(*b*) $\mathbf{D}\boldsymbol{\omega} = \text{vect}(\dot{\mathbf{p}})$ where

$$\mathbf{D} = \frac{1}{2}[\text{tr}(\mathbf{P})\mathbf{1} - \mathbf{P}] = \frac{1}{2}\begin{bmatrix} 10 & 0 & 0 \\ 30 & -20 & 0 \\ -10 & -10 & 30 \end{bmatrix}, \quad \text{vect}(\dot{\mathbf{P}}) = \frac{1}{2}\begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

Hence,

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Matrix is lower-triangular. Use forward substitution: $\omega_1 = 2$, $3\omega_1 - 2\omega_2 = 2 \Rightarrow -2\omega_2 = -4 \Rightarrow \omega_2 = 2$. Then, $-\omega_1 - \omega_2 + 3\omega_3 = 2 \Rightarrow 3\omega_3 = 6 \Rightarrow \omega_3 = 2$

$$\Rightarrow \quad \boldsymbol{\omega} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \text{ rad/s}$$

9 Geometry of General Serial Robots

9.1 A reflection $\mathbf{H}$ onto a plane Π of a unit normal $\mathbf{n}$ can be expressed as

$$\mathbf{H} = \mathbf{1} - 2\mathbf{n}\mathbf{n}^T \tag{149}$$

Now, if the reflection plane is normal to $\mathbf{f}$, which is not necessarily of unit magnitude, then vector $\mathbf{h}$ is mapped by $\mathbf{H}$ into $\mathbf{h}'$ given by

$$\mathbf{h}' = (\mathbf{1} - 2\,\mathbf{f}\,\mathbf{f}^T/\|\mathbf{f}\|^2)\mathbf{h} = \frac{(\mathbf{f}\cdot\mathbf{f})\mathbf{h} - 2(\mathbf{f}\cdot\mathbf{h})\mathbf{f}}{\|\mathbf{f}\|^2}$$

Notice that $\|\mathbf{f}\|^2\mathbf{h}'$ is the left-hand side of eq.(8.22f). By the same token, we can write, for $\mathbf{i}'$

$$\mathbf{i}' = (\mathbf{1} - 2\,\mathbf{g}\,\mathbf{g}^T/\|\mathbf{g}\|^2)\mathbf{i} = \frac{(\mathbf{g}\cdot\mathbf{g})\mathbf{i} - 2\,(\mathbf{g}\cdot\mathbf{i})\mathbf{g}}{\|\mathbf{g}\|^2}$$

Obviously, the right-hand side of eq.(8.22f) is equal to $\|\mathbf{g}\|^2\mathbf{i}'$

9.3 See Maple worksheet in Appendix 1.

9.5 Assume that we have an orthogonal matrix $\mathbf{H}$ chosen as a product of 12 Householder reflections[5] that will render the matrix $\mathbf{S}$ in an upper triangular form $\mathbf{U}$. That is,

$$\mathbf{H}\mathbf{S} = \mathbf{U}$$

Equation (8.51d) can be written as

$$\mathbf{S}^T\mathbf{H}^T\mathbf{H}\tilde{\mathbf{x}}_{45} = \mathbf{0}_{12} \Longrightarrow (\mathbf{H}\mathbf{S})^T\mathbf{H}\tilde{\mathbf{x}}_{45} = \mathbf{0}_{12} \tag{150}$$

where $\mathbf{0}_{12}$ is the 12-dimensional zero vector. Then, we can write eq.(150) as

$$\mathbf{U}^T\mathbf{v} = \mathbf{0}_{12}$$

with

$$\mathbf{v} = \mathbf{H}\,\tilde{\mathbf{x}}_{45} \tag{151}$$

[5] See Appendix B.

Since $\mathbf{S}$ is singular, the 12th row of $\mathbf{U}$ is full of zeros, and so is the 12th column of $\mathbf{U}^T$. As a consequence, the nullspace of $\mathbf{U}^T$ is spanned by the unit vector $\mathbf{n} = [\mathbf{0}_{11}^T \ 1]^T$. Consequently, vector $\mathbf{v}$ is a multiple of $\mathbf{n}$, i.e., $\mathbf{v} = \alpha\mathbf{n}$, where α is a scalar, as yet to be determined, which is done below. From eq.(151),

$$\tilde{\mathbf{x}}_{45} = \mathbf{H}^T\mathbf{v} = \alpha\mathbf{H}^T\mathbf{n} = \alpha \left[\ h_{12,1} \ldots h_{12,12} \ \right]^T$$

where $h_{12,i}$ denotes, as usual, the ith component of the 12th row of $\mathbf{H}$. Upon comparison of the above expression of $\mathbf{x}_{45}$ with its definition in eq.(8.27a), we find α as

$$\alpha h_{12,12} = 1 \Rightarrow \alpha = \frac{1}{h_{12,12}}$$

9.7 We have four equations according to eq.(9.70a). By selecting any two of them, we end up with a system of two nonlinear equations $\mathbf{f}$ in two unknowns $\mathbf{x}$. The Jacobian matrix $\mathbf{F}$ is given by

$$\mathbf{F} \equiv \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

According to the Newton-Raphson method, we have

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \mathbf{F}_i^{-1}\mathbf{f}_i, \quad i = 0, 1, 2, \ldots$$

where $\mathbf{F}_i$ and $\mathbf{f}_i$ represent $\mathbf{F}$ and $\mathbf{f}$ evaluated at $\mathbf{x}_i$, respectively.

With an initial guess $\mathbf{x}_0$ close enough to a root, the Newton-Raphson method may converge to that root rapidly. Given a tolerance ϵ, the criterion to stop the iteration is

$$\|\mathbf{x}_{i+1} - \mathbf{x}_i\|_\infty < \epsilon$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm. The Maple code implementing this calculation is given in Appendix 2.

By monitoring the condition number of $\mathbf{F}$ based on the Frobenius norm, we observe that the Newton-Raphson method converges faster when the condition number is smaller. The condition number introduced in Section 5.8, with the Frobenius-norm condition number discussed in eqs.(5.79)–(5.82).

10 Kinematics of Complex Robotic Mechanical Systems

10.1 For the parallel manipulator of Fig. 9.7, the matrix mapping joint forces into wrenches acting on the moving platform can be obtained by relating the power generated by the actuators and the power consumed by the load. From eq.(9.102a),

$$\dot{\mathbf{b}} = \mathbf{K}\mathbf{t}$$

where $\mathbf{K}$ is the Jacobian of the manipulator given in eq.(9.102b). Under static, conservative conditions, the power delivered by the actuators equals that developed by the load, i.e., $\Pi_a = \Pi_L$, where

$$\Pi_a = \dot{\mathbf{b}}^T\boldsymbol{\tau}$$
$$\Pi_L = \mathbf{t}^T\mathbf{w}$$

with $\dot{\mathbf{b}}$ being the vector of actuated joint rates, $\boldsymbol{\tau}$ the vector of actuated joint torques, $\mathbf{t}$ the twist of the moving platform, and $\mathbf{w}$ the wrench acting on the moving platform. Then,

$$\dot{\mathbf{b}}^T\boldsymbol{\tau} = \mathbf{t}^T\mathbf{w}$$
$$\mathbf{t}^T\mathbf{K}^T\boldsymbol{\tau} = \mathbf{t}^T\mathbf{w}$$

which is valid for every possible motion, i.e., for every possible twist $\mathbf{t}$, and hence, the above equation leads to

$$\mathbf{w} = \mathbf{K}^T\boldsymbol{\tau}$$

10.8 Matrix $\mathbf{\Theta}$ is given as

$$\mathbf{\Theta} = \begin{bmatrix} \alpha\cos\psi + (1/2)\sin\psi & -\alpha\cos\psi + (1/2)\sin\psi \\ \rho\,[-\alpha\sin\psi + (1/2)\cos\psi - \delta] & \rho\,[\alpha\sin\psi + (1/2)\cos\psi + \delta] \end{bmatrix}$$

with the definitions below:

$$\alpha \equiv \frac{a+b}{l}, \quad \delta \equiv \frac{d}{l}, \quad \rho \equiv \frac{r}{d}$$

Upon expanding the determinant of $\mathbf{\Theta}$ and imposing the singularity condition, we have

$$\det(\mathbf{\Theta}) = \rho[\alpha + \delta\sin\psi] = 0$$

Hence,

$$\sin\psi = -\frac{\alpha}{\delta} \Longrightarrow \frac{\alpha}{\delta} \leq 1$$

which yields

$$a + b \leq d \tag{152}$$

If the above condition holds, then the angle ψ rendering $\mathbf{\Theta}$ singular can be evaluated as

$$\psi = \sin^{-1}\left(-\frac{a+b}{d}\right)$$

However, notice that many a caster wheel exhibits the value $d = r$; most commonly, one has has $d < r$. In the extreme case $d = r$, condition (152) implies that $a + b$ is smaller than the radius of the wheels, a rather unlikely design!

10.9 To determine the conditions for matrix $\mathbf{\Theta}$ to be isotropic, any of $\mathbf{\Theta}^T\mathbf{\Theta}$ and $\mathbf{\Theta}\mathbf{\Theta}^T$ must be proportional to the 2×2 identity matrix. Below we calculate these two matrices:

I. $$\mathbf{\Theta}^T\mathbf{\Theta} = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix}$$

where

$$\begin{aligned} \alpha_1 &= (1-\rho^2)[(\alpha^2 - 1/4)\cos\psi + \alpha\sin\psi]\cos\psi - \delta\rho^2(\cos\psi - 2\alpha\sin\psi) \\ &\quad + \rho^2(\alpha^2 + \delta^2) + 1/4 \\ \alpha_2 &= (1-\rho^2)[(\alpha^2 - 1/4)\cos\psi - \alpha\sin\psi]\cos\psi + \delta\rho^2(\cos\psi + 2\alpha\sin\psi) \\ &\quad + \rho^2(\alpha^2 + \delta^2) + 1/4 \\ \alpha_3 &= (\alpha^2 + 1/4)(\rho^2 - 1)\cos^2\psi - 2\alpha\delta\rho^2\sin\psi - \rho^2(\alpha^2 + \delta^2) + 1/4 \end{aligned}$$

and

II. $$\mathbf{\Theta}\mathbf{\Theta}^T = \begin{bmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{bmatrix}$$

where

$$\begin{aligned} \beta_1 &= (2\alpha^2 - 1/2)\cos^2\psi + 1/2 \\ \beta_2 &= \rho^2[2(\alpha\sin\psi + \delta)^2 + (1/2)\cos^2\psi] \\ \beta_3 &= -(1/2)\rho(4\alpha^2\sin\psi + 4\alpha\delta - \sin\psi)\cos\psi\ . \end{aligned}$$

Apparently, matrix $\boldsymbol{\Theta\Theta}^T$ is simpler, and hence, we work with this matrix. Upon imposing the isotropy condition

$$\boldsymbol{\Theta\Theta}^T = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

we have $g_1 \equiv \beta_1 - \beta_2 = 0$ while $\beta_1 > 0$ and $g_2 \equiv \beta_3 = 0$, which yield the corresponding conditions:

$$g_1 \equiv (1+\rho^2)(2\alpha^2 - 1/2)\cos^2\psi - 4\alpha\delta\rho^2\sin\psi - 2(\alpha^2+\delta^2)\rho^2 + 1/2 = 0 \tag{153a}$$

$$g_2 \equiv \left[(4\alpha^2 - 1)\sin\psi + 4\alpha\delta\right]\cos\psi = 0 \tag{153b}$$

with α, δ, and ρ defined as in the book:

$$\alpha \equiv \frac{a+b}{l}, \quad \delta \equiv \frac{d}{l}, \quad \rho \equiv \frac{r}{d}.$$

From eq.(153b) we have two cases:

(i) $\cos\psi = 0$ and

(ii) $\cos\psi \neq 0$, which requires that $(4\alpha^2 - 1)\sin\psi + 4\alpha\delta = 0$

In case (i), $\sin\psi = \pm 1$. However, for stable equilibrium we must have[6] $\sin\psi = +1$, which leads to $\psi = \pi/2$, i.e., the posture adopted by the robot when travelling on a straight course. Then, from eq.(153a) we derive

$$-2(\alpha+\delta)^2\rho^2 + \frac{1}{2} = 0$$

and, since $\alpha,\ \delta \geq 0$, the foregoing relation implies

$$\alpha + \delta = \frac{1}{2\rho}$$

or, in terms of the architecture parameters,

$$\frac{a+b+d}{l} = \frac{d}{2r}$$

Substituting this condition into matrix $\boldsymbol{\Theta\Theta}^T$, we obtain

$$\boldsymbol{\Theta\Theta}^T = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apparently, in this case matrix $\boldsymbol{\Theta}$ is isotropic, and has the form

$$\boldsymbol{\Theta} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

In case (ii), we have

$$\sin\psi = \frac{4\alpha\delta}{1-4\alpha^2} \tag{154}$$

with $1 - 4\alpha^2 > 4\alpha\delta$ while $\alpha < 1/2$, and hence,[7]

$$\cos^2\psi = \frac{(1-4\alpha^2)^2 - 16\alpha^2\delta^2}{(1-4\alpha^2)^2} \tag{155}$$

[6] The axis of the caster wheel always *lags* the vertical axis of its bracket.

[7] We do not consider the case $|1 - 4\alpha^2| = 4\alpha\delta$ because this takes us back to case (i).

Then, upon substituting the above expressions into matrix $\mathbf{\Theta}\mathbf{\Theta}^T$, we obtain

$$\mathbf{\Theta}\mathbf{\Theta}^T = \frac{1-4(\alpha^2-\delta^2)}{1-4\alpha^2}\begin{bmatrix} 2\alpha^2 & 0 \\ 0 & (1/2)\rho^2 \end{bmatrix}$$

Apparently, to make this matrix proportional to 2×2 identical matrix, we need $\rho = 2\alpha$. Substituting eqs.(154) and (155) and $\rho = 2\alpha$ into matrix $\mathbf{\Theta}$, we obtain

$$\mathbf{\Theta} = \frac{\alpha}{(1-4\alpha^2)}\begin{bmatrix} \sqrt{S}+2\delta & -\sqrt{S}+2\delta \\ \sqrt{S}-2\delta & \sqrt{S}+2\delta \end{bmatrix}$$

with

$$S \equiv 16\alpha^2(\alpha^2-\delta^2) - 8\alpha^2 + 1$$

For example, if $\alpha = 1/3$, $\rho = 2\alpha = 2/3$ and $\delta = (1-4\alpha^2)/8\alpha\delta = 5/24$, then $\psi = \pi/6$ and $\mathbf{\Theta}$ has the form

$$\mathbf{\Theta} = \frac{1}{12}\begin{bmatrix} 2\sqrt{3}+3 & -2\sqrt{3}+3 \\ 2\sqrt{3}-3 & 2\sqrt{3}+3 \end{bmatrix}$$

A Maple worksheet supporting the calculations involved is included in Appendix 3.

10.12 (a) Upon inversion of eq.(8.128a), we obtain,

$$\dot{\boldsymbol{\theta}}_a = \mathbf{U}\dot{\boldsymbol{\theta}}_u$$

with $\mathbf{U} = \mathbf{\Theta}^{-1}$, i.e.,

$$\mathbf{U} = \frac{1}{2\rho(\alpha+\delta\sin\psi)}\begin{bmatrix} \rho[2(\alpha\sin\psi+\delta)+\cos\psi] & 2\alpha\cos\psi-\sin\psi \\ \rho[2(\alpha\sin\psi+\delta)-\cos\psi] & 2\alpha\cos\psi+\sin\psi \end{bmatrix}$$

(b) Let

$$\mathbf{u}_1 = \begin{bmatrix} \dfrac{2(\alpha\sin\psi+\delta)+\cos\psi}{2(\alpha+\delta\sin\psi)} \\ \dfrac{2\alpha\cos\psi-\sin\psi}{2\rho(\alpha+\delta\sin\psi)} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \dfrac{2(\alpha\sin\psi+\delta)-\cos\psi}{2(\alpha+\delta\sin\psi)} \\ \dfrac{2\alpha\cos\psi+\sin\psi}{2\rho(\alpha+\delta\sin\psi)} \end{bmatrix}$$

their gradient with respect to $\boldsymbol{\theta}_u$ being

$$\nabla\mathbf{u}_i = \begin{bmatrix} \dfrac{\partial\mathbf{u}_i}{\partial\theta_3} & \dfrac{\partial\mathbf{u}_i}{\partial\psi} \end{bmatrix}, \quad i = 1, 2$$

where

$$\frac{\partial\mathbf{u}_1}{\partial\theta_3} = \mathbf{0}$$

and

$$\frac{\partial\mathbf{u}_1}{\partial\psi} = \frac{1}{D}\begin{bmatrix} \rho[2(\alpha^2+\delta^2)\cos\psi-(\alpha\sin\psi+\delta)] \\ -\alpha[2(\alpha\sin\psi+\delta)+\cos\psi] \end{bmatrix}$$

$$D \equiv 2\rho(\alpha+\delta\sin\psi)^2$$

whence it is apparent that $\nabla\mathbf{u}_1$ is not symmetric, and hence, no function $U_1(\theta_3, \psi)$ exists whose gradient is $\mathbf{u}_1$. Likewise,

$$\frac{\partial\mathbf{u}_2}{\partial\theta_3} = \mathbf{0}$$

and

$$\frac{\partial\mathbf{u}_2}{\partial\psi} = \frac{1}{D}\begin{bmatrix} \rho[2(\alpha^2-\delta^2)\cos\psi+(\alpha\sin\psi+\delta)] \\ -\alpha[2(\alpha\sin\psi+\delta)-\cos\psi] \end{bmatrix},$$

Apparently, neither $\nabla\mathbf{u}_2$ is symmetric, the conclusion being that none of the two constraints in $\dot{\boldsymbol{\theta}}_a = \mathbf{U}\dot{\boldsymbol{\theta}}_u$ is holonomic.

10.13 Expressions for ω and $\dot{\mathbf{c}}$ are derived in Subsection 10.5.2, and are reproduced below:

$$\omega = -\frac{a}{3r}\sum_1^3 \dot{\theta}_i, \quad \dot{\mathbf{c}} = -\frac{2a}{3}\sum_1^3 \dot{\theta}_i \mathbf{f}_i$$

Apparently, the first equation leads to the integral

$$\psi = -\frac{a}{3r}\sum_1^3 \theta_i \tag{156}$$

where ψ is the angular displacement of the platform with respect to a horizontal line fixed to an inertial frame, under the assumption that, when $\theta_i = 0$, for $i = 1, 2, 3$, $\psi = 0$. Now, the second equation is cast in the form

$$\mathbf{F}\dot{\boldsymbol{\theta}} = -\frac{3}{2a}\dot{\mathbf{c}}, \quad \mathbf{F} \equiv [\,\mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3\,] \equiv \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix}$$

Thus, the second constraint equation is integrable iff there is a 2-dimensional vector function $\boldsymbol{\phi}(\boldsymbol{\theta}) = [\,\phi_1,\, \phi_2\,]^T$ such that

$$\mathbf{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}}$$

For example, we should have, for the (1, 1) and (1, 2) entries of $\mathbf{F}$,

$$\lambda_1 = \frac{\partial \phi_1}{\partial \theta_1}, \quad \lambda_2 = \frac{\partial \phi_1}{\partial \theta_2}$$

where

$$\lambda_1 = \cos\psi, \quad \lambda_2 = \cos\left(\psi + \frac{2\pi}{3}\right) = -\frac{1}{2}(\cos\psi + \sqrt{3}\sin\psi)$$

Let us assume that the foregoing equations hold, while ψ is given by eq.(156), i.e.,

$$\cos\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) = \frac{\partial \phi_1}{\partial \theta_1}$$

$$\frac{1}{2}\left[-\cos\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) + \sqrt{3}\sin\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right)\right] = \frac{\partial \phi_1}{\partial \theta_2}$$

Let us now take the partial derivative of the first of the foregoing equations with respect to θ_2 and of the second with respect to θ_1:

$$-\frac{a}{3r}\sin\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) = \frac{\partial^2 \phi_1}{\partial \theta_1 \theta_2}$$

$$\frac{a}{6r}\left[\sin\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) + \sqrt{3}\cos\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right)\right] = \frac{\partial^2 \phi_1}{\partial \theta_2 \theta_1}$$

For the above assumption to hold, the left-hand sides of the two foregoing equations must be equal, i.e.,

$$-\sin\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) = \frac{1}{2}\left[\sin\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) + \sqrt{3}\cos\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right)\right]$$

However, this equation leads to

$$\tan\left(\frac{\theta_1+\theta_2+\theta_3}{3r}a\right) = -\frac{\sqrt{3}}{3}$$

and hence,

$$\frac{a}{3r}(\theta_1+\theta_2+\theta_3) = -\frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$$

which contradicts the integral of the first constraint, the conclusion being that the second constraint equation is not integrable. This result is a consequence of the nonholonomy of the system at hand.

11 Trajectory Planning: Continuous-Path Operations

11.1 (a) Many solutions are possible for the location of the robot base. Obviously, the path must lie within the workspace of the robot. Additionally, two alternatives are possible: either the welding is done from outside or from within the helix. The workspace of the PUMA 560 is displayed in Fig. 31, along with a projection of the path. For an *interior operation*, we could have a path Γ^0, with coordinates in m:

$$x = 0.3\cos\vartheta$$
$$y = 0.3\sin\vartheta$$
$$z = \frac{0.8\vartheta}{\pi}$$

Here, the center[8] O_H of the helicoidal path has $\mathcal{F}_2$ coordinates $(0, 0, 0)$. For an *exterior* operation, we have a path Γ^1, with coordinates in m as well:

$$x = 0.3\cos\vartheta - 0.5$$
$$y = 0.3\sin\vartheta - 0.5$$
$$z = \frac{0.8\vartheta}{\pi} + 0.33$$

with the center of the helicoidal path located at a point of $\mathcal{F}_2$ coordinates $(-0.5, -0.5, 0.33)$ m. Now, let us determine the Frenet-Serret vectors, which are independent of where the center O_H is located. First, we determine the velocity along the helix:

$$\dot{x} = -0.3\dot{\vartheta}\sin\vartheta$$
$$\dot{y} = 0.3\dot{\vartheta}\cos\vartheta$$
$$\dot{z} = \frac{0.8\dot{\vartheta}}{\pi}$$

and the corresponding acceleration:

$$\ddot{x} = -0.3\dot{\vartheta}^2\cos\vartheta - 0.3\ddot{\vartheta}\sin\vartheta$$
$$\ddot{y} = 0.3\ddot{\vartheta}\cos\vartheta - 0.3\dot{\vartheta}^2\sin\vartheta$$
$$\ddot{z} = \frac{0.8\ddot{\vartheta}}{\pi}$$

The constant-speed condition (as in Example 11.3.1) leads to:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_0^2$$

where v_0 is the constant speed along the helix. Upon substitution of numerical values, the foregoing condition becomes

$$0.3^2\dot{\vartheta}^2 + \left(\frac{0.8}{\pi}\right)^2\dot{\vartheta}^2 = (0.050)^2$$
$$\dot{\vartheta} = 0.1271$$

and hence, with $c = 0.1271\ \text{s}^{-1}$, we obtain

$$\dot{x} = -0.3c\sin(ct)$$
$$\dot{y} = 0.3c\cos(ct)$$
$$\dot{z} = \frac{0.8c}{\pi}$$

[8]The center of the helicoidal path is defined as the intersection of its axis with a plane parallel to the X_2-Y_2 plane and containing the bottom end of the path.

Following Example 11.3.1, since this part is independent of the robot,

$$\mathbf{e}_t \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \equiv \frac{\dot{\mathbf{r}}}{\dot{s}} = \frac{c}{v_o}\begin{bmatrix} -a\sin ct \\ a\cos ct \\ b \end{bmatrix} \quad \text{and} \quad \mathbf{e}_n = -\begin{bmatrix} \cos ct \\ \sin ct \\ 0 \end{bmatrix}$$

Thus, the binormal vector $\mathbf{e}_b$ is calculated simply as the cross product of the first two vectors of the Frenet-Serret triad:

$$\mathbf{e}_b \equiv \mathbf{e}_t \times \mathbf{e}_n = -\frac{c}{v_o}\begin{bmatrix} -b\sin ct \\ b\cos ct \\ -a \end{bmatrix}$$

The orientation matrix $\mathbf{Q}$ of the nozzle is given by

$$\mathbf{Q} \equiv [\, \mathbf{e}_t \quad \mathbf{e}_n \quad \mathbf{e}_b \,]$$

Hence,

$$\mathbf{Q} = \frac{c}{v_o}\begin{bmatrix} -a\sin ct & -(v_o/c)\cos ct & b\sin ct \\ a\cos ct & -(v_o/c)\sin ct & -b\cos ct \\ b & 0 & a \end{bmatrix}$$

Now, the center of the wrist is located, with respect to the base, by the following vector:

$$\mathbf{c} = \mathbf{o}_H + \mathbf{p} + \mathbf{Q}\mathbf{c}_P$$

with $\mathbf{o}_H$ denoting the position vector of O_H, $\mathbf{p}$ that of an arbitrary point P on the helix, both given in $\mathcal{F}_2$, and $\mathbf{c}_P$, the position vector of P in the Frenet-Serret frame.

$$\mathbf{o}_H = \begin{bmatrix} -0.500 \\ -0.500 \\ 0.330 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} a\cos\vartheta \\ a\sin\vartheta \\ b\vartheta \end{bmatrix}, \quad \mathbf{c}_P = \begin{bmatrix} 0 \\ -0.050 \\ 0.0867 \end{bmatrix}$$

The joint trajectories appear in Figs. 32 and 33 as pertaining to paths Γ_0 and Γ_1, respectively.

(b) Now, once $\mathbf{c}$ is available, we proceed with the inverse kinematics of the PUMA 560 as done in Section 4.4. Once we obtain $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$ can be found from Section 4.5 as

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^{-1}\mathbf{t}$$

Shown in Figs. 34 and 35 are the plots of the time-histories of the joint rates, as pertaining to paths Γ_0 and Γ_1, respectively.

(c) The joint accelerations can be found from Section 4.6 as

$$\ddot{\boldsymbol{\theta}} = \mathbf{J}^{-1}(\dot{\mathbf{t}} - \dot{\mathbf{J}}\dot{\theta})$$

Plotted in Figs. 36 and 37 are the joint-acceleration time-histories for paths Γ_0 and Γ_1, respectively.

11.2 (a) Matrix representation of $\mathbf{S}(t)$ in $\mathcal{B}$. We have

$$\begin{aligned} \mathbf{e}_t &= -0.6\sin\varphi\mathbf{i}_o + 0.6\cos\varphi\mathbf{j}_o + 0.8\mathbf{k}_o \\ \mathbf{e}_n &= -\cos\varphi\mathbf{i}_o - \sin\varphi\mathbf{j}_o \\ \mathbf{e}_b &= 0.8\sin\varphi\mathbf{i}_o - 0.8\cos\varphi\mathbf{j}_o + 0.6\mathbf{k}_o \end{aligned}$$

and hence, $\mathbf{S}(t) = [\, \mathbf{e}_t \quad \mathbf{e}_n \quad \mathbf{e}_b \,]$ expressed in $\{\mathbf{i}_o, \mathbf{j}_o, \mathbf{k}_o\}$ is found from Definition 2.2.1 as

$$\mathbf{S}(t) = \begin{bmatrix} -0.6\sin\varphi & -\cos\varphi & 0.8\sin\varphi \\ 0.6\cos\varphi & -\sin\varphi & -0.8\cos\varphi \\ 0.8 & 0 & 0.6 \end{bmatrix}$$

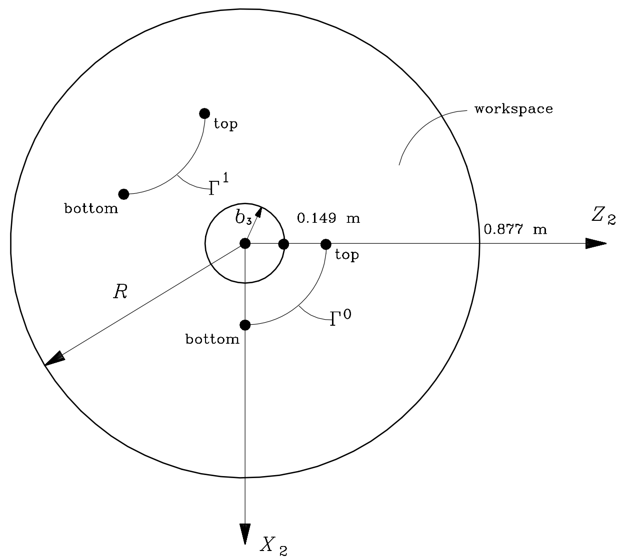

Figure 31: Top view of the workspace of the PUMA 560

(b) Now we determine likewise the matrix representation of $\mathbf{R}$ in $\mathcal{F}$ from the relation

$$\mathbf{A} \equiv [\,\mathbf{i}_7 \quad \mathbf{j}_7 \quad \mathbf{k}_7\,] = \begin{bmatrix} 0.933 & 0.067 & -0.354 \\ 0.067 & 0.933 & 0.354 \\ 0.354 & -0.354 & 0.866 \end{bmatrix}$$

whence $[\mathbf{R}]_{\mathcal{F}}$, defined as $[\,\mathbf{e}_t \quad \mathbf{e}_n \quad \mathbf{e}_b\,]$ in $\mathcal{F}_7$, is

$$[\mathbf{R}]_{\mathcal{F}} = \mathbf{A}^T = \begin{bmatrix} 0.933 & 0.067 & 0.354 \\ 0.067 & 0.933 & -0.354 \\ -0.354 & 0.354 & 0.866 \end{bmatrix}$$

(c) We have, from a result displayed in eqs.(4.8a) and (4.9a),

$$[\mathbf{Q}(t)]_{\mathcal{B}} = [\mathbf{R}]_{\mathcal{B}}[\mathbf{S}(t)]_{\mathcal{B}} = [\mathbf{S}(t)]_{\mathcal{B}}[\mathbf{R}]_{\mathcal{F}}\underbrace{[\mathbf{S}(t)]_{\mathcal{B}}^T[\mathbf{S}(t)]_{\mathcal{B}}}_{\mathbf{1}} = [\mathbf{S}(t)]_{\mathcal{B}}[\mathbf{R}]_{\mathcal{F}}$$

i.e.,

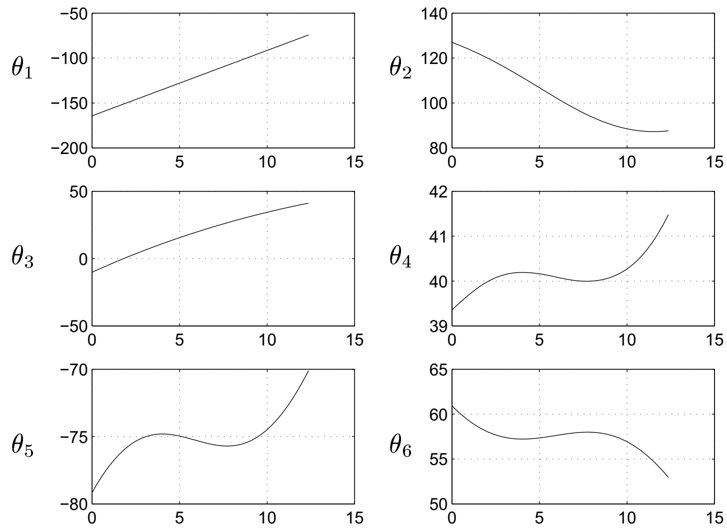

Figure 32: Joint trajectories vs. time (s) for path Γ_0, in degrees

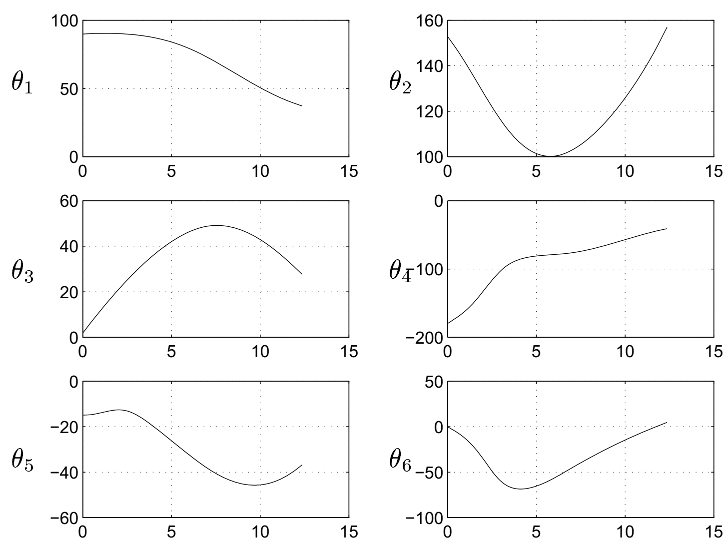

Figure 33: Joint trajectories vs. time (s) for path Γ_1, in degrees.

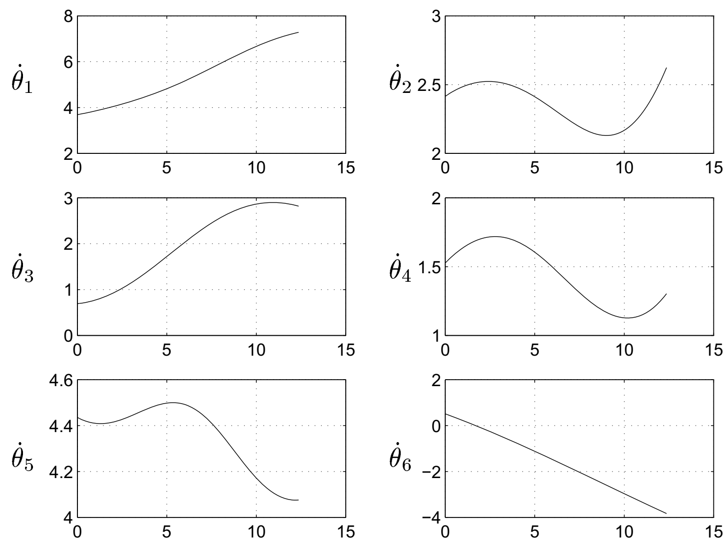

Figure 34: Joint velocities vs. time (s) for path Γ_0, in rad/s.

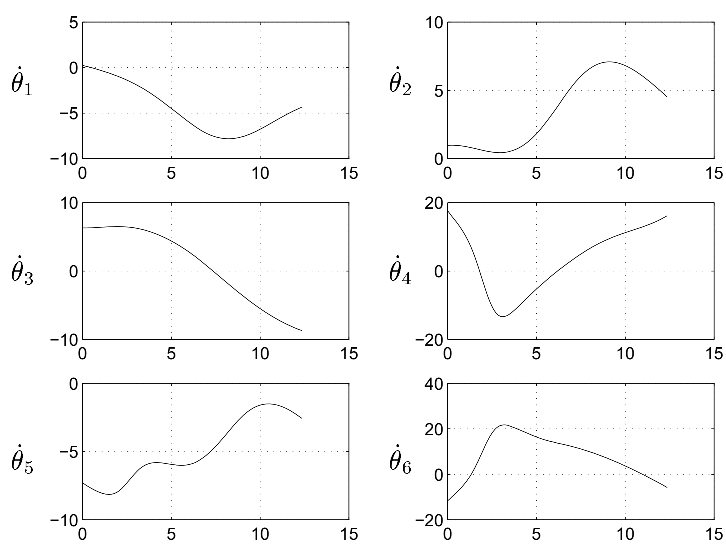

Figure 35: Joint velocities vs. time (s) for path Γ_1, in rad/s.

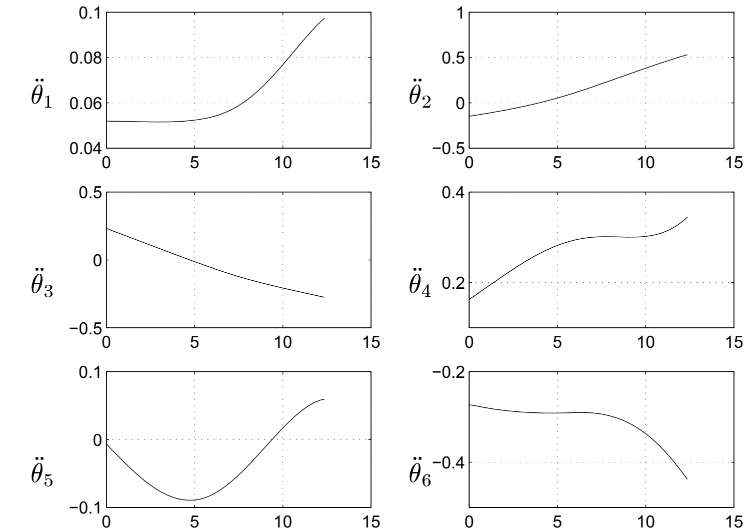

Figure 36: Joint accelerations vs. time (s) for path Γ_0, in rad/s^2.

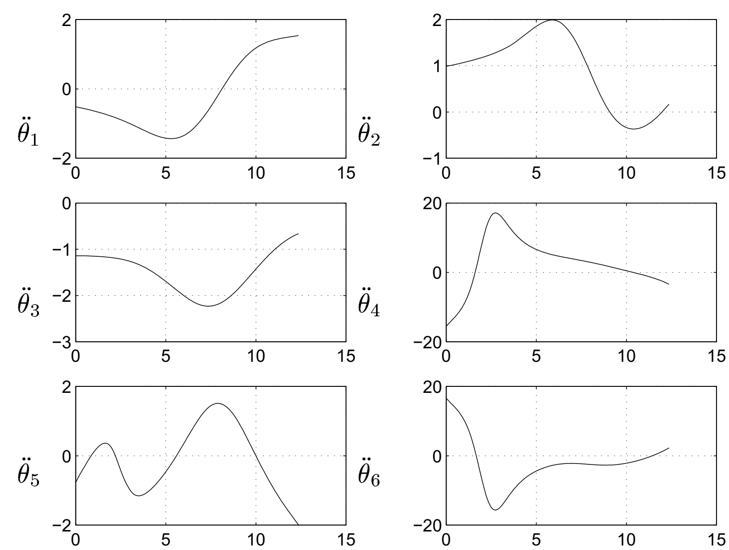

Figure 37: Joint accelerations vs. time (s) for path Γ_1, in rad/s^2.

$$
[\mathbf{Q}(t)]_{\mathcal{B}} = \begin{bmatrix} -0.6s\varphi & -c\varphi & 0.8s\varphi \\ 0.6c\varphi & -s\varphi & -0.8c\varphi \\ 0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 0.933 & 0.067 & 0.354 \\ 0.067 & 0.933 & -0.354 \\ -0.354 & 0.354 & 0.866 \end{bmatrix}
$$

$$
= \begin{bmatrix} -0.843s\varphi - 0.067c\varphi & 0.243s\varphi - 0.933c\varphi & 0.4804s\varphi + 0.354c\varphi \\ 0.843c\varphi - 0.067s\varphi & -0.243c\varphi - 0.933s\varphi & -0.4804c\varphi + 0.354s\varphi \\ 0.534 & 0.266 & 0.8028 \end{bmatrix}
$$

where $s\varphi = \sin\varphi$ and $c\varphi = \cos\varphi$.

(d) The Darboux vector $\boldsymbol{\delta}$ is given as

$$
\boldsymbol{\delta} = \tau \mathbf{e}_t + \kappa \mathbf{e}_b
$$

In the base frame $\mathcal{B}$, we have

$$
\mathbf{e}_t = \begin{bmatrix} -0.6\sin\varphi \\ 0.6\cos\varphi \\ 0.8 \end{bmatrix}, \quad \mathbf{e}_n = \begin{bmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{bmatrix}, \quad \mathbf{e}_b = \begin{bmatrix} 0.8\sin\varphi \\ -0.8\cos\varphi \\ 0.6 \end{bmatrix}
$$

From eq.(11.5a), we can find the curvature κ:

$$
\dot{\mathbf{e}}_t = 0.6\dot{\varphi} \begin{bmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{bmatrix} = 0.6\dot{\varphi}\mathbf{e}_n
$$

whence $\kappa = 0.6$. Now we determine τ:

$$
\dot{\mathbf{e}}_b = 0.8\dot{\varphi} \begin{bmatrix} \sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} = -0.8\dot{\varphi} \begin{bmatrix} -\sin\varphi \\ -\cos\varphi \\ 0 \end{bmatrix}
$$

$$
\frac{\dot{\mathbf{e}}_b}{\dot{\varphi}} = -0.8\mathbf{e}_n = -\tau \mathbf{e}_n
$$

So, $\tau = 0.8$. Finally,

$$
\boldsymbol{\delta} = 0.8\mathbf{e}_t + 0.6\mathbf{e}_b
$$

and

$$
\dot{\boldsymbol{\delta}} = 0.8\dot{\mathbf{e}}_t + 0.6\dot{\mathbf{e}}_b = (0.8)(0.6)\dot{\varphi}\mathbf{e}_n - (0.6)(0.8)\dot{\varphi}\mathbf{e}_n = \mathbf{0}
$$

11.3 (a) We have

$$
x = 2t, \quad y = t^2, \quad z = \frac{t^3}{3}
$$

Then, the position vector $\mathbf{r}$ of any point on the curve with its first and second-derivatives are

$$
\mathbf{r} = \begin{bmatrix} 2t \\ t^2 \\ t^3/3 \end{bmatrix}, \quad \dot{\mathbf{r}} = \begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix}, \quad \ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 2 \\ 2t \end{bmatrix}
$$

Now, the Frenet-Serret triad is readily evaluated as

$$
\mathbf{e}_t = \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} = \frac{1}{t^2+2} \begin{bmatrix} 2 \\ 2t \\ t^2 \end{bmatrix}, \quad \mathbf{e}_b = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|} = \frac{1}{t^2+2} \begin{bmatrix} t^2 \\ -2t \\ 2 \end{bmatrix},
$$

$$
\mathbf{e}_n = \mathbf{e}_b \times \mathbf{e}_t = \frac{1}{t^2+2^2} \begin{bmatrix} -2t^3+4t \\ -t^4+4 \\ 2t^3+4t \end{bmatrix}
$$

and hence, the orientation matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = [\ \mathbf{e}_t \quad \mathbf{e}_n \quad \mathbf{e}_b\ ] = \frac{1}{t^2+2}\begin{bmatrix} 2 & -2t & t^2 \\ 2t & 2-t^2 & -2t \\ t^2 & 2t & 2 \end{bmatrix}$$

Hence,

$$\text{vec}(\mathbf{Q}) = \frac{1}{2}\begin{bmatrix} \mathbf{Q}(3,2) - \mathbf{Q}(2,3) \\ \mathbf{Q}(1,3) - \mathbf{Q}(3,1) \\ \mathbf{Q}(2,1) - \mathbf{Q}(1,2) \end{bmatrix} = \frac{1}{t^2+2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \cos\phi = \frac{\text{tr}(\mathbf{Q}) - 1}{2} = \frac{2-t^2}{t^2+2}$$

the unit vector $\mathbf{e}$ and angle ϕ sought then being

$$\mathbf{e} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \phi = \tan^{-1}\left(\frac{\sqrt{2}}{2-t^2}\right)$$

(b) The expressions for the curvature and torsion in terms of time are readily evaluated as

$$\kappa = \frac{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|}{\|\dot{\mathbf{r}}\|^3} = \frac{2}{t^2+2}, \quad \tau = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}} \cdot \dddot{\mathbf{r}}}{\|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\|^2} = \frac{2}{t^2+2}$$

Moreover, $\boldsymbol{\omega} = \dot{s}\boldsymbol{\delta}$, and hence, all we need is $\dot{s}$ and $\boldsymbol{\delta}$, which are readily computed as

$$\dot{s} = \|\dot{\mathbf{r}}\| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2$$

and

$$\boldsymbol{\delta} = \tau\mathbf{e}_t + \kappa\mathbf{e}_b = \frac{2}{t^2+2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\boldsymbol{\omega} = \dot{s}\boldsymbol{\delta} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Finally, straightforward differentiation of the above expression for $\boldsymbol{\omega}$ with respect to time yields

$$\dot{\boldsymbol{\omega}} = \mathbf{0}$$

11.7 The path is given as

$$\Gamma : \mathbf{r} = r\begin{bmatrix} \lambda + \cos\varphi \\ \sin\varphi \\ 1/\hat{\varphi} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with $\hat{\varphi}$ defined in Example 11.3.2. To obtain the spline approximation of Γ, we make use of eqs.(11.47d & e). For $N = 5$, we have the supporting points displayed in Table 4. Hence,

$$\Delta\sigma_1 = 0.2348$$
$$\Delta\sigma_2 = 0.2279$$
$$\Delta\sigma_3 = 0.2279$$
$$\Delta\sigma_4 = 0.2348$$

while α_k, $\alpha_{i,j}$, β_k, and $\beta_{i,j}$ are displayed in Table 5.

Table 4: Supporting points for $N = 5$

φ	$0°$	$90°$	$180°$	$270°$	$360°$
x	0.45	0.30	0.15	0.30	0.45
y	0.00	0.15	0.00	-0.15	0.00
z	0.3969	0.4975	0.5809	0.4975	0.3969

Table 5: Spline parameters for $N = 5$

	α_k	$\alpha_{i,j}$	β_k	$\beta_{i,j}$
1	0.2348	0.4627	4.2589	8.5178
2	0.2279	0.4558	4.3879	8.6468
3	0.2279	0.4627	4.3879	8.7758
4	0.2348	0.4696	4.2589	8.6468

Now, we assemble the $\mathbf{A}$ and $\mathbf{C}$ matrices. Since the path is closed, we have a periodic spline. So, $\mathbf{A}$ and $\mathbf{C}$ are given by eqs.(11.59a & b). For $N = 5$, we obtain

$$\mathbf{A} = \begin{bmatrix} 0.9392 & 0.2348 & 0 & 0.2348 \\ 0.2348 & 0.9254 & 0.2279 & 0 \\ 0 & 0.2279 & 0.9116 & 0.2279 \\ 0.2348 & 0 & 0.2279 & 0.9254 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -8.5178 & 4.2589 & 0 & 4.2589 \\ 4.2589 & -8.6468 & 4.3879 & 0 \\ 0 & 4.3879 & -8.7758 & 4.3879 \\ 4.2589 & 0 & 4.3879 & -8.6468 \end{bmatrix}$$

Matrix $\mathbf{P}$ can be found using eq.(11.58):

$$\mathbf{P} = \begin{bmatrix} 0.45 & 0 & 0.3969 \\ 0.30 & 0.15 & 0.4975 \\ 0.15 & 0 & 0.5809 \\ 0.30 & -0.15 & 0.4975 \end{bmatrix}$$

From eq.(11.60), we obtain $\mathbf{P}''$ and, with $\mathbf{P}$, we obtain the spline coefficients, $\mathbf{a}_k$, $\mathbf{b}_k$, $\mathbf{c}_k$ and $\mathbf{d}_k$. Thus,

$$\mathbf{P}'' = 6\mathbf{A}^{-1}\mathbf{C}\mathbf{P} = \begin{bmatrix} -8.0389 & 0 & 5.8816 \\ -0.2483 & -8.4077 & -0.8104 \\ 8.7839 & 0 & -4.4127 \\ -0.2483 & 8.4077 & -0.8104 \end{bmatrix}$$

With $\mathbf{P}''$ and $\mathbf{P}$, we obtain the spline parameters from eq.(11.53a). The Frenet-Serret vectors, $\mathbf{e}_b$, $\mathbf{e}_b$, $\mathbf{e}_t$ at each supporting point are

$$\mathbf{e}_t = \frac{\mathbf{r}'(\sigma_k)}{\|\mathbf{r}'(\sigma_k)\|}$$

$$\mathbf{e}_b = \frac{\mathbf{r}'(\sigma_k) \times \mathbf{r}''(\sigma_k)}{\mathbf{r}'(\sigma_k) \times \mathbf{r}''(\sigma_k)}$$

$$\mathbf{e}_n = \mathbf{e}_b \times \mathbf{e}_t$$

where

$$\begin{aligned}
\mathbf{r}(\sigma) &= \mathbf{a}_k(\sigma-\sigma_k)^3 + \mathbf{b}_k(\sigma-\sigma_k)^2 + \mathbf{c}_k(\sigma-\sigma_k) + d_k \\
\mathbf{r}'(\sigma) &= 3\mathbf{a}_k(\sigma-\sigma_k)^2 + 2\mathbf{b}_k(\sigma-\sigma_k) + \mathbf{c}_k \\
\mathbf{r}''(\sigma) &= 6\mathbf{a}_k(\sigma-\sigma_k) + 2\mathbf{b}_k \\
\mathbf{r}'''(\sigma) &= 6\mathbf{a}_k
\end{aligned}$$

The orientation matrix is given as

$$\mathbf{Q} = [\, \mathbf{e}_b \quad \mathbf{e}_t \quad \mathbf{e}_n \,]$$

In order to obtain the time-histories of the joint angles, we need $\mathbf{Q}$ as a function of time. The time required to complete this task is found from $T = l/v_o$, where l is the length of the curve. We have

$$s(\sigma) = \int_0^{\sigma_{\text{final}}} \|\mathbf{r}'(\sigma)\| d\sigma$$

and hence,

$$l = \sum_{i=1}^{N-1} \int_{\sigma_i}^{\sigma_{i+1}} \|\mathbf{p}'(\sigma)\| d\sigma$$

Note that $ds/d\sigma = \|\mathbf{r}'(\sigma)\|$, so that we can write

$$\dot{s} = \dot{\sigma}\|\mathbf{r}'(\sigma)\| = v_o$$

Finally,

$$\dot{\sigma} = \frac{v_o}{\|\mathbf{r}'(\sigma)\|}$$

Unlike Example 11.5.1, $\mathbf{r}'(\sigma)$ is derived from the spline approximation of Γ. Next, we integrate numerically the above equation to obtain $\sigma(t)$, while letting $\sigma(0) = 0$, which thus yields $\mathbf{Q}$ and $\mathbf{p}$ as functions of time.

Below we compute the pose, twist and twist-rate at each spline supporting point. First, the pose at each supporting point is nothing but

$$\begin{aligned}
\mathbf{Q}(\sigma_k) &= [\, \mathbf{e}_b(\sigma_k) \quad \mathbf{e}_n(\sigma_k) \quad \mathbf{e}_k(\sigma_k) \,] \\
\mathbf{p}(\sigma_k) &= \mathbf{d}_k
\end{aligned}$$

Second, the twist $\mathbf{t}_k = [\, \boldsymbol{\omega}_k^T \quad \mathbf{v}_k^T \,]^T$, where $\boldsymbol{\omega}_k$ is the angular-velocity vector and $\mathbf{v}_k = \dot{\mathbf{p}}_k$ at $t = t_k$. We have, from eq.(11.12),

$$\boldsymbol{\omega}_k = \dot{s}\boldsymbol{\delta}_k$$

where $\dot{s} = v_o$ and $\boldsymbol{\delta}_k$ is the Darboux vector at the kth supporting point, i.e.,

$$\begin{aligned}
\boldsymbol{\delta}_k &= \tau_k(\mathbf{e}_t)_k + \kappa_k(\mathbf{e}_b)_k \\
\kappa_k &= \frac{\|\mathbf{c}_k \times 2\mathbf{b}_k\|}{\|\mathbf{c}_k\|^3} \\
\tau_k &= \frac{\mathbf{c}_k \times 2\mathbf{b}_k \cdot 6\mathbf{a}_k}{\|\mathbf{c}_k \times 2\mathbf{b}_k\|}
\end{aligned}$$

Therefore,

$$\dot{\mathbf{p}}_k = v_o(\mathbf{e}_t)_k(\sigma_k)$$

Now, for the twist rate, $\dot{\mathbf{t}}_k = [\, \dot{\boldsymbol{\omega}}_k^T \quad \dot{\mathbf{v}}_k^T \,]^T$, we have

$$\dot{\mathbf{v}}_k = \ddot{\mathbf{p}}_k = \dot{\sigma}^2 2\mathbf{b}_k$$

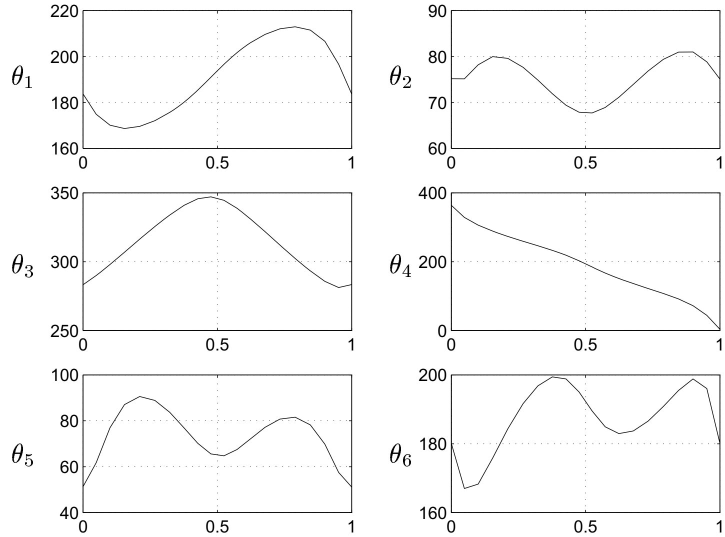

Figure 38: Joint trajectories vs. $\sigma_k/\sigma_{\max}$ for $N = 20$, in degrees.

and

$$\dot{\boldsymbol{\omega}}_k = \ddot{s}\boldsymbol{\delta}_k + \dot{s}\dot{\boldsymbol{\delta}}_k, \quad \ddot{s} = \dot{v}_o = 0$$

Hence,

$$\dot{\boldsymbol{\omega}}_k = v_o[\dot{\tau}_k(\mathbf{e}_t)_k(\sigma_k) + \dot{\kappa}_k(\mathbf{e}_b)_k(\sigma_k)]$$

with

$$\dot{\kappa}_k = \frac{v_o(\mathbf{c}_k \times 2\mathbf{b}_k)\cdot(\mathbf{c}_k \times 6\mathbf{a}_k)}{\kappa_k}$$

$$\dot{\tau}_k = \frac{v_o\left[-2\tau_k(\mathbf{c}_k \times 6\mathbf{a}_k)\cdot(\mathbf{c}_k \times 2\mathbf{b}_k)\right]}{\kappa_k^2}$$

Finally, $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, and $\ddot{\boldsymbol{\theta}}$ at the supporting points are determined using inverse kinematics. Once again, we apply Algorithm 11.5.2 with iterations performed between supporting points. The joint angle, velocity and acceleration trajectories are plotted in Figs. 38, 39 and 40, respectively, for $N = 20$.

Now, for comparison, we fit a periodic cubic spline using $\boldsymbol{\theta}$ at the supporting points. We refer to Section 11.4 to obtain the $\mathbf{A}$ and $\mathbf{C}$ matrices. Thus,

$$\ddot{\boldsymbol{\theta}} = 6\mathbf{A}^{-1}\mathbf{C}\boldsymbol{\theta}$$

The differences in joint accelerations for $N = 5, 15, 20$ are plotted in Figs. 41, 42 and 43, respectively.

12 Dynamics of Complex Robotic Mechanical Systems

12.1 The system mass and angular velocity matrices $\mathbf{M}$ and $\mathbf{W}$ are, in this case, of $6r \times 6r$. Moreover, $\mathbf{T}$ is of $6r \times n$. We can thus express the $6r$-dimensional twist vector $\mathbf{t}$ of the whole system and its rate of change as

$$\mathbf{t} = \mathbf{T}\dot{\boldsymbol{\theta}}_a, \qquad \dot{\mathbf{t}} = \mathbf{T}\ddot{\boldsymbol{\theta}}_a + \dot{\mathbf{T}}\dot{\boldsymbol{\theta}}_a \tag{157}$$