Fundamentals of Robotic Mechanical Systems

Chapter 3: Fundamentals of Rigid-Body Mechanics

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Overview

- Introduction
- General Rigid-Body Motion and Its Associated Screw
  - The Screw of a Rigid-Body Motion
  - The Plücker Coordinates of a Line
  - The Pose of a Rigid Body
- Rotation of a Rigid Body About a Fixed Point
- General Instantaneous Motion of a Rigid Body
  - The Instant Screw of a Rigid-Body Motion
  - The Twist of a Rigid Body
- Acceleration Analysis of Rigid-Body Motions
- Rigid-Body Motion Referred to Moving Coordinate Axes
- Static Analysis of Rigid Bodies
- Dynamics of Rigid Bodies

General Rigid-Body Motion and Its Associated Screw

Point A : a reference point of rigid body $\mathcal{B}$ with the position vector $\mathbf{a}$ in the original configuration and $\mathbf{a}'$ in the displaced configuration, A'.

Point P : an arbitrary point of $\mathcal{B}$ with the position vector $\mathbf{p}$ in the original configuration and $\mathbf{p}'$ in the displaced configuration, P'.

Determine $\mathbf{p}'$ with given $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{p}$, and the rotation matrix $\mathbf{Q}$:

$$\mathbf{p}' - \mathbf{a}' = \mathbf{Q}(\mathbf{p} - \mathbf{a}) \quad \Rightarrow \quad \mathbf{p}' = \mathbf{a}' + \mathbf{Q}(\mathbf{p} - \mathbf{a}) \tag{3.2}$$

General Rigid-Body Motion and Its Associated Screw (Cont'd)

Displacements of A and P :

$$\mathbf{d}_A \equiv \mathbf{a}' - \mathbf{a}, \quad \mathbf{d}_P \equiv \mathbf{p}' - \mathbf{p} \tag{3.3}$$

From eqs.(3.2) and (3.3) :

$$\begin{aligned} \mathbf{d}_P &= \mathbf{p}' - \mathbf{p} = \mathbf{a}' - \mathbf{p} + \mathbf{Q}(\mathbf{p} - \mathbf{a}) \\ &= \mathbf{a}' - \mathbf{a} - \mathbf{p} + \mathbf{Q}(\mathbf{p} - \mathbf{a}) + \mathbf{a} \end{aligned}$$

$$\Rightarrow \quad \mathbf{d}_P = \mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p} - \mathbf{a}) \tag{3.5}$$

General Rigid-Body Motion and Its Associated Screw (Cont'd)

Theorem

The component of the displacements of all the points of a rigid body undergoing a general motion along the axis of the underlying rotation is a constant.

Proof : Multiply both sides of eq.(3.5) by $\mathbf{e}^T$:

$$\mathbf{e}^T\mathbf{d}_P = \mathbf{e}^T\mathbf{d}_A + \mathbf{e}^T(\mathbf{Q}-\mathbf{1})(\mathbf{p}-\mathbf{a})$$

$$\mathbf{Q}\mathbf{e} = \mathbf{e} \quad \rightarrow \quad \mathbf{e}^T\mathbf{Q} - \mathbf{e}^T = \mathbf{0}^T \quad \rightarrow \quad \mathbf{e}^T(\mathbf{Q}-\mathbf{1}) = \mathbf{0}^T$$

$$\Rightarrow \quad \mathbf{e}^T\mathbf{d}_P = \mathbf{e}^T\mathbf{d}_A \equiv d_0 \tag{3.6}$$

General Rigid-Body Motion and Its Associated Screw (Cont'd)

Theorem (Mozzi, 1763 ; Chasles, 1830)

Given a rigid body undergoing a general motion, a set of its points located on a line $\mathcal{L}$ undergo identical displacements of minimum magnitude. Moreover, line $\mathcal{L}$ and the minimum-magnitude displacement are parallel to the axis of the rotation involved.

Proof :

$$\mathbf{d}_P = \mathbf{d}_{\parallel} + \mathbf{d}_{\perp} \tag{3.7a}$$

where

$$\mathbf{d}_{\parallel} = \mathbf{e}\mathbf{e}^T\mathbf{d}_P = d_0\mathbf{e}, \quad \mathbf{d}_{\perp} = (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_P \tag{3.7b}$$

$$\|\mathbf{d}_P\|^2 = \|\mathbf{d}_{\parallel}\|^2 + \|\mathbf{d}_{\perp}\|^2 = d_0^2 + \|\mathbf{d}_{\perp}\|^2$$

General Rigid-Body Motion and Its Associated Screw (Cont'd)

Minimize displacement $\|\mathbf{d}_P\|$: $\quad \mathbf{d}_\perp = \mathbf{0} \Rightarrow \mathbf{d}_P = \alpha\mathbf{e}$

$\Rightarrow$ All points of minimum-magnitude displacement lie in a line parallel to the axis of rotation of $\mathbf{Q}$, q.e.d.

If P^* is a point of minimum-magnitude displacement, then

$$\mathbf{d}_\perp^* \equiv (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_{P^*} = (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A + (\mathbf{1} - \mathbf{e}\mathbf{e}^T)(\mathbf{Q} - \mathbf{1})(\mathbf{p}^* - \mathbf{a}) = \mathbf{0}$$

$$\Rightarrow \quad (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}^* - \mathbf{a}) = \mathbf{0}$$

Above equation holds if $\alpha\mathbf{e}$ is added to $\mathbf{p}^*$,
$\Rightarrow$ Screw motion of axes $\mathcal{L}$ and pitch p :

$$p \equiv \frac{d_0}{\phi} = \frac{\mathbf{d}_P^T\mathbf{e}}{\phi} \qquad \text{or} \qquad p \equiv \frac{2\pi d_0}{\phi} \tag{3.8}$$

The Screw of a Rigid-Body Motion

The screw axis $\mathcal{L}$: specified by P_0 and $\mathbf{e}$.
If P_0 is closest to the origin, then

$$\mathbf{e}^T \mathbf{p}_0 = 0 \tag{3.9}$$

and its displacement $\mathbf{d}_0$ is identical to $\mathbf{d}_{\|}$:

$$(\mathbf{Q} - \mathbf{1})\mathbf{d}_0 = \mathbf{0}$$

where, from eq.(3.5),

$$\mathbf{d}_0 = \mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}_0 - \mathbf{a}) \tag{3.10a}$$

and, from eq.(3.7b),

$$\mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}_0 - \mathbf{a}) = \mathbf{d}_{\|} \equiv \mathbf{e}\mathbf{e}^T \mathbf{d}_0$$

The Screw of a Rigid-Body Motion
(Cont'd)

From Theorem 1

$$\mathbf{e}^T\mathbf{d}_0 = \mathbf{e}^T\mathbf{d}_A \quad \Rightarrow \quad \mathbf{d}_A + (\mathbf{Q}-\mathbf{1})(\mathbf{p}_0 - \mathbf{a}) = \mathbf{e}\mathbf{e}^T\mathbf{d}_A$$

or

$$(\mathbf{Q}-\mathbf{1})\mathbf{p}_0 = (\mathbf{Q}-\mathbf{1})\mathbf{a} - (\mathbf{1}-\mathbf{e}\mathbf{e}^T)\mathbf{d}_A \qquad (3.10b)$$

Adjoin eqs.(3.9) and (3.10b) :

$$\mathbf{A}\mathbf{p}_0 = \mathbf{b} \qquad (3.11)$$

where

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{Q}-\mathbf{1} \\ \mathbf{e}^T \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} (\mathbf{Q}-\mathbf{1})\mathbf{a} - (\mathbf{1}-\mathbf{e}\mathbf{e}^T)\mathbf{d}_A \\ 0 \end{bmatrix} \qquad (3.12)$$

The Screw of a Rigid-Body Motion
(Cont'd)

$$\mathbf{A}^T\mathbf{A}\mathbf{p}_0 = \mathbf{A}^T\mathbf{b} \tag{3.13}$$

Substituting into eq.(3.12) expression

$$\mathbf{Q} = \mathbf{e}\mathbf{e}^T + \cos\phi(\mathbf{1} - \mathbf{e}\mathbf{e}^T) + \sin\phi\mathbf{E}$$

yields $\quad \mathbf{A}^T\mathbf{A} = 2(1-\cos\phi)\mathbf{1} - (1 - 2\cos\phi)\mathbf{e}\mathbf{e}^T$

or $\quad (\mathbf{A}^T\mathbf{A})^{-1} = \alpha\mathbf{1} + \beta\mathbf{e}\mathbf{e}^T.$

Since $(\mathbf{A}^T\mathbf{A})(\mathbf{A}^T\mathbf{A})^{-1} = \mathbf{1}$,

$$\alpha = \frac{1}{2(1-\cos\phi)}, \quad \beta = \frac{1 - 2\cos\phi}{2(1-\cos\phi)} \tag{3.16}$$

The Screw of a Rigid-Body Motion
(Cont'd)

$$\Rightarrow \quad (\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{2(1-\cos\phi)}\mathbf{1} + \frac{1-2\cos\phi}{2(1-\cos\phi)}\mathbf{e}\mathbf{e}^T \tag{3.17}$$

On the other hand,

$$\mathbf{A}^T\mathbf{b} = (\mathbf{Q}-\mathbf{1})^T[(\mathbf{Q}-\mathbf{1})\mathbf{a} - \mathbf{d}_A] \tag{3.18}$$

Upon solving eq.(3.13) for $\mathbf{p}_0$ and substituting relations (3.17) and (3.18) into the expression thus resulting, one finally obtains

$$\mathbf{p}_0 = \frac{(\mathbf{Q}-\mathbf{1})^T(\mathbf{Q}\mathbf{a}-\mathbf{a}')}{2(1-\cos\phi)}, \quad \text{for } \phi \neq 0 \tag{3.19}$$

The Plücker Coordinates of a Line

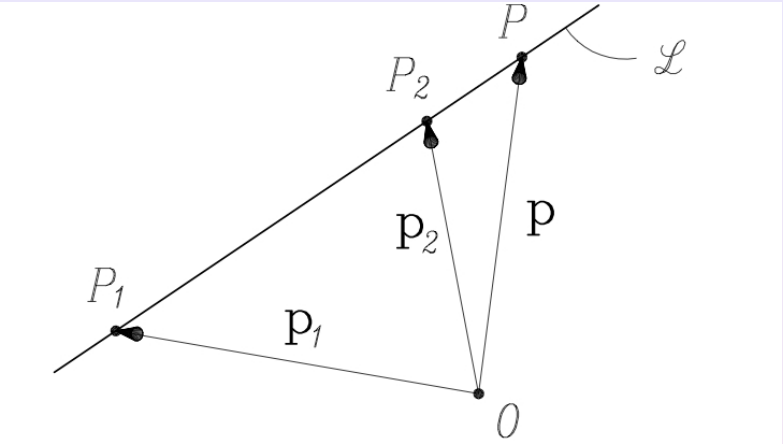

FIGURE: A line $\mathcal{L}$ passing through two points.

$$(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p} - \mathbf{p}_1) = \mathbf{0}$$

$$\Rightarrow \quad (\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{p} + \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{0}$$

The Plücker Coordinates of a Line
(Cont'd)

$$(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{p} + \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{P}_2 - \mathbf{P}_1 & \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \mathbf{0} \tag{3.21}$$

We define

$$\boldsymbol{\gamma}_{\mathcal{L}} \equiv \begin{bmatrix} \mathbf{p}_2 - \mathbf{p}_1 \\ \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) \end{bmatrix} \tag{3.22}$$

Components of $\boldsymbol{\gamma}_{\mathcal{L}}$: Plücker coordinates of $\mathcal{L}$.

The Plücker Coordinates of a Line
(Cont'd)

Let

$$\mathbf{e} \equiv \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|}, \quad \mathbf{n} \equiv \mathbf{p}_1 \times \mathbf{e} \tag{3.23}$$

$$\Rightarrow \quad \boldsymbol{\gamma}_{\mathcal{L}} = \|\mathbf{p}_2 - \mathbf{p}_1\| \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \end{bmatrix}$$

Vector $\mathbf{e}$ determines the direction of $\mathcal{L}$, while $\mathbf{n}$, the moment of $\mathcal{L}$, determines its location.

Plücker array $\boldsymbol{\kappa}$:

$$\boldsymbol{\kappa} = \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \end{bmatrix} \tag{3.24}$$

The Plücker Coordinates of a Line
(Cont'd)

There are only four independent Plücker coordinates of line $\mathcal{L}$ since :

$$\mathbf{e} \cdot \mathbf{e} = 1, \quad \mathbf{n} \cdot \mathbf{e} = 0 \tag{3.25}$$

That is,

(i) the sum of the squares of the first three components of a Plücker array is unity, and

(ii) the unit vector of a line is normal to the moment of the line

$\Rightarrow$ Set of Plücker arrays does not form a vector space

The Plücker Coordinates of a Line
(Cont'd)

$$\mathbf{n}_A \equiv (\mathbf{p} - \mathbf{a}) \times \mathbf{e}, \quad \mathbf{n}_B \equiv (\mathbf{p} - \mathbf{b}) \times \mathbf{e} \tag{3.26}$$

$$\Rightarrow \quad \boldsymbol{\kappa}_A \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_A \end{bmatrix}, \quad \boldsymbol{\kappa}_B \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_B \end{bmatrix} \tag{3.27}$$

$$\mathbf{n}_B - \mathbf{n}_A = (\mathbf{a} - \mathbf{b}) \times \mathbf{e} \tag{3.28}$$

$$\Rightarrow \quad \boldsymbol{\kappa}_B = \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_A + (\mathbf{a} - \mathbf{b}) \times \mathbf{e} \end{bmatrix} \tag{3.29}$$

or $\boldsymbol{\kappa}_B = \mathbf{U}\boldsymbol{\kappa}_A$: Plücker-coordinate transfer formula

$$\mathbf{U} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{A} - \mathbf{B} & \mathbf{1} \end{bmatrix} \tag{3.31}$$

The Plücker Coordinates of a Line
(Cont'd)

$$\det(\mathbf{U}) = 1$$

$\Rightarrow$ $\mathbf{U}$ belongs to the unimodular group of 6×6 matrices

$$\Rightarrow \quad \mathbf{U}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{B} - \mathbf{A} & \mathbf{1} \end{bmatrix} \tag{3.33}$$

Note that from eq.(3.28) :

$$(\mathbf{a} - \mathbf{b})^T \mathbf{n}_B = (\mathbf{a} - \mathbf{b})^T \mathbf{n}_A \tag{3.34}$$

$\Rightarrow$ The moments of the same line $\mathcal{L}$ with respect to two points are not independent : they have the same component along the line joining the two points.

The Plücker Coordinates of a Line
(Cont'd)

A line at infinity : the orientation is undefined but a direction of its moment is defined

If we rewrite eq.(3.24) in the form

$$\boldsymbol{\kappa} = \|\mathbf{n}\| \begin{bmatrix} \mathbf{e}/\|\mathbf{n}\| \\ \mathbf{n}/\|\mathbf{n}\| \end{bmatrix}, \quad \mathbf{f} \equiv \mathbf{n}/\|\mathbf{n}\|$$

$$\text{then, } \lim_{\|\mathbf{n}\|\to\infty} \boldsymbol{\kappa} = \left(\lim_{\|\mathbf{n}\|\to\infty} \|\mathbf{n}\| \right) \left(\lim_{\|\mathbf{n}\|\to\infty} \begin{bmatrix} \mathbf{e}/\|\mathbf{n}\| \\ \mathbf{f} \end{bmatrix} \right)$$

$$\Rightarrow \lim_{\|\mathbf{n}\|\to\infty} \boldsymbol{\kappa} = \left(\lim_{\|\mathbf{n}\|\to\infty} \|\mathbf{n}\| \right) \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \Rightarrow \boldsymbol{\kappa}_\infty = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \tag{3.35}$$

The Pose of a Rigid Body

A rigid-body motion is fully described by six independent or seven dependent parameters.
Hence, if a rigid body undergoes a general motion of rotation $\mathbf{Q}$ and displacement $\mathbf{d}$ from a reference configuration $\mathcal{C}_0$, then pose array, or the pose, $\mathbf{s}$ of the body, is defined as a 7-dimensional array :

$$\mathbf{s} \equiv \begin{bmatrix} \mathbf{q}^T & q_0 & \mathbf{d}_A^T \end{bmatrix}^T \tag{3.36}$$

where $\mathbf{q}$ and q_0 are any four invariants of $\mathbf{Q}$.

The Pose of a Rigid Body (Cont'd)

For example, Euler-Rodrigues parameters :

$$\mathbf{q} \equiv \sin(\phi/2)\mathbf{e}, \quad q_0 \equiv \cos(\phi/2)$$

linear invariants : $\mathbf{q} \equiv (\sin\phi)\mathbf{e}, \qquad q_0 \equiv \cos\phi$

with $\|\mathbf{q}\|^2 + q_0^2 = 1$;

the natural invariants : $\mathbf{q} \equiv \mathbf{e}, \quad q_0 \equiv \phi$ with $\|\mathbf{e}\|^2 = 1$.

Pose estimation : the computation of the attitude of a rigid body, given by matrix $\mathbf{Q}$ or its invariants from coordinate measurements over a certain finite set of points.

The Pose of a Rigid Body (Cont'd)

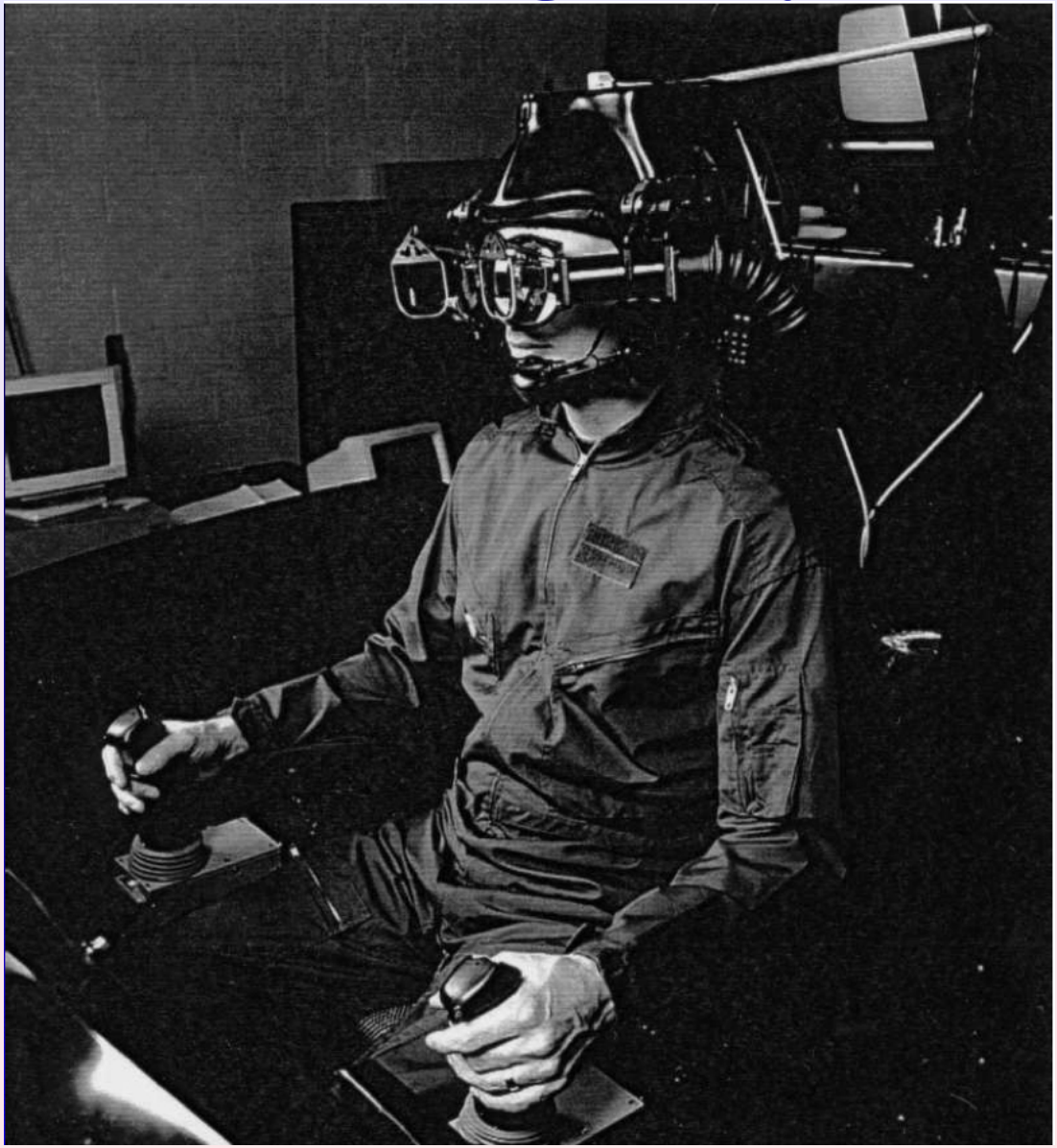

FIGURE: Helmet-mounted display system (CAE Electronics, St.-Laurent, Quebec, Canada.)

The Pose of a Rigid Body (Cont'd)

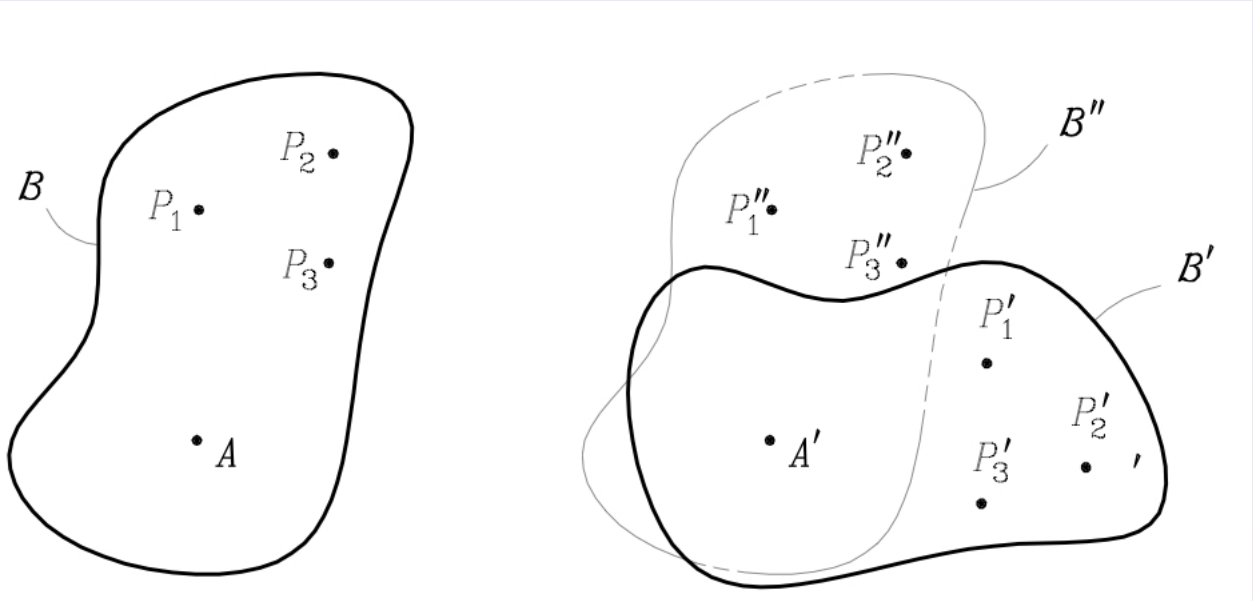

FIGURE: Decomposition of the displacement of a rigid body.

To determine matrix $\mathbf{Q}$, define P_1, P_2, and P_3 so that

$$\begin{aligned} \mathbf{e}_1 &\equiv \overrightarrow{AP_1}, \quad \mathbf{e}_2 \equiv \overrightarrow{AP_2}, \quad \mathbf{e}_3 \equiv \overrightarrow{AP_3} \\ \mathbf{e}_i \cdot \mathbf{e}_j &= \delta_{ij}, \quad i,j = 1,2,3, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \end{aligned}$$

where δ_{ij} is the Kronecker delta, defined as 1 if $i = j$ and 0 otherwise.

The Pose of a Rigid Body (Cont'd)

Let q_{ij} be the entries of $\mathbf{Q}$ in X, Y, Z with the origin at A and with X, Y, Z parallel to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. From Definition 2.2.1, $q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$, i.e.,

$$[\,\mathbf{Q}\,] = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 & \mathbf{e}_1 \cdot \mathbf{e}'_3 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 & \mathbf{e}_2 \cdot \mathbf{e}'_3 \\ \mathbf{e}_3 \cdot \mathbf{e}'_1 & \mathbf{e}_3 \cdot \mathbf{e}'_2 & \mathbf{e}_3 \cdot \mathbf{e}'_3 \end{bmatrix} \tag{3.42}$$

Note that all $\mathbf{e}_i$ and $\mathbf{e}'_i$ in eq.(3.42) must be represented in the same coordinate frame.

Then, use, for example, eq.(3.19) to determine the point of the screw axis that lies closest to the origin and compute the Plücker coordinates of the screw axis.

Rotation of a Rigid Body About a Fixed Point

Fully described by a rotation matrix $\mathbf{Q}$ that is proper orthogonal and is a smooth function of time. Then,

$$\mathbf{p}(t) = \mathbf{Q}(t)\mathbf{p}_0 \tag{3.43}$$

$$\Rightarrow \quad \dot{\mathbf{p}}(t) = \dot{\mathbf{Q}}(t)\mathbf{p}_0 \tag{3.44}$$

Solve eq.(3.43) for $\mathbf{p}_0$ and substitute the result into eq.(3.44) :

$$\dot{\mathbf{p}} = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{p} \tag{3.45}$$

The angular-velocity matrix :

$$\mathbf{\Omega} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T \tag{3.46}$$

Rotation of a Rigid Body About a Fixed Point (Cont'd)

Theorem

The angular-velocity matrix is skew symmetric.

The angular-velocity vector $\boldsymbol{\omega}$ of the rigid-body motion is defined as

$$\boldsymbol{\omega} \equiv \mathrm{vect}(\boldsymbol{\Omega}) \tag{3.47}$$

and hence, eq.(3.45) can be written as

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}\mathbf{p} = \boldsymbol{\omega} \times \mathbf{p} \tag{3.48}$$

$\Rightarrow$ The velocity of any point P of a body moving with a point O fixed is perpendicular to line OP.

General Instantaneous Motion of a Rigid Body (Cont'd)

$$\mathbf{p}(t) = \mathbf{a}(t) + \mathbf{Q}(t)(\mathbf{p}_0 - \mathbf{a}_0) \tag{3.49}$$

$$\Rightarrow \quad \dot{\mathbf{p}}(t) = \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)(\mathbf{p}_0 - \mathbf{a}_0) \tag{3.50}$$

Then, eq.(3.49) is solved for $(\mathbf{p}_0 - \mathbf{a}_0)$ and the result is substituted into eq.(3.50) :

$$\dot{\mathbf{p}} = \dot{\mathbf{a}} + \mathbf{\Omega}(\mathbf{p} - \mathbf{a}) \tag{3.51}$$

$$\Rightarrow \quad \dot{\mathbf{p}} = \dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}) \tag{3.52}$$

$$\Rightarrow \quad (\dot{\mathbf{p}} - \dot{\mathbf{a}}) \cdot (\mathbf{p} - \mathbf{a}) = 0 \tag{3.53}$$

Theorem

The relative velocity of two points of the same rigid body is perpendicular to the line joining them.

General Instantaneous Motion of a Rigid Body (Cont'd)

Upon dot-multiplying both sides of eq.(3.52) by $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} \cdot \dot{\mathbf{p}} = \boldsymbol{\omega} \cdot \dot{\mathbf{a}} = \text{constant} \quad \Rightarrow$$

Corollary

The projections of the velocities of all the points of a rigid body onto the angular-velocity vector are identical.

Theorem

Given a rigid body under general motion, a set of its points located on a line $\mathcal{L}'$ undergoes the identical minimum-magnitude velocity $\mathbf{v}_0$ parallel to the angular velocity.

Definition

The line containing the points of a rigid body undergoing minimum-magnitude velocities is called the instant screw axis (ISA) of the body under the given motion.

The Instant Screw of a Rigid-Body Motion

The instantaneous motion of a body is equivalent to that of the bolt of a screw of instantaneous axis $\mathcal{L}'$, the ISA, and is called an instantaneous screw.
Since $\mathbf{v}_0 \parallel \boldsymbol{\omega}$,

$$\mathbf{v}_0 = v_0 \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \tag{3.54}$$

$$\Rightarrow \text{the pitch: } p' \equiv \frac{v_0}{\|\boldsymbol{\omega}\|} \equiv \frac{\dot{\mathbf{p}} \cdot \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \quad \text{or} \quad p' \equiv \frac{2\pi v_0}{\|\boldsymbol{\omega}\|} \tag{3.55}$$

Plücker coordinates of ISA $\mathcal{L}'$:

$$\mathbf{p}_{\mathcal{L}'} \equiv \begin{bmatrix} \mathbf{e}' \\ \mathbf{n}' \end{bmatrix} \tag{3.56}$$

$$\text{with} \quad \mathbf{e}' \equiv \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}, \quad \mathbf{n}' \equiv \mathbf{p} \times \mathbf{e}' \tag{3.57}$$

The Instant Screw of a Rigid-Body Motion (Cont'd)

An instantaneous rigid-body motion is defined by a line $\mathcal{L}'$, a pitch p', and an amplitude $\|\boldsymbol{\omega}\|$:

$$-\infty \leq p' \leq +\infty$$

A line supplied with a pitch : a screw

A screw supplied with an amplitude representing the magnitude of an angular velocity : a twist.

The Instant Screw of a Rigid-Body Motion (Cont'd)

The ISA can be alternatively described in terms of $\mathbf{p}_0'$, its point lying closest to the origin. Let

$$\dot{\mathbf{a}} \equiv \dot{\mathbf{a}}_{\parallel} + \dot{\mathbf{a}}_{\perp} \tag{3.58}$$

where

$$\begin{aligned} \dot{\mathbf{a}}_{\parallel} &\equiv \dot{\mathbf{a}} \cdot \boldsymbol{\omega} \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \equiv \frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2} \dot{\mathbf{a}}, \\ \dot{\mathbf{a}}_{\perp} &\equiv \left(\mathbf{1} - \frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2} \right) \dot{\mathbf{a}} \equiv -\frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}^2 \dot{\mathbf{a}} \end{aligned} \tag{3.59}$$

if we use the identity in (2.39) :

$$\boldsymbol{\Omega}^2 \equiv \boldsymbol{\omega}\boldsymbol{\omega}^T - \|\boldsymbol{\omega}\|^2 \mathbf{1}, \tag{3.60}$$

The Instant Screw of a Rigid-Body Motion (Cont'd)

Substitute eq.(3.59) into eq.(3.52) :

$$\dot{\mathbf{p}} = \underbrace{\frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2}\dot{\mathbf{a}}}_{\dot{\mathbf{p}}_{\parallel}} \underbrace{-\frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}^2\dot{\mathbf{a}} + \boldsymbol{\Omega}(\mathbf{p}-\mathbf{a})}_{\dot{\mathbf{p}}_{\perp}} \tag{3.61}$$

$\frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2}\dot{\mathbf{a}}$: the axial component of $\dot{\mathbf{p}}$ ($\parallel$ to $\boldsymbol{\omega}$)

$-\frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}^2\dot{\mathbf{a}} + \boldsymbol{\Omega}(\mathbf{p}-\mathbf{a})$: the normal component of $\dot{\mathbf{p}}$ ($\perp$ to $\boldsymbol{\omega}$)

Is it possible, for an arbitrary motion, to find a certain point of position vector $\mathbf{p}$ whose velocity normal component vanishes ?

The Instant Screw of a Rigid-Body Motion (Cont'd)

$$\dot{\mathbf{p}}_{\perp} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\Omega}(\mathbf{p} - \mathbf{a}) - \frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}^2\,\dot{\mathbf{a}} = \mathbf{0} \tag{3.62}$$

$$\Rightarrow \quad \boldsymbol{\Omega}\mathbf{p} = \boldsymbol{\Omega}\left(\mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}\dot{\mathbf{a}}\right) \tag{3.63}$$

or

$$\boldsymbol{\Omega}(\mathbf{p} - \mathbf{r}) = \mathbf{0} \tag{3.64a}$$

with

$$\mathbf{r} \equiv \mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}\dot{\mathbf{a}} \tag{3.64b}$$

The Instant Screw of a Rigid-Body Motion (Cont'd)

A possible solution is

$$\mathbf{p} = \mathbf{r} = \mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}\dot{\mathbf{a}} \tag{3.65}$$

This solution is not unique :

$\alpha\boldsymbol{\omega} + \mathbf{p}$ also satisfies eq.(3.63)

$\Rightarrow$ eq.(3.63) determines a set of points lying on the ISA.

The Instant Screw of a Rigid-Body Motion (Cont'd)

For example, we can find the point of the ISA $\mathbf{p}_0'$ lying closest to the origin. Obviously :

$$\boldsymbol{\omega}^T \mathbf{p}_0' = 0 \tag{3.66}$$

If eq.(3.63) is rewritten for $\mathbf{p}_0'$ and eq.(3.66) is adjoined to it, then,

$$\mathbf{A}\mathbf{p}_0' = \mathbf{b} \tag{3.67}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} \boldsymbol{\Omega} \\ \boldsymbol{\omega}^T \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} \boldsymbol{\Omega}\mathbf{a} + (1/\|\boldsymbol{\omega}\|^2)\boldsymbol{\Omega}^2\dot{\mathbf{a}} \\ 0 \end{bmatrix} \tag{3.68}$$

The Instant Screw of a Rigid-Body Motion (Cont'd)

Multiply eq.(3.67) by $\mathbf{A}^T$: $\quad \mathbf{A}^T\mathbf{A}\mathbf{p}_0' = \mathbf{A}^T\mathbf{b}$ (3.69)

$$\Rightarrow \quad \mathbf{A}^T\mathbf{A} = \boldsymbol{\Omega}^T\boldsymbol{\Omega} + \boldsymbol{\omega}\boldsymbol{\omega}^T = -\boldsymbol{\Omega}^2 + \boldsymbol{\omega}\boldsymbol{\omega}^T \tag{3.70}$$

Use eq.(3.60) :

$$\boldsymbol{\Omega}^2 \equiv \boldsymbol{\omega}\boldsymbol{\omega}^T - \|\boldsymbol{\omega}\|^2\mathbf{1}$$

Hence, we have

$$\mathbf{A}^T\mathbf{A} = \|\boldsymbol{\omega}\|^2\mathbf{1}, \quad \mathbf{A}^T\mathbf{b} = \boldsymbol{\Omega}(\dot{\mathbf{a}} - \boldsymbol{\Omega}\mathbf{a}) \tag{3.71}$$

The Instant Screw of a Rigid-Body Motion (Cont'd)

Substitute eqs.(3.71) into eq.(3.69) and solve for $\mathbf{p}_0'$:

$$\mathbf{p}_0' = \frac{\boldsymbol{\Omega}(\dot{\mathbf{a}} - \boldsymbol{\Omega}\mathbf{a})}{\|\boldsymbol{\omega}\|^2} \equiv \frac{\boldsymbol{\omega} \times (\dot{\mathbf{a}} - \boldsymbol{\omega} \times \mathbf{a})}{\|\boldsymbol{\omega}\|^2} \tag{3.72}$$

$\Rightarrow$ the instantaneous screw is fully defined by an alternative set of 6 independent scalars : 3 components of $\boldsymbol{\omega}$ and 3 components of $\dot{\mathbf{a}}$.

The Twist of a Rigid Body

A pitch p added as a fifth feature to the Plücker array of a line $\Rightarrow$ a screw $\mathbf{s}$:

$$\mathbf{s} \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + p\mathbf{e} \end{bmatrix} \tag{3.73}$$

An amplitude A multiplying the screw produces a twist or a wrench as an 8-parameter array : the amplitude, the pitch, and the 6 Plücker coordinates of the associated line, with only 6 of them independent.

A twist defines completely the velocity field of a rigid body : 3 components of the angular velocity and 3 components of the velocity of any of the points of the body.

The Twist of a Rigid Body (Cont'd)

Upon multiplication of the screw in eq.(3.73) by A :

$$\mathbf{t} \equiv \begin{bmatrix} A\mathbf{e} \\ \mathbf{p} \times (A\mathbf{e}) + p(A\mathbf{e}) \end{bmatrix}$$

Here : $A\mathbf{e}$ is the angular velocity $\boldsymbol{\omega} \parallel \mathbf{e}$ of magnitude A.

The lower vector of $\mathbf{t}$, $\mathbf{v}$, is the velocity of point O of a rigid body moving with an angular velocity $\boldsymbol{\omega}$ so that point P moves with a velocity $p\boldsymbol{\omega} \parallel \boldsymbol{\omega}$:

$$\mathbf{v} = -\boldsymbol{\omega} \times \mathbf{p} + p\boldsymbol{\omega}$$

$$\Rightarrow \quad \text{the twist} \quad \mathbf{t} \equiv \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{bmatrix} \tag{3.74}$$

The Twist of a Rigid Body (Cont'd)

The screw of infinitely large pitch :

$$\lim_{p\to\infty}\begin{bmatrix}\mathbf{e}\\ \mathbf{p}\times\mathbf{e}+p\mathbf{e}\end{bmatrix} \equiv \lim_{p\to\infty}\left(p\begin{bmatrix}\mathbf{e}/p\\ (\mathbf{p}\times\mathbf{e})/p+\mathbf{e}\end{bmatrix}\right) = \left(\lim_{p\to\infty}p\right)\begin{bmatrix}\mathbf{0}\\ \mathbf{e}\end{bmatrix}$$

$$\text{The screw of infinite pitch} \quad \mathbf{s}_\infty \equiv \begin{bmatrix}\mathbf{0}\\ \mathbf{e}\end{bmatrix} \tag{3.75}$$

is identical to the Plücker array of the line at ∞ in a plane of unit normal $\mathbf{e}$.

The Twist of a Rigid Body (Cont'd)

The relationships between $\boldsymbol{\omega}$ and the time derivatives of the invariants of the associated rotation :

$$\boldsymbol{\nu} \equiv \begin{bmatrix} \mathbf{e} \\ \phi \end{bmatrix}, \ \boldsymbol{\lambda} \equiv \begin{bmatrix} (\sin\phi)\mathbf{e} \\ \cos\phi \end{bmatrix}, \ \boldsymbol{\eta} \equiv \begin{bmatrix} [\sin(\phi/2)]\mathbf{e} \\ \cos(\phi/2) \end{bmatrix},$$

$$\dot{\boldsymbol{\nu}} = \mathbf{N}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\lambda}} = \mathbf{L}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\eta}} = \mathbf{H}\boldsymbol{\omega} \tag{3.77a}$$

$$\mathbf{N} \equiv \begin{bmatrix} [\sin\phi/(2(1-\cos\phi))](\mathbf{1} - \mathbf{e}\mathbf{e}^T) - (1/2)\mathbf{E} \\ \mathbf{e}^T \end{bmatrix},$$

$$\mathbf{L} \equiv \begin{bmatrix} (1/2)[\mathrm{tr}(\mathbf{Q})\mathbf{1} - \mathbf{Q}] \\ -(\sin\phi)\mathbf{e}^T \end{bmatrix}, \ \mathbf{H} \equiv \frac{1}{2}\begin{bmatrix} \cos(\phi/2)\mathbf{1} - \sin(\phi/2)\mathbf{E} \\ -\sin(\phi/2)\mathbf{e}^T \end{bmatrix}$$

The Twist of a Rigid Body (Cont'd)

The inverse relations of (3.77a) :

$$\boldsymbol{\omega} = \widetilde{\mathbf{N}}\dot{\boldsymbol{\nu}} = \widetilde{\mathbf{L}}\dot{\boldsymbol{\lambda}} = \widetilde{\mathbf{H}}\dot{\boldsymbol{\eta}} \tag{3.78a}$$

where

$$\begin{aligned}
\widetilde{\mathbf{N}} &\equiv \left[(\sin\phi)\mathbf{1} + (1-\cos\phi)\mathbf{E} \quad \mathbf{e}\right], \\
\widetilde{\mathbf{L}} &\equiv \left[\mathbf{1} + \tfrac{\sin\phi}{1+\cos\phi}\mathbf{E} \quad -\tfrac{\sin\phi}{1+\cos\phi}\mathbf{e}\right], \\
\widetilde{\mathbf{H}} &\equiv 2\left[[\cos(\phi/2)]\mathbf{1} + [\sin(\phi/2)]\mathbf{E} \quad -[\sin(\phi/2)]\mathbf{e}\right]
\end{aligned}$$

Caveat : The angular velocity vector is **not** a time-derivative, i.e., no Cartesian vector exists whose time-derivative is the angular-velocity vector.

The Twist of a Rigid Body (Cont'd)

The relationships between the twist and the time-rate of change of the 7-dimensional pose array $\mathbf{s}$:

$$\dot{\mathbf{s}} = \mathbf{T}\mathbf{t} \tag{3.79}$$

where

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{F} & \mathbf{O}_{43} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \tag{3.80}$$

and

$$\mathbf{t} = \mathbf{S}\dot{\mathbf{s}} \tag{3.81a}$$

where

$$\mathbf{S} \equiv \begin{bmatrix} \widetilde{\mathbf{F}} & \mathbf{O} \\ \mathbf{O}_{34} & \mathbf{1} \end{bmatrix} \tag{3.81b}$$

The Twist of a Rigid Body (Cont'd)

The relation between the twists of the same rigid body at two different points, A and P :

$$\mathbf{t}_A = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_A \end{bmatrix}, \quad \mathbf{t}_P = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_P \end{bmatrix} \tag{3.82}$$

Equation (3.52), rewritten as $\mathbf{v}_P = \mathbf{v}_A + (\mathbf{a} - \mathbf{p}) \times \boldsymbol{\omega}$, combined with eq.(3.82) yields

$$\mathbf{t}_P = \mathbf{U}\mathbf{t}_A \quad \text{where} \quad \mathbf{U} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{A} - \mathbf{P} & \mathbf{1} \end{bmatrix} \tag{3.84}$$

where $\mathbf{A}$ and $\mathbf{P}$ denote the cross-product matrices of vectors $\mathbf{a}$ and $\mathbf{p}$, respectively. Equation (3.84) is called the twist-transfer formula.

Acceleration Analysis of Rigid-Body Motions (Cont'd)

Differentiate eq.(3.51) with respect to time :

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \dot{\mathbf{\Omega}}(\mathbf{p} - \mathbf{a}) + \mathbf{\Omega}(\dot{\mathbf{p}} - \dot{\mathbf{a}}) \tag{3.86}$$

Solving eq.(3.51) for $\dot{\mathbf{p}} - \dot{\mathbf{a}}$ and substituting the result into eq.(3.86), we obtain

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)(\mathbf{p} - \mathbf{a}) \tag{3.87}$$

where

$$\mathbf{W} \equiv \dot{\mathbf{\Omega}} + \mathbf{\Omega}^2 \tag{3.88}$$

is termed the angular-acceleration matrix.

Acceleration Analysis of Rigid-Body Motions (Cont'd)

The first term of the right-hand side of eq.(3.88) is skew-symmetric, the second one is symmetric. Thus,

$$\mathrm{vect}(\mathbf{W}) = \mathrm{vect}(\dot{\boldsymbol{\Omega}}) = \dot{\boldsymbol{\omega}} \tag{3.89}$$

the angular-acceleration vector. Moreover,

$$\begin{aligned} \mathrm{tr}(\mathbf{W}) &= \mathrm{tr}(\boldsymbol{\Omega}^2) = \mathrm{tr}(-\|\boldsymbol{\omega}\|^2\mathbf{1} + \boldsymbol{\omega}\boldsymbol{\omega}^T) \\ &= -\|\boldsymbol{\omega}\|^2\mathrm{tr}(\mathbf{1}) + \boldsymbol{\omega}\cdot\boldsymbol{\omega} = -2\|\boldsymbol{\omega}\|^2 \end{aligned} \tag{3.90}$$

Equation (3.87) can be written as

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \dot{\boldsymbol{\omega}} \times (\mathbf{p} - \mathbf{a}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a})] \tag{3.91}$$

Acceleration Analysis of Rigid-Body Motions (Cont'd)

The twist rate : $$\dot{\mathbf{t}} \equiv \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{v}} \end{bmatrix} \tag{3.92}$$

Differentiate eq.(3.79) : $\ddot{\mathbf{s}} = \mathbf{T}\dot{\mathbf{t}} + \dot{\mathbf{T}}\mathbf{t}$

$$\dot{\mathbf{T}} \equiv \begin{bmatrix} \dot{\mathbf{F}} & \mathbf{O}_{43} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.94}$$

and $\mathbf{F}$ is one of $\mathbf{N}$, $\mathbf{L}$, or $\mathbf{H}$, accordingly. For the inverse relationship, differentiate (3.81a) :

$$\dot{\mathbf{t}} = \mathbf{S}\ddot{\mathbf{s}} + \dot{\mathbf{S}}\dot{\mathbf{s}} \quad \text{where} \quad \dot{\mathbf{S}} = \begin{bmatrix} \dot{\tilde{\mathbf{F}}} & \mathbf{O} \\ \mathbf{O}_{34} & \mathbf{O} \end{bmatrix} \tag{3.96}$$

Acceleration Analysis of Rigid-Body Motions (Cont'd)

Let

$$\boldsymbol{\lambda} \equiv \begin{bmatrix} \mathbf{u} \\ u_0 \end{bmatrix}, \quad \boldsymbol{\eta} \equiv \begin{bmatrix} \mathbf{r} \\ r_0 \end{bmatrix} \tag{3.97a}$$

where

$$\begin{aligned} \mathbf{u} &\equiv \sin\phi\mathbf{e}, \quad u_0 \equiv \cos\phi, \\ \mathbf{r} &\equiv \sin\left(\frac{\phi}{2}\right)\mathbf{e}, \quad r_0 \equiv \cos\left(\frac{\phi}{2}\right) \end{aligned}$$

Acceleration Analysis of Rigid-Body Motions (Cont'd)

$$\dot{\mathbf{N}} = \frac{1}{4(1-\cos\phi)}\begin{bmatrix}\mathbf{B}\\ \dot{\mathbf{e}}\end{bmatrix} \qquad \dot{\mathbf{H}} = \frac{1}{2}\begin{bmatrix}\dot{r}_0\mathbf{1}-\dot{\mathbf{R}}\\ -\dot{\mathbf{r}}^T\end{bmatrix}$$

$$\dot{\mathbf{L}} = \frac{1}{2}\begin{bmatrix}\mathbf{1}\mathrm{tr}(\dot{\mathbf{Q}})-\dot{\mathbf{Q}}\\ -\boldsymbol{\omega}^T\mathrm{tr}(\mathbf{Q})+\boldsymbol{\omega}^T\mathbf{Q}^T\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}2(\boldsymbol{\omega}\cdot\mathbf{u})\mathbf{1}+\boldsymbol{\Omega}\mathbf{Q}\\ \boldsymbol{\omega}^T\mathrm{tr}(\mathbf{Q})-\boldsymbol{\omega}^T\mathbf{Q}^T\end{bmatrix}$$

where $\mathbf{R}$ denotes the cross-product matrix of $\mathbf{r}$,

$$\begin{aligned}\mathbf{B} \equiv\ & -2(\mathbf{e}\cdot\boldsymbol{\omega})\mathbf{1}+2(3-\cos\phi)(\mathbf{e}\cdot\boldsymbol{\omega})\mathbf{e}\mathbf{e}^T-2(1+\sin\phi)\boldsymbol{\omega}\mathbf{e}^T\\ & -(2\cos\phi+\sin\phi)\mathbf{e}\boldsymbol{\omega}^T-(\sin\phi)[\boldsymbol{\Omega}-(\mathbf{e}\cdot\boldsymbol{\omega})\mathbf{E}]\end{aligned}$$

and

$$\mathrm{tr}(\dot{\mathbf{Q}}) \equiv \mathrm{tr}(\boldsymbol{\Omega}\mathbf{Q}) \equiv -2\boldsymbol{\omega}^T\mathbf{u}$$

Acceleration Analysis of Rigid-Body Motions (Cont'd)

We also have :

$$\dot{\tilde{\mathbf{N}}} = \left[\dot{\phi}(\cos\phi)\mathbf{1} + \dot{\phi}(\sin\theta)\mathbf{E} \quad \dot{\mathbf{e}}\right]$$

$$\dot{\tilde{\mathbf{L}}} = \left[\mathbf{V}/D \quad \dot{\mathbf{u}}\right] \qquad \dot{\tilde{\mathbf{H}}} = \left[\dot{r}_0\mathbf{1} + \dot{\mathbf{R}} \quad -\dot{\mathbf{r}}\right]$$

where

$$\begin{aligned} \mathbf{V} &\equiv \dot{\mathbf{U}} - (\dot{\mathbf{u}}\mathbf{u}^T + \mathbf{u}\dot{\mathbf{u}}^T) - \frac{\dot{u}_0}{D}(\mathbf{U} - \mathbf{u}\mathbf{u}^T) \\ D &\equiv 1 + u_0 \end{aligned}$$

with $\mathbf{U}$ denoting the cross-product matrix of $\mathbf{u}$.

Rigid-Body Motion Referred to Moving Coordinate Axes

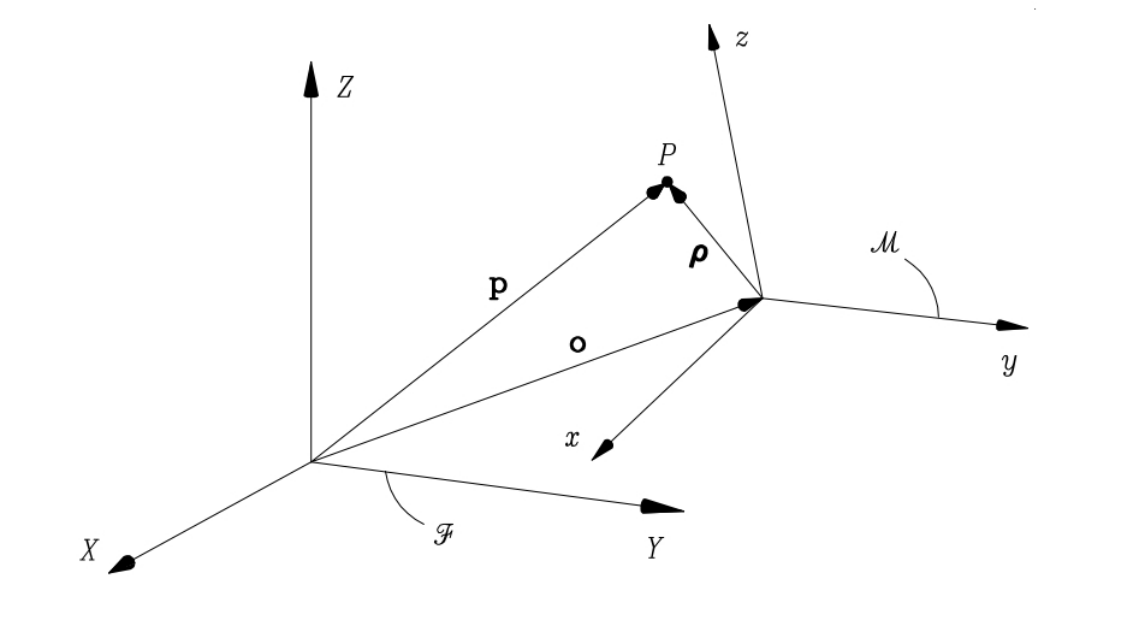

FIGURE: Fixed and moving coordinate frames.

$\mathcal{F}$: the fixed coordinate frame X, Y, Z
$\mathcal{M}$: the moving coordinate frame $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$

Rigid-Body Motion Referred to Moving Coordinate Axes (Cont'd)

$$[\,\mathbf{p}\,]_{\mathcal{F}} = [\,\mathbf{o}\,]_{\mathcal{F}} + [\,\boldsymbol{\rho}\,]_{\mathcal{F}} \tag{3.100}$$

$$[\,\boldsymbol{\rho}\,]_{\mathcal{F}} = [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} \tag{3.101}$$

Substitute eq.(3.101) into eq.(3.100) :

$$[\,\mathbf{p}\,]_{\mathcal{F}} = [\,\mathbf{o}\,]_{\mathcal{F}} + [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} \tag{3.102}$$

$$\Rightarrow [\,\dot{\mathbf{p}}\,]_{\mathcal{F}} = [\,\dot{\mathbf{o}}\,]_{\mathcal{F}} + [\,\dot{\mathbf{Q}}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} + [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\dot{\boldsymbol{\rho}}\,]_{\mathcal{M}} \tag{3.103}$$

From the definition of $\boldsymbol{\Omega}$ in eq.(3.46) :

$$[\,\dot{\mathbf{Q}}\,]_{\mathcal{F}} = [\,\boldsymbol{\Omega}\,]_{\mathcal{F}}[\,\mathbf{Q}\,]_{\mathcal{F}} \tag{3.104}$$

Rigid-Body Motion Referred to Moving Coordinate Axes (Cont'd)

Hence,

$$[\,\dot{\mathbf{p}}\,]_{\mathcal{F}} = [\,\dot{\mathbf{o}}\,]_{\mathcal{F}} + [\,\mathbf{\Omega}\,]_{\mathcal{F}}[\,\mathbf{Q}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} + [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\dot{\boldsymbol{\rho}}\,]_{\mathcal{M}} \tag{3.105}$$

Differentiate eq.(3.105) with respect to time :

$$\begin{aligned}[\,\ddot{\mathbf{p}}\,]_{\mathcal{F}} &= [\,\ddot{\mathbf{o}}\,]_{\mathcal{F}} + [\,\dot{\mathbf{\Omega}}\,]_{\mathcal{F}}[\,\mathbf{Q}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} + [\,\mathbf{\Omega}\,]_{\mathcal{F}}[\,\dot{\mathbf{Q}}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} \\ &\quad + [\,\mathbf{\Omega}\,]_{\mathcal{F}}[\,\mathbf{Q}\,]_{\mathcal{F}}[\,\dot{\boldsymbol{\rho}}\,]_{\mathcal{M}} + [\,\dot{\mathbf{Q}}\,]_{\mathcal{F}}[\,\dot{\boldsymbol{\rho}}\,]_{\mathcal{M}} + [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\ddot{\boldsymbol{\rho}}\,]_{\mathcal{M}}\end{aligned}$$

Substitute (3.104) into the above equation :

$$\begin{aligned}[\,\ddot{\mathbf{p}}\,]_{\mathcal{F}} &= [\,\ddot{\mathbf{o}}\,]_{\mathcal{F}} + (\,[\,\dot{\mathbf{\Omega}}\,]_{\mathcal{F}} + [\,\mathbf{\Omega}^2\,]_{\mathcal{F}}\,)[\,\mathbf{Q}\,]_{\mathcal{F}}[\,\boldsymbol{\rho}\,]_{\mathcal{M}} \\ &\quad + 2[\,\mathbf{\Omega}\,]_{\mathcal{F}}[\,\mathbf{Q}\,]_{\mathcal{F}}[\,\dot{\boldsymbol{\rho}}\,]_{\mathcal{M}} + [\,\mathbf{Q}\,]_{\mathcal{F}}[\,\ddot{\boldsymbol{\rho}}\,]_{\mathcal{M}}\end{aligned}$$

Static Analysis of Rigid Bodies

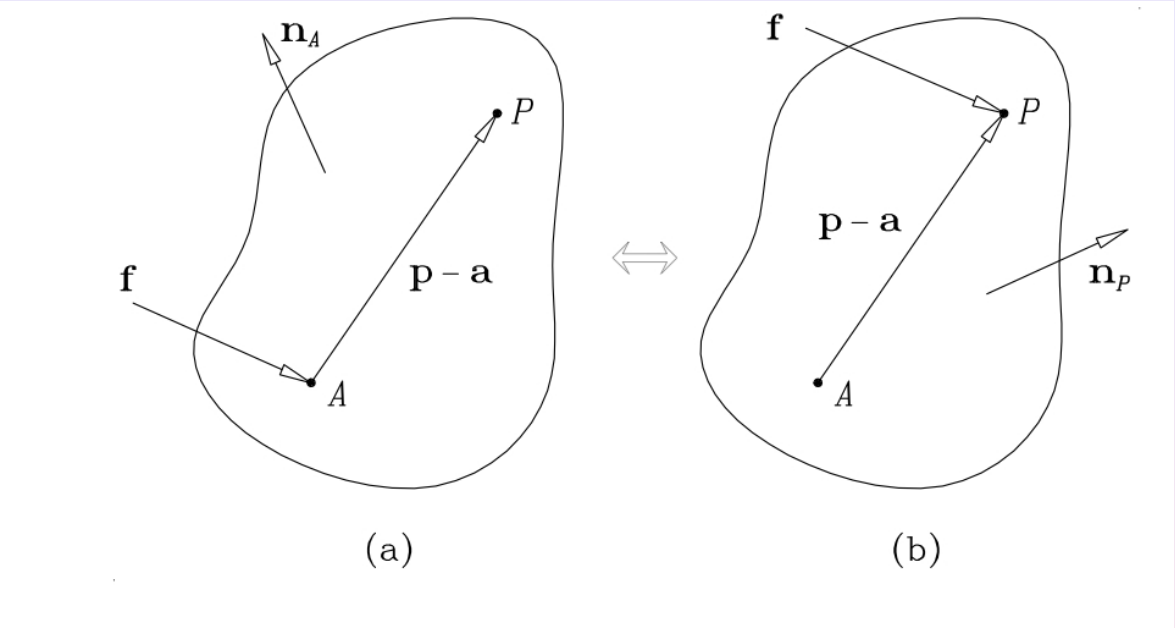

FIGURE: Equivalent systems of force and moment acting on a rigid body.

$$\mathbf{n}_P = \mathbf{n}_A + (\mathbf{a} - \mathbf{p}) \times \mathbf{f} \tag{3.108}$$

$$\mathbf{n}_P = \mathbf{n}_A + \mathbf{f} \times (\mathbf{p} - \mathbf{a}) \tag{3.109}$$

Static Analysis of Rigid Bodies (Cont'd)

Similar to Theorem 3 :

Theorem

For a given system of forces and moments acting on a rigid body, if the resultant force is applied at any point of a particular line $\mathcal{L}''$, then the resultant moment is of minimum magnitude. Moreover, that minimum-magnitude moment is parallel to the resultant force.

The minimum-magnitude moment $\mathbf{n}_0$:

$$\mathbf{n}_0 = n_0 \frac{\mathbf{f}}{\|\mathbf{f}\|}, \quad n_0 \equiv \frac{\mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|} \tag{3.110}$$

Static Analysis of Rigid Bodies (Cont'd)

The pitch of the wrench, p'' :

$$p'' \equiv \frac{n_0}{\|\mathbf{f}\|} = \frac{\mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \quad \text{or} \quad p'' = \frac{2\pi \mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \tag{3.111}$$

The Plücker array of the wrench axis :

$$\mathbf{p}_{\mathcal{L}''} \equiv \begin{bmatrix} \mathbf{e}'' \\ \mathbf{n}'' \end{bmatrix}, \quad \mathbf{e}'' = \frac{\mathbf{f}}{\|\mathbf{f}\|}, \quad \mathbf{n}'' = \mathbf{p} \times \mathbf{e}'' \tag{3.112}$$

The wrench axis is fully specified, then, by $\mathbf{f}$ and P_0'' lying closest to the origin

$$\mathbf{p}_0'' = \frac{1}{\|\mathbf{f}\|^2} \mathbf{f} \times (\mathbf{n}_A - \mathbf{f} \times \mathbf{a}) \tag{3.113}$$

Static Analysis of Rigid Bodies (Cont'd)

Similar to Theorem 1 :

Theorem

The projection of the resultant moment of a system of moments and forces acting on a rigid body that arises when the resultant force is applied at an arbitrary point of the body onto the wrench axis is constant.

To derive the wrench array $\mathbf{w}$, we need to multiply the screw of eq.(3.73) by an amplitude A with units of force.
We shall be able to obtain the power developed by the wrench on the body moving with the twist $\mathbf{t}$ but the inner product of these two arrays would be meaningless.

Static Analysis of Rigid Bodies (Cont'd)

We redefine the wrench as a linear transformation of $\mathbf{s}$:

$$\mathbf{\Gamma} \equiv \begin{bmatrix} \mathbf{O} & \mathbf{1} \\ \mathbf{1} & \mathbf{O} \end{bmatrix} \Rightarrow \quad \mathbf{w} \equiv A\mathbf{\Gamma}\mathbf{s} \equiv \begin{bmatrix} \mathbf{p} \times (A\mathbf{e}) + p(A\mathbf{e}) \\ A\mathbf{e} \end{bmatrix}$$

The first 3 components of $\mathbf{w}$ represent the moment of a force of magnitude A acting along $\mathbf{e}$ with respect to P plus a moment parallel to $\mathbf{e}$ and of magnitude pA. The last 3 components pertain to a force of magnitude A and parallel to $\mathbf{e}$:

$$\mathbf{f} \equiv A\mathbf{e}, \quad \mathbf{n} \equiv \mathbf{p} \times \mathbf{f} + p\mathbf{f}$$

$$\Rightarrow \quad \mathbf{w} \equiv \begin{bmatrix} \mathbf{n} \\ \mathbf{f} \end{bmatrix} \quad \text{and} \quad \Pi = \mathbf{t}^T \mathbf{w} \tag{3.116}$$

Static Analysis of Rigid Bodies (Cont'd)

Let

$$\mathbf{w} = W\mathbf{\Gamma}\mathbf{s}_w \ , \quad \mathbf{t} = T\mathbf{s}_t \tag{3.117}$$

The two screws $\mathbf{s}_w$ and $\mathbf{s}_t$ are reciprocal if

$$(\mathbf{\Gamma}\mathbf{s}_w)^T\mathbf{s}_t \equiv \mathbf{s}_w^T\mathbf{\Gamma}^T\mathbf{s}_t = 0 \tag{3.118}$$

or

$$\mathbf{s}_w^T\mathbf{\Gamma}\mathbf{s}_t = 0 \quad \text{and} \quad \mathbf{s}_t^T\mathbf{\Gamma}\mathbf{s}_w = 0 \tag{3.119}$$

For arbitrary points A and P

$$\mathbf{w}_A \equiv \begin{bmatrix} \mathbf{n}_A \\ \mathbf{f} \end{bmatrix}, \quad \mathbf{w}_P \equiv \begin{bmatrix} \mathbf{n}_P \\ \mathbf{f} \end{bmatrix} \tag{3.120}$$

Static Analysis of Rigid Bodies (Cont'd)

$$\Rightarrow \quad \mathbf{w}_P = \mathbf{V}\mathbf{w}_A \quad \text{where} \quad \mathbf{V} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{A}-\mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{3.121}$$

The last relation is termed the wrench-transfer formula.

Multiplying the transpose of eq.(3.84) by eq.(3.121) yields

$$\mathbf{t}_P^T \mathbf{w}_P = \mathbf{t}_A^T \mathbf{U}^T \mathbf{V} \mathbf{w}_A \tag{3.123}$$

where

$$\mathbf{U}^T\mathbf{V} = \begin{bmatrix} \mathbf{1} & -\mathbf{A}+\mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{A}-\mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \mathbf{1}_{6\times 6} \tag{3.124}$$

Thus, $\mathbf{t}_P^T \mathbf{w}_P = \mathbf{t}_A^T \mathbf{w}_A$. Also note that $\mathbf{V}^{-1} = \mathbf{U}^T$.

Dynamics of Rigid Bodies (Cont'd)

The rigid body mass :

$$m = \int_{\mathcal{B}} \rho d\mathcal{B} \tag{3.126}$$

where ρ is a mass density and $\mathcal{B}$ denotes the region occupied by the body.

The mass first moment

$$\mathbf{q}_O \equiv \int_{\mathcal{B}} \rho \mathbf{p} d\mathcal{B} \tag{3.127}$$

The mass second moment (symmetric matrix)

$$\mathbf{I}_O \equiv \int_{\mathcal{B}} \rho [\, (\mathbf{p} \cdot \mathbf{p})\mathbf{1} - \mathbf{p}\mathbf{p}^T \,] d\mathcal{B} \tag{3.128}$$

is called the moment-of-inertia matrix of the body with respect to O.

Theorem

The moment of inertia of a rigid body with respect to a point O is positive definite.

Dynamics of Rigid Bodies (Cont'd)

Proof : For any vector $\boldsymbol{\omega}$, the quadratic form $\boldsymbol{\omega}^T \mathbf{I}_O \boldsymbol{\omega}$ is positive :

$$\boldsymbol{\omega}^T \mathbf{I}_O \boldsymbol{\omega} = \int_{\mathcal{B}} \rho[\, \|\mathbf{p}\|^2 \|\boldsymbol{\omega}\|^2 - (\mathbf{p} \cdot \boldsymbol{\omega})^2 \,] d\mathcal{B} \tag{3.129}$$

Substituting $\quad \mathbf{p} \cdot \boldsymbol{\omega} = \|\mathbf{p}\| \|\boldsymbol{\omega}\| \cos(\mathbf{p}, \boldsymbol{\omega}) \quad (3.130)$

into eq.(3.129) leads to :

$$\begin{aligned} \boldsymbol{\omega}^T \mathbf{I}_O \boldsymbol{\omega} &= \int_{\mathcal{B}} \rho \|\mathbf{p}\|^2 \|\boldsymbol{\omega}\|^2 [\, 1 - \cos^2(\mathbf{p}, \boldsymbol{\omega}) \,] d\mathcal{B} \\ &= \int_{\mathcal{B}} \rho \|\mathbf{p}\|^2 \|\boldsymbol{\omega}\|^2 \sin^2(\mathbf{p}, \boldsymbol{\omega}) d\mathcal{B} \end{aligned}$$

Dynamics of Rigid Bodies (Cont'd)

The kinetic energy of a rigid body : $T \equiv \int_{\mathcal{B}} \frac{1}{2}\rho\|\dot{\mathbf{p}}\|^2 d\mathcal{B}$

where $\dot{\mathbf{p}} = \boldsymbol{\omega} \times \mathbf{p} = -\mathbf{P}\boldsymbol{\omega}$

$$\begin{aligned} \Rightarrow \|\dot{\mathbf{p}}\|^2 &= (\mathbf{P}\boldsymbol{\omega})^T\mathbf{P}\boldsymbol{\omega} = \boldsymbol{\omega}^T\mathbf{P}^T\mathbf{P}\boldsymbol{\omega} = -\boldsymbol{\omega}^T\mathbf{P}^2\boldsymbol{\omega} \\ &= \boldsymbol{\omega}^T(\|\mathbf{p}\|^2\mathbf{1} - \mathbf{p}\mathbf{p}^T)\boldsymbol{\omega} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{B}} \rho\boldsymbol{\omega}^T(\|\mathbf{p}\|^2\mathbf{1} - \mathbf{p}\mathbf{p}^T)\boldsymbol{\omega} d\mathcal{B} \\ &= \frac{1}{2}\boldsymbol{\omega}^T\left[\int_{\mathcal{B}} \rho(\|\mathbf{p}\|^2\mathbf{1} - \mathbf{p}\mathbf{p}^T)d\mathcal{B}\right]\boldsymbol{\omega} \\ \Rightarrow \quad T &= \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}_O\boldsymbol{\omega} \end{aligned} \tag{3.134}$$

Dynamics of Rigid Bodies (Cont'd)

The mass center of a rigid body is defined as a point C of position vector $\mathbf{c}$:

$$\mathbf{c} \equiv \frac{\mathbf{q}_O}{m} \tag{3.135}$$

The centroidal mass moment of inertia :

$$\Rightarrow \quad \mathbf{I}_C \equiv \int_{\mathcal{B}} \rho[\, \|\mathbf{r}\|^2 \mathbf{1} - \mathbf{r}\mathbf{r}^T \,] d\mathcal{B} \tag{3.136}$$

where $\mathbf{r} \equiv \mathbf{p} - \mathbf{c}$

The Theorem of Parallel Axes :

$$\mathbf{I}_O = \mathbf{I}_C + m(\|\mathbf{c}\|^2 \mathbf{1} - \mathbf{c}\mathbf{c}^T) \tag{3.138}$$

Dynamics of Rigid Bodies (Cont'd)

For a body acted upon by a wrench of force $\mathbf{f}$ applied at its mass center, and a moment $\mathbf{n}_C$,

the Newton equation :
$$\mathbf{f} = m\ddot{\mathbf{c}} \tag{3.139a}$$

the Euler equation :
$$\mathbf{n}_C = \mathbf{I}_C\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega} \tag{3.139b}$$

The momentum $\mathbf{m}$ and the angular momentum $\mathbf{h}_C$:
$$\mathbf{m} \equiv m\dot{\mathbf{c}}, \quad \mathbf{h}_C \equiv \mathbf{I}_C\boldsymbol{\omega} \tag{3.140}$$

Moreover,
$$\dot{\mathbf{m}} = m\ddot{\mathbf{c}}, \quad \dot{\mathbf{h}}_C = \mathbf{I}_C\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega} \tag{3.141}$$

$$\Rightarrow \quad \mathbf{f} = \dot{\mathbf{m}}, \quad \mathbf{n}_C = \dot{\mathbf{h}}_C \tag{3.142}$$

Dynamics of Rigid Bodies (Cont'd)

The inertia dyad :

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_C & \mathbf{O} \\ \mathbf{O} & m\mathbf{1} \end{bmatrix} \tag{3.143}$$

Now the Newton-Euler equations can be written as

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t} = \mathbf{w} \tag{3.144}$$

where

$$\mathbf{W} \equiv \begin{bmatrix} \mathbf{\Omega} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.145}$$

is termed the angular-velocity dyad. Note that

$$\mathbf{W}\mathbf{t} = \mathbf{0} \tag{3.146}$$

Dynamics of Rigid Bodies (Cont'd)

The momentum screw about the rigid body mass center

$$\boldsymbol{\mu} \equiv \begin{bmatrix} \mathbf{I}_C \boldsymbol{\omega} \\ m\dot{\mathbf{c}} \end{bmatrix} = \mathbf{M}\mathbf{t} \tag{3.147}$$

$$\Rightarrow \quad \dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\boldsymbol{\mu} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t} \tag{3.148}$$

The kinetic energy

$$T = \frac{1}{2} m \|\dot{\mathbf{c}}\|^2 + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I}_C \boldsymbol{\omega} = \frac{1}{2} \mathbf{t}^T \mathbf{M} \mathbf{t} \tag{3.150}$$

The Newton-Euler equations

$$\dot{\boldsymbol{\mu}} = \mathbf{w} \tag{3.151}$$