta(\lambda)}-2\right)\left(\lambda^2-\frac{r}{b^2}\,(\theta(0)-\theta(\lambda))^2\right), \tag{100}$$

then $\alpha^*\in(0,1)$ *is a local minimum of the Nash value* $J(u^\alpha, z)$*. That is,*

$$\partial_\alpha J(u^{\alpha^*}, z)=0, \quad \textit{and} \quad \partial^2_{\alpha\alpha}J(u^{\alpha^*}, z)\geq 0. \tag{101}$$

Remark 4. *The purpose of this proposition is to provide sufficient conditions which a new agent joining this infinite horizon game at GMFG equilibrium can verify in order to choose, locally, a node with the smallest cost. The only global information needed is the initial means* m^α*. Note that whenever* $k=0$*, all* $\alpha^*\in(0,1)$ *satisfy condition* (95)*.*

Proof 8. *Let (A1) to (A6) be in force. It follows from the previous proposition that the Nash value function is given by*

$$J(u^\alpha, z)=\left(\nu^2+\frac{\sigma^2}{\rho}\right)\pi+(\bar{m}+k\alpha)^2\,\pi + (A_\theta+B_\theta\alpha)\sum_{\ell=1}^{L} f_\ell(\alpha)\langle m, f_\ell\rangle + C_\theta\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\langle m, f_\ell\rangle\right)^2, \tag{102}$$

its first derivative is given by

$$\partial_\alpha J(u^\alpha, z)=2k\pi\,(\bar{m}+k\alpha) + (A_\theta+B_\theta\alpha)\sum_{\ell=1}^{L}\partial_\alpha f_\ell(\alpha)\langle m, f_\ell\rangle + 2C_\theta\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\langle m, f_\ell\rangle\right)\left(\sum_{\ell=1}^{L}\partial_\alpha f_\ell(\alpha)\langle m, f_\ell\rangle\right)^2 + B_\theta\left(\sum_{\ell=1}^{L} f_\ell(\alpha)\langle m, f_\ell\rangle\right), \tag{103, 104}$$

and its second derivative is given by

$$\partial^2_{\alpha\alpha} J(u^\alpha, z) = 2B_\theta \sum_{\ell=1}^{L} \partial_\alpha f_\ell(\alpha)\langle m, f_\ell\rangle + 2C_\theta \left(\sum_{\ell=1}^{L} \partial_\alpha f_\ell(\alpha)\langle m, f_\ell\rangle\right)^2 + 2C_\theta \left(\sum_{\ell=1}^{L} f_\ell(\alpha)\langle m, f_\ell\rangle\right)\left(\sum_{\ell=1}^{L} \partial^2_{\alpha\alpha} f_\ell(\alpha)\langle m, f_\ell\rangle\right) + 2k^2\pi + (A_\theta + B_\theta\alpha)\sum_{\ell=1}^{L} \partial^2_{\alpha\alpha} f_\ell(\alpha)\langle m, f_\ell\rangle.$$

Clearly, all $\alpha^* \in (0,1)$ satisfying (95) and (96), it follows that $\partial_\alpha J(u^{\alpha^*}, z) = 0$, and moreover, if ((97)) is satisied by $\alpha^* \in (0,1)$, it follows that $\partial^2_{\alpha\alpha} J(u^{\alpha^*}, z) \geq 0$. The proof is complete.

7. Numerical Illustrations

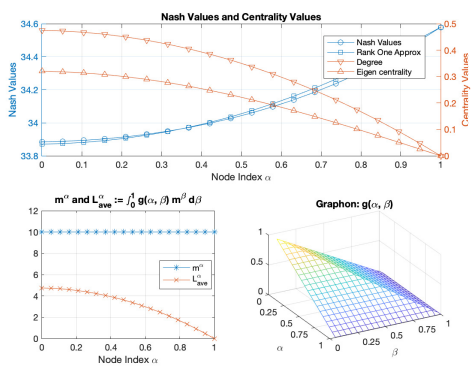

Figure 1: Nash values with homogeneous initial mean.

The uniform attachment graphon is used as an example. It is not of finite rank, but it admits good finite rank approximations due to its nice spectral properties (see Gao et al. (2023)). We also plot the graphon's eigen centrality and degree centrality in order to contrast them with Nash values. We observe that Nash values are sensitive to the homogeneity of the initial conditions. We recall the definitions of eigen centrality and degree centrality.

Definition 2. *Let $g(\cdot,\cdot)$ be a graphon, we have:*

1. *Degree centrality: For every node $\alpha \in [0,1]$, the degree centrality of $g(\cdot,\cdot)$ is defined by*

$$d(\alpha) := \int_0^1 g(\alpha,\beta)d\beta. \tag{105}$$

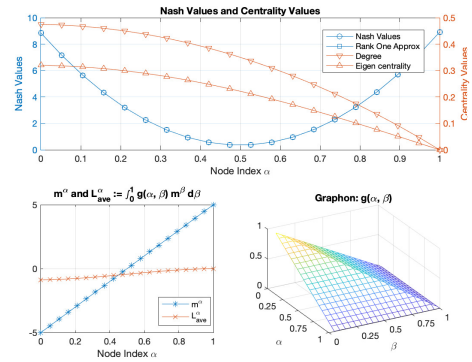

Figure 2: Nash values with inhomogenous initial mean

2. *Eigen centrality: Consider $f_1(\cdot)$ the graphon's eigenfunction associated with the largest eigenvalue. For every node $\alpha \in [0,1]$, the eigen centrality of $g(\cdot,\cdot)$ is defined by*

$$e(\alpha) := f_1(\alpha). \tag{106}$$

The following parameters are used for numerical examples: $a = \rho = 0.5, \nu = 1, \sigma = 0.15, b = 1, m = 10$.

In Fig. 1, the numerical result is illustrated for the case with homogeneous means for the initial conditions across different node indices. The top figure illustrates Nash values, the eigen centrality and the degree centrality. In addition, Nash values for the games with the limit graphon are compared with those approximately given by the rank-one approximation of the limit graphon, and due to the spectral property of the uniform attachment graphon (see Gao et al. (2023)), the rank-one approximation error is small. As observed in Foguen-Tchuendom et al. (2022a) the node with local minimal Nash value corresponds to the node with the maximum degree.

In Fig. 2, the numerical result is illustrated for the case with inhomogeneous means for the initial conditions across different node indices. The initial means m^α are assumed to be an affine function of the index α. We note that in this situation, the minimum Nash value node is no longer the node with the maximum degree.

8. Conclusion

This work established the Nash values for linear quadratic graphon mean field games with infinite time horizon, and analyzed its properties with respect to the variations of nodal index. Further studies should include the properties of the Nash values for nonlinear

graphon mean field game problems (see e.g. the control affine cases in Caines et al. (2023)), investigate the situations where the Nash values influence the individual decisions in the large dynamic network games, and build further relations and comparisons with centrality notions for graphons (see Gao (2022); Avella-Medina et al. (2018)).

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