Fundamentals of Robotic Mechanical Systems

Chapter 3: Fundamentals of Rigid-Body Mechanics

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Point $A$ : a reference point of rigid body $B$ with the position vector $a$ in the original configuration and $a'$ in the displaced configuration, $A'$.

Point $P$ : an arbitrary point of $B$ with the position vector $p$ in the original configuration and $p'$ in the displaced configuration, $P'$.

Determine $p'$ with given $a$, $a'$, $p$, and the rotation matrix $Q$ :

\[ p' - a' = Q(p - a) \Rightarrow p' = a' + Q(p - a) \]  

(3.2)
Displacements of $A$ and $P$:

\[ d_A \equiv a' - a, \quad d_P \equiv p' - p \]  

(3.3)

From eqs. (3.2) and (3.3):

\[ d_P = p' - p = a' - p + Q(p - a) \]
\[ = a' - a - p + Q(p - a) + a \]

\[ \Rightarrow d_P = d_A + (Q - 1)(p - a) \]  

(3.5)
Theorem

The component of the displacements of all the points of a rigid body undergoing a general motion along the axis of the underlying rotation is a constant.

Proof: Multiply both sides of eq. (3.5) by $e^T$:

$$e^T d_P = e^T d_A + e^T (Q - 1)(p - a)$$

$$Qe = e \quad \Rightarrow \quad e^T Q - e^T = 0^T \quad \Rightarrow \quad e^T (Q - 1) = 0^T$$

$$\Rightarrow \quad e^T d_P = e^T d_A \equiv d_0 \quad (3.6)$$
Theorem (Mozzi, 1763; Chasles, 1830)

Given a rigid body undergoing a general motion, a set of its points located on a line \( \mathcal{L} \) undergo identical displacements of minimum magnitude. Moreover, line \( \mathcal{L} \) and the minimum-magnitude displacement are parallel to the axis of the rotation involved.

**Proof:**

\[ d_P = d_\parallel + d_\perp \]  

where

\[ d_\parallel = ee^T d_P = d_0 e, \quad d_\perp = (1 - ee^T) d_P \]

\[ \|d_P\|^2 = \|d_\parallel\|^2 + \|d_\perp\|^2 = d_0^2 + \|d_\perp\|^2 \]
General Rigid-Body Motion and Its Associated Screw (Cont’d)

Minimize displacement $\|d_P\|$:  
\[
  d_\perp = 0 \Rightarrow d_P = \alpha \mathbf{e}
\]

⇒ All points of minimum-magnitude displacement lie in a line parallel to the axis of rotation of $Q$, q.e.d.

If $P^*$ is a point of minimum-magnitude displacement, then

\[
d^*_\perp \equiv (1 - ee^T)d_{P^*} = (1 - ee^T)d_A + (1 - ee^T)(Q - I)(p^* - a) = 0
\]

⇒ \((1 - ee^T)d_A + (Q - I)(p^* - a) = 0\)

Above equation holds if $\alpha \mathbf{e}$ is added to $p^*$,
⇒ Screw motion of axes $\mathcal{L}$ and pitch $p$:

\[
p \equiv \frac{d_0}{\phi} = \frac{d^T_P \mathbf{e}}{\phi} \quad \text{or} \quad p \equiv \frac{2\pi d_0}{\phi} \quad (3.8)
\]
The Screw of a Rigid-Body Motion

The screw axis $\mathcal{L}$ : specified by $P_0$ and $e$.

If $P_0$ is closest to the origin, then

$$e^T p_0 = 0 \quad (3.9)$$

and its displacement $d_0$ is identical to $d_\parallel$ :

$$(Q - 1)d_0 = 0$$

where, from eq.(3.5),

$$d_0 = d_A + (Q - 1)(p_0 - a) \quad (3.10a)$$

and, from eq.(3.7b),

$$d_A + (Q - 1)(p_0 - a) = d_\parallel \equiv ee^T d_0$$
The Screw of a Rigid-Body Motion  
(Cont’d)

From Theorem 1

\[ e^T d_0 = e^T d_A \Rightarrow d_A + (Q - 1)(p_0 - a) = ee^T d_A \]

or

\[ (Q - 1)p_0 = (Q - 1)a - (1 - ee^T)d_A \quad (3.10b) \]

Adjoin eqs. (3.9) and (3.10b):

\[ Ap_0 = b \quad (3.11) \]

where

\[ A \equiv \begin{bmatrix} Q - 1 \\ e^T \end{bmatrix}, \quad b \equiv \begin{bmatrix} (Q - 1)a - (1 - ee^T)d_A \\ 0 \end{bmatrix} \quad (3.12) \]
The Screw of a Rigid-Body Motion
(Cont’d)

\[
A^T Ap_0 = A^T b
\]  
(3.13)

Substituting into eq.(3.12) expression

\[
Q = ee^T + \cos \phi (1 - ee^T) + \sin \phi E
\]

yields

\[
A^T A = 2(1 - \cos \phi) 1 - (1 - 2 \cos \phi) ee^T
\]

or

\[
(A^T A)^{-1} = \alpha 1 + \beta ee^T.
\]

Since \((A^T A)(A^T A)^{-1} = 1\),

\[
\alpha = \frac{1}{2(1 - \cos \phi)}, \quad \beta = \frac{1 - 2 \cos \phi}{2(1 - \cos \phi)}
\]  
(3.16)
The Screw of a Rigid-Body Motion
(Cont’d)

\[ \Rightarrow (A^T A)^{-1} = \frac{1}{2(1 - \cos \phi)} 1 + \frac{1 - 2 \cos \phi}{2(1 - \cos \phi)} ee^T \] (3.17)

On the other hand,

\[ A^T b = (Q - 1)^T [(Q - 1)a - d_A] \] (3.18)

Upon solving eq.(3.13) for \( p_0 \) and substituting relations (3.17) and (3.18) into the expression thus resulting, one finally obtains

\[ p_0 = \frac{(Q - 1)^T (Qa - a^\prime)}{2(1 - \cos \phi)}, \quad \text{for } \phi \neq 0 \] (3.19)
The Plücker Coordinates of a Line

**Figure:** A line $L$ passing through two points.

\[(p_2 - p_1) \times (p - p_1) = 0\]

\[\Rightarrow (p_2 - p_1) \times p + p_1 \times (p_2 - p_1) = 0\]
The Plücker Coordinates of a Line
(Cont’d)

\[(P_2 - P_1)p + p_1 \times (p_2 - p_1) = 0\]

\[
\begin{bmatrix}
P_2 - P_1 & p_1 \times (p_2 - p_1)
\end{bmatrix}
\begin{bmatrix}
p_1
\end{bmatrix} = 0
\] \hspace{1cm} (3.21)

We define

\[\gamma_L \equiv \begin{bmatrix}
p_2 - p_1
p_1 \times (p_2 - p_1)
\end{bmatrix}\] \hspace{1cm} (3.22)

Components of \[\gamma_L\] : Plücker coordinates of \(L\).
The Plücker Coordinates of a Line
(Cont’d)

Let

\[ e \equiv \frac{p_2 - p_1}{\|p_2 - p_1\|}, \quad n \equiv p_1 \times e \]  

(3.23)

\[ \Rightarrow \gamma_L = \|p_2 - p_1\| \begin{bmatrix} e \\ n \end{bmatrix} \]

Vector \( e \) determines the direction of \( L \), while \( n \), the moment of \( L \), determines its location.

Plücker array \( \kappa \):

\[ \kappa = \begin{bmatrix} e \\ n \end{bmatrix} \]  

(3.24)
The Plücker Coordinates of a Line (Cont’d)

There are only four independent Plücker coordinates of line \( L \) since:

\[
e \cdot e = 1, \quad n \cdot e = 0
\]  

That is,

(i) the sum of the squares of the first three components of a Plücker array is unity, and

(ii) the unit vector of a line is normal to the moment of the line

\( \Rightarrow \) Set of Plücker arrays does not form a vector space
The Plücker Coordinates of a Line (Cont’d)

\[ n_A \equiv (p - a) \times e, \quad n_B \equiv (p - b) \times e \quad (3.26) \]

\[ \Rightarrow \quad \kappa_A \equiv \begin{bmatrix} e \\ n_A \end{bmatrix}, \quad \kappa_B \equiv \begin{bmatrix} e \\ n_B \end{bmatrix} \quad (3.27) \]

\[ n_B - n_A = (a - b) \times e \quad (3.28) \]

\[ \Rightarrow \quad \kappa_B = \begin{bmatrix} e \\ n_A + (a - b) \times e \end{bmatrix} \quad (3.29) \]

or \( \kappa_B = U \kappa_A \): Plücker-coordinate transfer formula

\[ U \equiv \begin{bmatrix} 1 & 0 \\ A - B & 1 \end{bmatrix} \quad (3.31) \]
The Plücker Coordinates of a Line
(Cont’d)

\[
\det(U) = 1
\]

⇒ \( U \) belongs to the unimodular group of \( 6 \times 6 \) matrices

\[
U^{-1} = \begin{bmatrix}
1 & 0 \\
B - A & 1
\end{bmatrix} \tag{3.33}
\]

Note that from eq.(3.28) :

\[
(a - b)^T n_B = (a - b)^T n_A \tag{3.34}
\]

⇒ The moments of the same line \( \mathcal{L} \) with respect to two points are not independent: they have the same component along the line joining the two points.
The Plücker Coordinates of a Line (Cont’d)

A line at infinity: the orientation is undefined but a direction of its moment is defined

If we rewrite eq.(3.24) in the form

\[ \kappa = \|n\| \begin{bmatrix} e/\|n\| \\ n/\|n\| \end{bmatrix}, \quad f \equiv n/\|n\| \]

then,

\[ \lim_{\|n\| \to \infty} \kappa = \left( \lim_{\|n\| \to \infty} \|n\| \right) \left( \lim_{\|n\| \to \infty} \begin{bmatrix} e/\|n\| \\ f \end{bmatrix} \right) \]

\[ \Rightarrow \lim_{\|n\| \to \infty} \kappa = \left( \lim_{\|n\| \to \infty} \|n\| \right) \begin{bmatrix} 0 \\ f \end{bmatrix} \Rightarrow \kappa_\infty = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad (3.35) \]
A rigid-body motion is fully described by six independent or seven dependent parameters. Hence, if a rigid body undergoes a general motion of rotation $Q$ and displacement $d$ from a reference configuration $C_0$, then pose array, or the pose, $s$ of the body, is defined as a 7-dimensional array:

$$s \equiv [q^T \quad q_0^T \quad d_{A}^T]^T \quad (3.36)$$

where $q$ and $q_0$ are any four invariants of $Q$. 
For example, Euler-Rodrigues parameters:

\[ q \equiv \sin(\phi/2)e, \quad q_0 \equiv \cos(\phi/2) \]

linear invariants: \[ q \equiv (\sin \phi)e, \quad q_0 \equiv \cos \phi \]

with \[ \|q\|^2 + q_0^2 = 1 \];

the natural invariants: \[ q \equiv e, \quad q_0 \equiv \phi \] with \[ \|e\|^2 = 1 \].

**Pose estimation**: the computation of the attitude of a rigid body, given by matrix \( Q \) or its invariants from coordinate measurements over a certain finite set of points.
The Pose of a Rigid Body (Cont’d)

**Figure:** Helmet-mounted display system (CAE Electronics, St.-Laurent, Quebec, Canada.)
The Pose of a Rigid Body (Cont’d)

**Figure**: Decomposition of the displacement of a rigid body.

To determine matrix $\mathbf{Q}$, define $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ so that

$$
\mathbf{e}_1 \equiv \overrightarrow{A\mathbf{P}_1}, \quad \mathbf{e}_2 \equiv \overrightarrow{A\mathbf{P}_2}, \quad \mathbf{e}_3 \equiv \overrightarrow{A\mathbf{P}_3}
$$

$$
\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2
$$

where $\delta_{ij}$ is the Kronecker delta, defined as 1 if $i = j$ and 0 otherwise.
Let $q_{ij}$ be the entries of $Q$ in $X, Y, Z$ with the origin at $A$ and with $X, Y, Z$ parallel to $e_1, e_2, e_3$. From Definition 2.2.1, $q_{ij} = e_i \cdot e'_j$, i.e.,

$$[Q] = \begin{bmatrix}
  e_1 \cdot e'_1 & e_1 \cdot e'_2 & e_1 \cdot e'_3 \\
  e_2 \cdot e'_1 & e_2 \cdot e'_2 & e_2 \cdot e'_3 \\
  e_3 \cdot e'_1 & e_3 \cdot e'_2 & e_3 \cdot e'_3
\end{bmatrix}$$  \hfill (3.42)

Note that all $e_i$ and $e'_i$ in eq.(3.42) must be represented in the same coordinate frame.

Then, use, for example, eq.(3.19) to determine the point of the screw axis that lies closest to the origin and compute the Plücker coordinates of the screw axis.
Rotation of a Rigid Body About a Fixed Point

Fully described by a rotation matrix $Q$ that is proper orthogonal and is a smooth function of time. Then,

$$p(t) = Q(t)p_0 \quad (3.43)$$

$$\Rightarrow \dot{p}(t) = \dot{Q}(t)p_0 \quad (3.44)$$

Solve eq.(3.43) for $p_0$ and substitute the result into eq.(3.44) :

$$\dot{p} = \dot{Q}Q^T p \quad (3.45)$$

The angular-velocity matrix :

$$\Omega \equiv \dot{Q}Q^T \quad (3.46)$$
Theorem

The angular-velocity matrix is skew-symmetric.

The angular-velocity vector $\mathbf{\omega}$ of the rigid-body motion is defined as

$$\mathbf{\omega} \equiv \text{vect}(\mathbf{\Omega}) \quad (3.47)$$

and hence, eq.(3.45) can be written as

$$\dot{\mathbf{p}} = \mathbf{\Omega p} = \mathbf{\omega} \times \mathbf{p} \quad (3.48)$$

$\Rightarrow$ The velocity of any point $P$ of a body moving with a point $O$ fixed is perpendicular to line $OP$. 

General Instantaneous Motion of a Rigid Body (Cont’d)

\[ \mathbf{p}(t) = \mathbf{a}(t) + \mathbf{Q}(t)(\mathbf{p}_0 - \mathbf{a}_0) \]  
(3.49)

\[ \Rightarrow \dot{\mathbf{p}}(t) = \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)(\mathbf{p}_0 - \mathbf{a}_0) \]  
(3.50)

Then, eq.(3.49) is solved for \((\mathbf{p}_0 - \mathbf{a}_0)\) and the result is substituted into eq.(3.50):

\[ \dot{\mathbf{p}} = \dot{\mathbf{a}} + \Omega(\mathbf{p} - \mathbf{a}) \]  
(3.51)

\[ \Rightarrow \dot{\mathbf{p}} = \dot{\mathbf{a}} + \omega \times (\mathbf{p} - \mathbf{a}) \]  
(3.52)

\[ \Rightarrow (\dot{\mathbf{p}} - \dot{\mathbf{a}}) \cdot (\mathbf{p} - \mathbf{a}) = 0 \]  
(3.53)

**Theorem**

The relative velocity of two points of the same rigid body is perpendicular to the line joining them.
Upon dot-multiplying both sides of eq. (3.52) by $\omega$:

$$\omega \cdot \dot{p} = \omega \cdot \dot{a} = \text{constant}$$

**Corollary**

The projections of the velocities of all the points of a rigid body onto the angular-velocity vector are identical.

**Theorem**

Given a rigid body under general motion, a set of its points located on a line $L'$ undergoes the identical minimum-magnitude velocity $v_0$ parallel to the angular velocity.

**Definition**

The line containing the points of a rigid body undergoing minimum-magnitude velocities is called the *instant screw axis* (ISA) of the body under the given motion.
The Instant Screw of a Rigid-Body Motion

The instantaneous motion of a body is equivalent to that of the bolt of a screw of instantaneous axis $\mathcal{L}'$, the ISA, and is called an instantaneous screw. Since $v_0 \parallel \omega$,

$$v_0 = v_0 \frac{\omega}{\|\omega\|}$$ (3.54)

$\Rightarrow$ the pitch: $p' \equiv \frac{v_0}{\|\omega\|} \equiv \frac{\dot{p} \cdot \omega}{\|\omega\|^2}$ or $p' \equiv \frac{2\pi v_0}{\|\omega\|}$ (3.55)

Plücker coordinates of ISA $\mathcal{L}'$:

$$\mathbf{p}_{\mathcal{L}'} \equiv \begin{bmatrix} e' \\ n' \end{bmatrix}$$ (3.56)

with $e' \equiv \frac{\omega}{\|\omega\|}$, $n' \equiv \mathbf{p} \times e'$ (3.57)
An instantaneous rigid-body motion is defined by a line $\mathcal{L}'$, a pitch $p'$, and an amplitude $\|\omega\|:

$$-\infty \leq p' \leq +\infty$$

A line supplied with a pitch: a **screw**

A screw supplied with an amplitude representing the magnitude of an angular velocity: a **twist**.
The Instant Screw of a Rigid-Body Motion (Cont’d)

The ISA can be alternatively described in terms of $p'_0$, its point lying closest to the origin. Let

$$\dot{a} \equiv \dot{a}_\parallel + \dot{a}_\perp$$  \hspace{1cm} (3.58)

where

$$\dot{a}_\parallel \equiv \dot{a} \cdot \omega \frac{\omega}{\|\omega\|^2} \equiv \frac{\omega \omega^T}{\|\omega\|^2} \dot{a},$$

$$\dot{a}_\perp \equiv \left(1 - \frac{\omega \omega^T}{\|\omega\|^2}\right) \dot{a} \equiv -\frac{1}{\|\omega\|^2} \Omega^2 \dot{a}$$  \hspace{1cm} (3.59)

if we use the identity in (2.39):

$$\Omega^2 \equiv \omega \omega^T - \|\omega\|^2 1,$$  \hspace{1cm} (3.60)
Substitute eq.(3.59) into eq.(3.52):

\[
\dot{p} = \frac{\omega \omega^T}{\|\omega\|^2} \dot{\mathbf{a}} - \frac{1}{\|\omega\|^2} \Omega^2 \dot{\mathbf{a}} + \Omega (p - a) \\
\dot{p}_\parallel - \dot{p}_\perp
\]

\[
\frac{\omega \omega^T}{\|\omega\|^2} \dot{\mathbf{a}} : \text{ the axial component of } \dot{p} (\parallel \text{ to } \omega)
\]

\[
- \frac{1}{\|\omega\|^2} \Omega^2 \dot{\mathbf{a}} + \Omega (p - a) : \text{ the normal component of } \dot{p} (\perp \text{ to } \omega)
\]

Is it possible, for an arbitrary motion, to find a certain point of position vector \( p \) whose velocity normal component vanishes?
\[ \dot{p}_\perp = 0 \quad \Rightarrow \quad \Omega (p - a) - \frac{1}{\|\omega\|^2} \Omega^2 \dot{a} = 0 \]  

(3.62)

\[ \Rightarrow \quad \Omega p = \Omega \left( a + \frac{1}{\|\omega\|^2} \Omega \dot{a} \right) \]  

(3.63)

or

\[ \Omega (p - r) = 0 \]  

(3.64a)

with

\[ r \equiv a + \frac{1}{\|\omega\|^2} \Omega \dot{a} \]  

(3.64b)
A possible solution is

\[ p = r = a + \frac{1}{\|\omega\|^2} \Omega \dot{a} \]  

(3.65)

This solution is not unique:

\[ \alpha \omega + p \]  

also satisfies eq.(3.63)

\[ \Rightarrow \]  

eq.(3.63) determines a set of points lying on the ISA.
For example, we can find the point of the ISA $p'_0$ lying closest to the origin. Obviously:

$$\omega^T p'_0 = 0 \quad (3.66)$$

If eq.(3.63) is rewritten for $p'_0$ and eq.(3.66) is adjoined to it, then,

$$A p'_0 = b \quad (3.67)$$

where

$$A \equiv \begin{bmatrix} \Omega \\ \omega^T \end{bmatrix}, \quad b \equiv \begin{bmatrix} \Omega a + \frac{1}{||\omega||^2} \Omega^2 \dot{a} \\ 0 \end{bmatrix} \quad (3.68)$$
Multiply eq.(3.67) by $A^T$:  
\[ A^T A p'_0 = A^T b \]  
(3.69)  

\[ \Rightarrow A^T A = \Omega^T \Omega + \omega \omega^T = -\Omega^2 + \omega \omega^T \]  
(3.70)  

Use eq.(3.60):  
\[ \Omega^2 \equiv \omega \omega^T - \|\omega\|^2 \mathbf{1} \]  

Hence, we have  
\[ A^T A = \|\omega\|^2 \mathbf{1}, \quad A^T b = \Omega(\dot{a} - \Omega a) \]  
(3.71)
Substitute eqs. (3.71) into eq. (3.69) and solve for $p'_0$:

$$p'_0 = \frac{\Omega (\dot{\mathbf{a}} - \Omega \mathbf{a})}{\|\omega\|^2} \equiv \frac{\omega \times (\dot{\mathbf{a}} - \omega \times \mathbf{a})}{\|\omega\|^2} \quad (3.72)$$

$\Rightarrow$ the instantaneous screw is fully defined by an alternative set of 6 independent scalars: 3 components of $\omega$ and 3 components of $\dot{\mathbf{a}}$. 
A pitch $p$ added as a fifth feature to the Plücker array of a line $\Rightarrow$ a screw $s$:

$$s \equiv \begin{bmatrix} e \\ p \times e + pe \end{bmatrix}$$  \hspace{1cm} (3.73)

An amplitude $A$ multiplying the screw produces a twist or a wrench as an 8-parameter array: the amplitude, the pitch, and the 6 Plücker coordinates of the associated line, with only 6 of them independent.

A twist defines completely the velocity field of a rigid body: 3 components of the angular velocity and 3 components of the velocity of any of the points of the body.
The Twist of a Rigid Body (Cont’d)

Upon multiplication of the screw in eq.(3.73) by $A$:

$$t \equiv \begin{bmatrix} Ae \\ p \times (Ae) + p(Ae) \end{bmatrix}$$

Here: $Ae$ is the angular velocity $\omega \parallel e$ of magnitude $A$.

The lower vector of $t$, $v$, is the velocity of point $O$ of a rigid body moving with an angular velocity $\omega$ so that point $P$ moves with a velocity $p\omega \parallel \omega$:

$$v = -\omega \times p + p\omega$$

$$\Rightarrow \text{ the twist } \quad t \equiv \begin{bmatrix} \omega \\ v \end{bmatrix}$$

(3.74)
The Twist of a Rigid Body (Cont’d)

The screw of infinitely large pitch:

\[
\lim_{p \to \infty} \left[p \times e + pe \right] \equiv \lim_{p \to \infty} \left(p \left(p \times e + pe \right) \right)
\]

\[
= \left( \lim_{p \to \infty} p \right) \left[0 \right]
\]

The screw of infinite pitch \( s_\infty \equiv \left[0 \right] \) (3.75)

is identical to the Plücker array of the line at \( \infty \) in a plane of unit normal \( e \).
The relationships between $\omega$ and the time derivatives of the invariants of the associated rotation:

$$\nu \equiv \begin{bmatrix} e \\ \phi \end{bmatrix}, \quad \lambda \equiv \begin{bmatrix} (\sin \phi)e \\ \cos \phi \end{bmatrix}, \quad \eta \equiv \begin{bmatrix} [\sin(\phi/2)]e \\ \cos(\phi/2) \end{bmatrix},$$

$$\dot{\nu} = N\omega, \quad \dot{\lambda} = L\omega, \quad \dot{\eta} = H\omega \quad (3.77a)$$

$$N \equiv \begin{bmatrix} [\sin \phi/(2(1 - \cos \phi))](1 - ee^T) - (1/2)E \\
\end{bmatrix},$$

$$L \equiv \begin{bmatrix} (1/2)[\text{tr}(Q)1 - Q] \\
-(\sin \phi)e^T \end{bmatrix}, \quad H \equiv \frac{1}{2} \begin{bmatrix} \cos(\phi/2)1 - \sin(\phi/2)E \\
-\sin(\phi/2)e^T \end{bmatrix}$$
The inverse relations of (3.77a):

\[ \omega = \tilde{N} \dot{\nu} = \tilde{L} \dot{\lambda} = \tilde{H} \dot{\eta} \] (3.78a)

where

\[ \tilde{N} \equiv \begin{bmatrix} (\sin \phi) \mathbf{1} + (1 - \cos \phi) \mathbf{E} & \mathbf{e} \end{bmatrix}, \]

\[ \tilde{L} \equiv \begin{bmatrix} \mathbf{1} + \frac{\sin \phi}{1 + \cos \phi} \mathbf{E} & - \frac{\sin \phi}{1 + \cos \phi} \mathbf{e} \end{bmatrix}, \]

\[ \tilde{H} \equiv 2 \begin{bmatrix} [\cos(\phi/2)] \mathbf{1} + [\sin(\phi/2)] \mathbf{E} & - [\sin(\phi/2)] \mathbf{e} \end{bmatrix} \]

Caveat: The angular velocity vector is not a time-derivative, i.e., no Cartesian vector exists whose time-derivative is the angular-velocity vector.
The relationships between the twist and the time-rate of change of the 7-dimensional pose array \( s \):

\[
\dot{s} = Tt \tag{3.79}
\]

where

\[
T \equiv \begin{bmatrix} F & O_{43} \\ O_{34} & 1 \end{bmatrix} \tag{3.80}
\]

and

\[
t = S\dot{s} \tag{3.81a}
\]

where

\[
S \equiv \begin{bmatrix} \tilde{F} & O \\ O_{34} & 1 \end{bmatrix} \tag{3.81b}
\]
The relation between the twists of the same rigid body at two different points, $A$ and $P$:

$$t_A = \begin{bmatrix} \omega \\ v_A \end{bmatrix}, \quad t_P = \begin{bmatrix} \omega \\ v_P \end{bmatrix}$$  (3.82)

Equation (3.52), rewritten as $v_P = v_A + (a - p) \times \omega$, combined with eq.(3.82) yields

$$t_P = Ut_A$$  where  $$U \equiv \begin{bmatrix} 1 & O \\ A - P & 1 \end{bmatrix}$$  (3.84)

where $A$ and $P$ denote the cross-product matrices of vectors $a$ and $p$, respectively. Equation (3.84) is called the twist-transfer formula.
Differentiate eq.(3.51) with respect to time:

\[ \ddot{p} = \ddot{a} + \dot{\Omega}(p - a) + \Omega(p - a) \]  \hspace{1cm} (3.86)

Solving eq.(3.51) for \( \dot{p} - \dot{a} \) and substituting the result into eq.(3.86), we obtain

\[ \ddot{p} = \ddot{a} + (\dot{\Omega} + \Omega^2)(p - a) \]  \hspace{1cm} (3.87)

where

\[ W \equiv \dot{\Omega} + \Omega^2 \]  \hspace{1cm} (3.88)

is termed the angular-acceleration matrix.
The first term of the right-hand side of eq.(3.88) is skew-symmetric, the second one is symmetric. Thus,

\[ \text{vect}(W) = \text{vect}(\dot{\Omega}) = \dot{\omega} \] (3.89)

the angular-acceleration vector. Moreover,

\[ \text{tr}(W) = \text{tr}(\Omega^2) = \text{tr}(-\|\omega\|^2 \mathbf{1} + \omega \omega^T) \]
\[ = -\|\omega\|^2 \text{tr}(\mathbf{1}) + \omega \cdot \omega = -2\|\omega\|^2 \] (3.90)

Equation (3.87) can be written as

\[ \ddot{p} = \ddot{a} + \dot{\omega} \times (p - a) + \omega \times [\omega \times (p - a)] \] (3.91)
The twist rate:

\[ \dot{t} \equiv \begin{bmatrix} \dot{\omega} \\ \dot{v} \end{bmatrix} \]  \hspace{1cm} (3.92)

Differentiate eq.(3.79):

\[ \ddot{s} = T\dot{t} + \dot{T}t \]

\[ T \equiv \begin{bmatrix} \hat{F} & O_{43} \\ O & O \end{bmatrix} \]  \hspace{1cm} (3.94)

and \( F \) is one of \( N, L, \) or \( H, \) accordingly. For the inverse relationship, differentiate (3.81a):

\[ \dot{t} = S\ddot{s} + \dot{S}\dot{s} \quad \text{where} \quad \dot{S} = \begin{bmatrix} \hat{F} & O \\ O_{34} & O \end{bmatrix} \]  \hspace{1cm} (3.96)
Let

\[ \lambda \equiv \begin{bmatrix} u \\ u_0 \end{bmatrix}, \quad \eta \equiv \begin{bmatrix} r \\ r_0 \end{bmatrix} \]  

where

\[ u \equiv \sin \phi \mathbf{e}, \quad u_0 \equiv \cos \phi, \]

\[ r \equiv \sin \left( \frac{\phi}{2} \right) \mathbf{e}, \quad r_0 \equiv \cos \left( \frac{\phi}{2} \right) \]  

(3.97a)
Acceleration Analysis of Rigid-Body Motions (Cont’d)

\[
\dot{\mathbf{N}} = \frac{1}{4(1 - \cos \phi)} \begin{bmatrix} \mathbf{B} \\ \dot{\mathbf{e}} \end{bmatrix} \quad \dot{\mathbf{H}} = \frac{1}{2} \begin{bmatrix} \dot{\mathbf{r}}_0 \mathbf{1} - \dot{\mathbf{R}} \\ -\dot{\mathbf{r}}^T \end{bmatrix}
\]

\[
\dot{\mathbf{L}} = \frac{1}{2} \begin{bmatrix} 1 \text{tr}(\dot{\mathbf{Q}}) - \dot{\mathbf{Q}} \\ -\omega^T \text{tr}(\mathbf{Q}) + \omega^T \mathbf{Q}^T \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 2(\omega \cdot \mathbf{u}) \mathbf{1} + \Omega \mathbf{Q} \\ \omega^T \text{tr}(\mathbf{Q}) - \omega^T \mathbf{Q}^T \end{bmatrix}
\]

where \( \mathbf{R} \) denotes the cross-product matrix of \( \mathbf{r} \),

\[
\mathbf{B} \equiv -2(e \cdot \omega) \mathbf{1} + 2(3 - \cos \phi)(e \cdot \omega)ee^T - 2(1 + \sin \phi)\omega e^T \\
- (2 \cos \phi + \sin \phi)e\omega^T - (\sin \phi)[\mathbf{\Omega} - (e \cdot \omega) \mathbf{E}]
\]

and

\[
\text{tr}(\dot{\mathbf{Q}}) \equiv \text{tr}(\Omega \mathbf{Q}) \equiv -2\omega^T \mathbf{u}
\]
We also have:

\[
\begin{align*}
\dot{\tilde{N}} &= \left[ \dot{\phi}(\cos \phi) \mathbf{1} + \dot{\phi}(\sin \theta) \mathbf{E} \right] \mathbf{e} \\
\dot{\tilde{L}} &= \left[ \mathbf{V}/D \right] \dot{\mathbf{u}} \\
\dot{\tilde{H}} &= \left[ \dot{r}_0 \mathbf{1} + \mathbf{R} \right] - \mathbf{r}
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{V} &\equiv \dot{\mathbf{U}} - (\dot{\mathbf{u}}\mathbf{u}^T + \mathbf{u}\dot{\mathbf{u}}^T) - \frac{\dot{\mathbf{u}}_0}{D} (\mathbf{U} - \mathbf{u}\mathbf{u}^T) \\
D &\equiv 1 + u_0
\end{align*}
\]

with \( \mathbf{U} \) denoting the cross-product matrix of \( \mathbf{u} \).
Rigid-Body Motion Referred to Moving Coordinate Axes

**Figure**: Fixed and moving coordinate frames.

\(\mathcal{F}\) : the **fixed** coordinate frame \(X, Y, Z\)  

\(\mathcal{M}\) : the **moving** coordinate frame \(\mathcal{X}, \mathcal{Y}, \mathcal{Z}\)
Rigid-Body Motion Referred to Moving Coordinate Axes (Cont’d)

\[ [p]_F = [o]_F + [\rho]_F \] (3.100)

\[ [\rho]_F = [Q]_F [\rho]_M \] (3.101)

Substitute eq.(3.101) into eq.(3.100):

\[ [\dot{p}]_F = [\dot{o}]_F + [Q]_F [\dot{\rho}]_M \] (3.102)

From the definition of \( \Omega \) in eq.(3.46):

\[ [\dot{Q}]_F = [\Omega]_F [Q]_F \] (3.104)
Hence,

\[
[p]_F = [o]_F + [\Omega]_F[Q]_F[p]_M + [Q]_F[\dot{\rho}]_M
\]  \tag{3.105}

Differentiate eq.(3.105) with respect to time:

\[
[\ddot{p}]_F = [\ddot{o}]_F + \left([\hat{\Omega}]_F[Q]_F[p]_M + [\Omega]_F[\dot{Q}]_F[p]_M + \Omega]_F[Q]_F[\dot{\rho}]_M + [\dot{Q}]_F[\dot{\rho}]_M + [Q]_F[\ddot{\rho}]_M
\]

Substitute (3.104) into the above equation:

\[
[\ddot{p}]_F = [\ddot{o}]_F + \left([\hat{\Omega}]_F + [\Omega^2]_F\right)[Q]_F[p]_M + 2[\Omega]_F[Q]_F[\dot{\rho}]_M + [Q]_F[\ddot{\rho}]_M
\]
Static Analysis of Rigid Bodies

**Figure:** Equivalent systems of force and moment acting on a rigid body.

\[ \mathbf{n}_P = \mathbf{n}_A + (\mathbf{a} - \mathbf{p}) \times \mathbf{f} \]  \hspace{1cm} (3.108)

\[ \mathbf{n}_P = \mathbf{n}_A + \mathbf{f} \times (\mathbf{p} - \mathbf{a}) \]  \hspace{1cm} (3.109)
Similar to Theorem 3:

**Theorem**

*For a given system of forces and moments acting on a rigid body, if the resultant force is applied at any point of a particular line $L''$, then the resultant moment is of minimum magnitude. Moreover, that minimum-magnitude moment is parallel to the resultant force.*

The minimum-magnitude moment $n_0$:

$$n_0 = n_0 \frac{f}{\|f\|}, \quad n_0 \equiv \frac{n_P \cdot f}{\|f\|} \quad (3.110)$$
The pitch of the wrench, \( p'' \):

\[
p'' \equiv \frac{n_0}{\|f\|} = \frac{n_P \cdot f}{\|f\|^2} \quad \text{or} \quad p'' = \frac{2\pi n_P \cdot f}{\|f\|^2}
\] (3.111)

The Plücker array of the wrench axis:

\[
p_{\mathcal{L}''} \equiv \begin{bmatrix} e'' \\ n'' \end{bmatrix}, \quad e'' = \frac{f}{\|f\|}, \quad n'' = p \times e''
\] (3.112)

The wrench axis is fully specified, then, by \( f \) and \( P_0'' \) lying closest to the origin

\[
p_0'' = \frac{1}{\|f\|^2} f \times (n_A - f \times a)
\] (3.113)
Similar to Theorem 1:

**Theorem**

_The projection of the resultant moment of a system of moments and forces acting on a rigid body that arises when the resultant force is applied at an arbitrary point of the body onto the wrench axis is constant._

To derive the wrench array $\mathbf{w}$, we need to multiply the screw of eq.(3.73) by an amplitude $A$ with units of force. We shall be able to obtain the power developed by the wrench on the body moving with the twist $\mathbf{t}$ but the inner product of these two arrays would be meaningless.
We redefine the wrench as a linear transformation of \( s \):

\[
\Gamma \equiv \begin{bmatrix} O & 1 \\ 1 & O \end{bmatrix} \Rightarrow \ w \equiv A\Gamma s \equiv \begin{bmatrix} \mathbf{p} \times (A\mathbf{e}) + p(A\mathbf{e}) \\ A\mathbf{e} \end{bmatrix}
\]

The first 3 components of \( w \) represent the moment of a force of magnitude \( A \) acting along \( \mathbf{e} \) with respect to \( P \) plus a moment parallel to \( \mathbf{e} \) and of magnitude \( pA \). The last 3 components pertain to a force of magnitude \( A \) and parallel to \( \mathbf{e} \):

\[
f \equiv A\mathbf{e}, \quad n \equiv \mathbf{p} \times f + pf
\]

\[
\Rightarrow \ w \equiv \begin{bmatrix} n \\ f \end{bmatrix} \quad \text{and} \quad \Pi = t^T w \tag{3.116}
\]
Let

\[ \mathbf{w} = \mathbf{W} \Gamma \mathbf{s}_w, \quad \mathbf{t} = \mathbf{T} \mathbf{s}_t \]  \hspace{1cm} (3.117)

The two screws \( \mathbf{s}_w \) and \( \mathbf{s}_t \) are reciprocal if

\[ (\Gamma \mathbf{s}_w)^T \mathbf{s}_t \equiv \mathbf{s}_w^T \Gamma^T \mathbf{s}_t = 0 \]  \hspace{1cm} (3.118)

or

\[ \mathbf{s}_w^T \Gamma \mathbf{s}_t = 0 \quad \text{and} \quad \mathbf{s}_t^T \Gamma \mathbf{s}_w = 0 \]  \hspace{1cm} (3.119)

For arbitrary points \( A \) and \( P \)

\[ \mathbf{w}_A \equiv \begin{bmatrix} \mathbf{n}_A \\ \mathbf{f} \end{bmatrix}, \quad \mathbf{w}_P \equiv \begin{bmatrix} \mathbf{n}_P \\ \mathbf{f} \end{bmatrix} \]  \hspace{1cm} (3.120)
⇒ \( w_P = Vw_A \) where \( V \equiv \begin{bmatrix} 1 & A - P \\ 0 & 1 \end{bmatrix} \) (3.121)

The last relation is termed the wrench-transfer formula.

Multiplying the transpose of eq.(3.84) by eq.(3.121) yields

\[ t_P^T w_P = t_A^T U^T V w_A \] (3.123)

where

\[
U^T V = \begin{bmatrix} 1 & -A + P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & A - P \\ 0 & 1 \end{bmatrix} = 1_{6\times6}
\] (3.124)

Thus, \( t_P^T w_P = t_A^T w_A \). Also note that \( V^{-1} = U^T \).
The rigid body mass:

\[ m = \int_B \rho \, d\mathcal{B} \quad (3.126) \]

where \( \rho \) is a mass density and \( \mathcal{B} \) denotes the region occupied by the body.

The mass first moment

\[ q_O \equiv \int_B \rho \mathbf{p} \, d\mathcal{B} \quad (3.127) \]

The mass second moment (symmetric matrix)

\[ I_O \equiv \int_B \rho [ (\mathbf{p} \cdot \mathbf{p}) \mathbf{1} - \mathbf{p} \mathbf{p}^T ] \, d\mathcal{B} \quad (3.128) \]

is called the moment-of-inertia matrix of the body with respect to \( O \).

**Theorem**

*The moment of inertia of a rigid body with respect to a point \( O \) is positive definite.*
Proof : For any vector $\omega$, the quadratic form $\omega^T I_O \omega$ is positive:

$$\omega^T I_O \omega = \int_B \rho \left[ \|p\|^2 \|\omega\|^2 - (p \cdot \omega)^2 \right] dB \quad (3.129)$$

Substituting $p \cdot \omega = \|p\| \|\omega\| \cos(p, \omega)$ (3.130) into eq.(3.129) leads to:

$$\omega^T I_O \omega = \int_B \rho \|p\|^2 \|\omega\|^2 \left[ 1 - \cos^2(p, \omega) \right] dB$$

$$= \int_B \rho \|p\|^2 \|\omega\|^2 \sin^2(p, \omega) dB$$
The kinetic energy of a rigid body: \[ T \equiv \int_B \frac{1}{2} \rho \| \dot{p} \|^2 d\mathbf{B} \]

where \( \dot{p} = \omega \times p = -P\omega \)

\[
\|\dot{p}\|^2 = (P\omega)^T P\omega = \omega^T P^T P \omega = -\omega^T P^2 \omega
= \omega^T (\|p\|^2 \mathbf{1} - pp^T) \omega
\]

Hence,

\[
T = \frac{1}{2} \int_B \rho \omega^T (\|p\|^2 \mathbf{1} - pp^T) \omega d\mathbf{B}
= \frac{1}{2} \omega^T \left[ \int_B \rho (\|p\|^2 \mathbf{1} - pp^T) d\mathbf{B} \right] \omega
\]

\[
\Rightarrow T = \frac{1}{2} \omega^T I_O \omega
\]

(3.134)
The mass center of a rigid body is defined as a point \( C \) of position vector \( c \):

\[
c \equiv \frac{qO}{m} \quad (3.135)
\]

The centroidal mass moment of inertia:

\[
I_C \equiv \int_B \rho \left[ \|r\|^2 1 - rr^T \right] dB \quad (3.136)
\]

where \( r \equiv p - c \)

The Theorem of Parallel Axes:

\[
I_O = I_C + m (\|c\|^2 1 - cc^T) \quad (3.138)
\]
Dynamics of Rigid Bodies (Cont’d)

For a body acted upon by a wrench of force $\mathbf{f}$ applied at its mass center, and a moment $\mathbf{n}_C$, the Newton equation:

$$\mathbf{f} = m\ddot{\mathbf{c}}$$  \hspace{1cm} (3.139a)

the Euler equation:

$$\mathbf{n}_C = I_C\dot{\omega} + \omega \times I_C\omega$$  \hspace{1cm} (3.139b)

The momentum $\mathbf{m}$ and the angular momentum $\mathbf{h}_C$:

$$\mathbf{m} \equiv m\dot{\mathbf{c}}, \quad \mathbf{h}_C \equiv I_C\dot{\omega}$$  \hspace{1cm} (3.140)

Moreover,

$$\dot{\mathbf{m}} = m\ddot{\mathbf{c}}, \quad \dot{\mathbf{h}}_C = I_C\ddot{\omega} + \omega \times I_C\omega$$  \hspace{1cm} (3.141)

$$\Rightarrow \quad \mathbf{f} = \dot{\mathbf{m}}, \quad \mathbf{n}_C = \dot{\mathbf{h}}_C$$  \hspace{1cm} (3.142)
The inertia dyad:

\[ M \equiv \begin{bmatrix} I_C & O \\ O & m_1 \end{bmatrix} \]  

(3.143)

Now the Newton-Euler equations can be written as

\[ M\dot{t} + WMt = w \]  

(3.144)

where

\[ W \equiv \begin{bmatrix} \Omega & O \\ O & O \end{bmatrix} \]  

(3.145)

is termed the angular-velocity dyad. Note that

\[ Wt = 0 \]  

(3.146)
Dynamics of Rigid Bodies (Cont’d)

The momentum screw about the rigid body mass center

\[ \mu \equiv \begin{bmatrix} I_C \omega \\ m \dot{c} \end{bmatrix} = Mt \]  \hspace{1cm} (3.147)

\[ \Rightarrow \dot{\mu} = Mt + W \mu = Mt + WMt \]  \hspace{1cm} (3.148)

The kinetic energy

\[ T = \frac{1}{2} m \| \dot{c} \|^2 + \frac{1}{2} \omega^T I_C \omega = \frac{1}{2} t^T Mt \]  \hspace{1cm} (3.150)

The Newton-Euler equations

\[ \dot{\mu} = w \]  \hspace{1cm} (3.151)