305-572A Mechanics of Robotic Systems I **OPEN BOOK**

Examiner: *Prof. Jorge Angeles* (Local: 6315) J. Angeles Date: December 11, 1998

Associate Examiner: *Ms. Svetlana Ostrovskaya* Ostrovskaya
(Local: 8203) Time: 2:00–5:00

Note: Although the Faculty of Engineering standard calculator is allowed, all problems can be handled without any calculator at all. You are encouraged to answer all questions by means of longhand calculations.

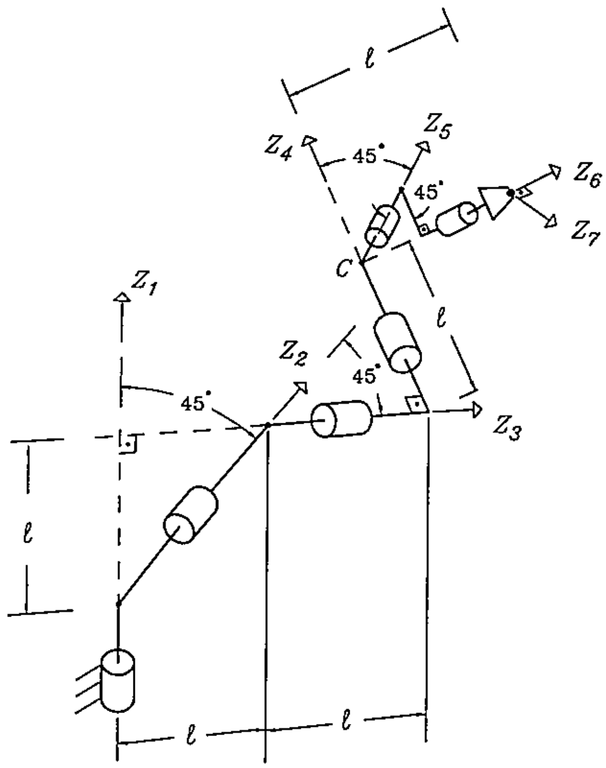

Figure 1: Arbitrary posture of an experimental six-revolute manipulator

1. An experimental six-revolute robot is shown in an arbitrary posture in Fig. 1.

(a) (10%) Produce a table with the Denavit-Hartenberg parameters of the manipulator, a_i, b_i, α_i, $\lambda_i \equiv \cos\alpha_i$, and $\mu_i \equiv \sin\alpha_i$, for $i = 1, \ldots, 6$.

(b) (15%) Find **all** arm inverse kinematic solutions for the positioning of point C, the centre of the spherical wrist.

(c) (5%) Find **all** wrist inverse kinematic solutions for the orientation of the EE. In how many postures can this manipulator produce the same EE pose?

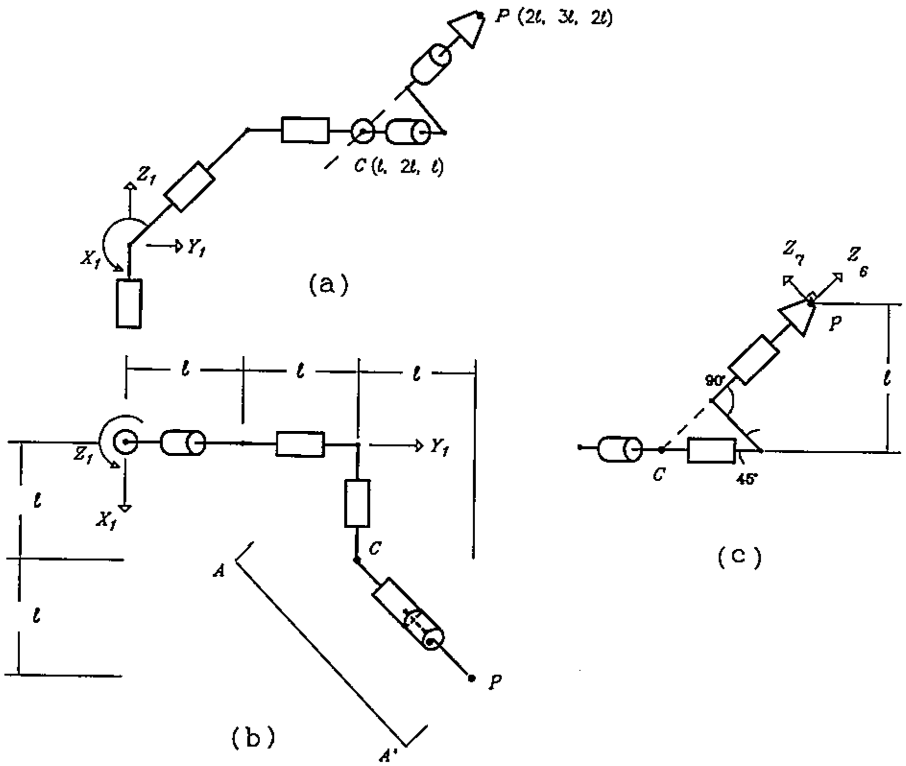

Figure 2: Experimental six-revolute manipulator in a given posture: (a) front view; (b) plant view; (c) AA' view

2. The manipulator of Problem 2 is now shown in Fig. 2 in a posture whereby its end-effector (EE) has the pose given by the position vector $[\mathbf{p}]_1 = [2\ell,\ 3\ell,\ 2\ell]^T$ and matrix $\mathbf{Q}$, the latter representing the orientation of its EE with respect to the frame $\mathcal{F}_1$ (X_1, Y_1, Z_1). In that posture, note that the axis of the fifth revolute is horizontal, while the axis of the sixth revolute makes an angle of 45° with this axis and lies in a vertical plane. Moreover, Fig. 2a is a front view of the manipulator; Fig. 2b is a plant view of the same; and Fig. 2c is the AA' view of the wrist.

(a) (5%) Find the orientation $\mathbf{Q}$ of the EE frame $\mathcal{F}_6$ with respect to $\mathcal{F}_1$, where all frames are defined according to the Denavit-Hartenberg notation.

(b) (5%) Find the direction of the axis of the rotation carrying frame $\mathcal{F}_1$ into $\mathcal{F}_6$, **without using a calculator.**

(c) (5%) Find the angle of the same rotation, **without using a calculator.** In which quadrant does the angle lie?

3. While the manipulator is in the posture of Fig. 2, performing a routing operation, it is known that a certain force $\mathbf{f}$, but no moment, is exerted on the EE at the operation point P.

(a) (15%) Sensors placed at the arm joints measure the torques exerted by the arm motors to balance the force $\mathbf{f}$. If the readouts of these sensors are all 5ℓ Nm, find the force $\mathbf{f}$.

(b) (5%) Find the corresponding readouts of the wrist torque sensors.

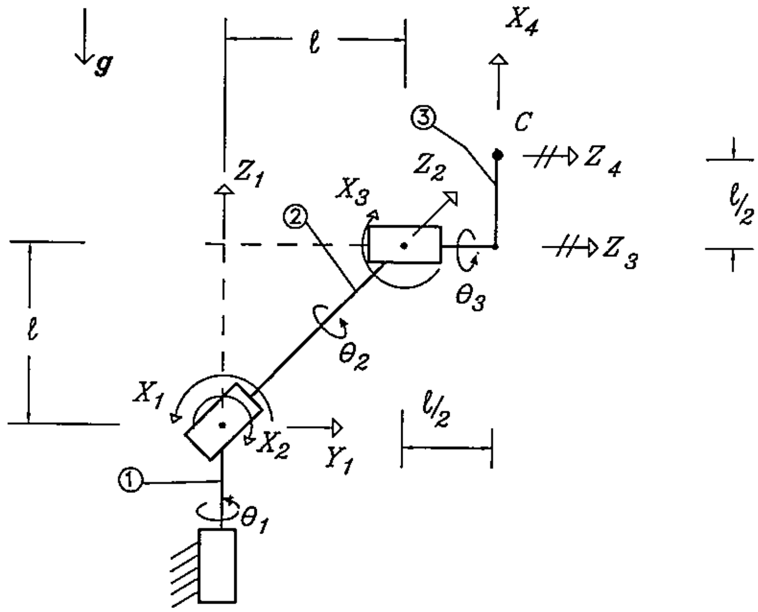

Figure 3: A three-revolute arm

4. Here we analyze the three-revolute manipulator of Fig. 3, shown in a posture with all revolute axes contained in the plane of the figure, which is the Y_1-Z_1 plane.

(a) (5%) When the motors produce unit joint rates (1 rad/s), find the twist of the third link in frame $\mathcal{F}_1$;

(b) (10%) when the motors produce, additionally, unit joint accelerations (1 rad/s^2), find the twist-rate of the third link in frame $\mathcal{F}_1$.

(c) (10%) The moment-of-inertia matrix of the third link, in the fourth frame, X_4, Y_4, Z_4, is given as

$$[\mathbf{I}_3]_4 = \frac{1}{4}m\ell^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $\mathbf{I}_3$ in $\mathcal{F}_1$;

(d) (10%) find the torque τ_3 that the third motor should exert in order to produce the motion of items (a) and (b), if point C is the mass centre of the third link. *Hint: Note that τ_1 and τ_2 are not required, and hence, the twists of links 1 and 2, and neither their twist-rates, are needed. Do not waste your time calculating them.*

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INSTRUCTIONS

Fill in the above carefully.

Write your answers on the RULED SIDE ONLY — Use the unruled side for rough work or calculations.

Do not write in the margin — If a page is accidentally left blank write "P.T.O." on it.

Do not tear pages from this book; all your writing must be handed in.

Put additional books inside first book when handing in.

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C

1) a) (10 %)

joint	a_i	b_i	α_i	λ_i	μ_i
1	0	0	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
2	0	$\sqrt{2}\,l$	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
3	0	l	$90°$	0	1
4	0	l	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
5	0	0	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
6	0	l	$90°$	0	1

b) (15 %). We set up the two eqns in θ_1 and θ_3, (4.19 a) & (4.20 a) of text:

$$A = 0, \quad B = 0, \quad C = -2\sqrt{2}\, l^2 \frac{\sqrt{2}}{2} = -2l^2, \quad D = 0$$

$$E = l^2 + 2l^2 + l^2 - \underbrace{(x_c^2 + y_c^2 + z_c^2)}_{\equiv\, d^2} + 2\sqrt{2}\, l^2 \frac{\sqrt{2}}{2}$$

$$= 6l^2 - d^2$$

$$F = \frac{\sqrt{2}}{2} y_c, \quad G = -\frac{\sqrt{2}}{2} x_c, \quad H = -l\frac{\sqrt{2}}{2}, \quad I = 0$$

$$J = \sqrt{2}\, l + l\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} z_c = \frac{3}{2}\sqrt{2}\, l - \frac{\sqrt{2}}{2} z_c$$

$$(4.19\,a) \Rightarrow -2l^2 c_3 + 6l^2 - d^2 = 0 \qquad (1)$$

$$(4.20\,a) \Rightarrow \frac{\sqrt{2}}{2} y_c c_1 - \frac{\sqrt{2}}{2} x_c s_1 - \frac{\sqrt{2}}{2} l c_3 + \frac{3}{2}\sqrt{2}\, l - \frac{\sqrt{2}}{2} z_c = 0 \qquad (2)$$

$$(1) \Rightarrow l c_3 = -\frac{d^2}{2l} + 3l \qquad (3)$$

$$(3) \text{ into } \sqrt{2}(2) \Rightarrow y_c c_1 - x_c s_1 + \frac{d^2}{2l} \cancel{-3l} + \cancel{3l} - z_c = 0 \qquad (4)$$

$$c_1 \equiv \frac{1-\tau_1^2}{1+\tau_1^2}, \quad s_1 \equiv \frac{2\tau_1}{1+\tau_1^2}, \quad \tau_1 \equiv \tan\left(\frac{\theta_1}{2}\right) \text{ into } (4):$$

$$y_c(1-\tau_1^2) - 2x_c\tau_1 + \left(\frac{d^2}{2l} - z_c\right)(1+\tau_1^2) = 0$$

or

$$\left(\frac{d^2}{2\ell} - z_c - y_c\right)\tau_1^2 - 2x_c\tau_1 + \frac{d^2}{2\ell} - z_c + y_c = 0$$

$$\Delta \equiv x_c^2 - \left(\frac{d^2}{2\ell} - z_c - y_c\right)\left(\frac{d^2}{2\ell} - z_c + y_c\right)$$

$$= x_c^2 - \left(\frac{d^2}{2\ell} - z_c\right)^2 + y_c^2$$

is discriminant of quadratic eqn., i.e.,

$$\Delta = x_c^2 + y_c^2 - z_c^2 + \frac{d^2}{\ell} z_c - \frac{d^4}{4\ell^2}$$

$$\tau_1 = \frac{x_c \pm \sqrt{\Delta}}{(d^2/2\ell) - y_c - z_c} \Rightarrow \text{two values of } \theta_1$$

(3) $\Rightarrow$ two values of θ_3

Now we derive two eqns in c_2 & s_2, eqs. (4.28a & b) of text:

$$A_{11} = \ell s_3, \quad A_{12} = \frac{\sqrt{2}}{2}\ell c_3 + \frac{\sqrt{2}}{2}\ell$$

$$(4.28a) \Rightarrow \ell\left[s_3 c_2 + \frac{\sqrt{2}}{2}(1 + c_3)s_2\right] = x_c c_1 + y_c s_1$$

$$(4.28b) \Rightarrow \ell\left[-\frac{\sqrt{2}}{2}(1+c_3)c_2 + s_3 s_2\right] = \frac{\sqrt{2}}{2}(-x_c s_1 + y_c c_1 + z_c)$$

or

$$\underbrace{\begin{bmatrix} s_3 & \frac{\sqrt{2}}{2}(1+c_3) \\ -\frac{\sqrt{2}}{2}(1+c_3) & s_3 \end{bmatrix}}_{\mathbf{A}}\begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = \frac{1}{\ell}\begin{bmatrix} x_c c_1 + y_c s_1 \\ \frac{\sqrt{2}}{2}(-x_c s_1 + y_c c_1 + z_c) \end{bmatrix}$$

$$\mathbf{A}; \quad \det(\mathbf{A}) = s_3^2 + \frac{1}{2}(1+c_3)^2 = s_3^2 + \frac{1}{2}c_3^2 + c_3 + \frac{1}{2}$$

$$\begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = \frac{2}{\ell(2s_3^2 + c_3^2 + 2c_3 + 1)}\begin{bmatrix} s_3 & -\frac{\sqrt{2}}{2}(1+c_3) \\ \frac{\sqrt{2}}{2}(1+c_3) & s_3 \end{bmatrix}\begin{bmatrix} x_c c_1 + y_c s_1 \\ \frac{\sqrt{2}}{2}(-x_c s_1 + y_c c_1 + z_c) \end{bmatrix}$$

i.e.,

$$c_2 = \frac{2}{\ell} \frac{s_3(x_c c_1 + y_c s_1) - \frac{1}{2}(1+c_3)(-x_c s_1 + y_c c_1 + z_c)}{2s_3^2 + c_3^2 + 2c_3 + 1}$$

$$s_2 = \frac{\sqrt{2}}{\ell} \frac{(1+c_3)(x_c c_1 + y_c s_1) + s_3(-x_c s_1 + y_c c_1 + z_c)}{2s_3^2 + c_3^2 + 2c_3 + 1}$$

$\Rightarrow$ one unique value of θ_2 for every pair of (θ_1, θ_3) values

$\Rightarrow$ Arm can be postured in four different modes for a given position of point C.

c) (5 pts) θ_4 is computed from eq. (4.36) of text:

$$\xi \frac{\sqrt{2}}{2} \sin\theta_4 - \eta \frac{\sqrt{2}}{2} \cos\theta_4 + \zeta \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

or $\xi \sin\theta_4 - \eta \cos\theta_4 + \zeta - 1 = 0$

$$\sin\theta_4 \equiv \frac{2\tau_4}{1+\tau_4^2}, \quad \cos\theta_4 \equiv \frac{1-\tau_4^2}{1+\tau_4^2}, \quad \tau_4 \equiv \tan\left(\frac{\theta_4}{2}\right)$$

$$\Rightarrow 2\xi\tau_4 - \eta(1-\tau_4^2) + (\zeta-1)(1+\tau_4^2) = 0$$

$$(\eta + \zeta - 1)\tau_4^2 + 2\xi\tau_4 - \eta + \zeta - 1 = 0$$

$$\Delta = \xi^2 - (\eta+\zeta-1)(-\eta+\zeta-1) = \xi^2 - (\zeta-1)^2 + \eta^2$$

$$= \underbrace{\xi^2 + \eta^2}_{1-\zeta^2} - (\zeta^2 - 2\zeta + 1) = -2\zeta^2 + 2\zeta = 2\zeta(1-\zeta)$$

$$\Rightarrow \tau_4 = \frac{-\xi \pm \sqrt{\Delta}}{\eta + \zeta - 1} \Rightarrow 2 \text{ values of } \theta_4$$

θ_5 is obtained from the two eqns in c_5 & s_5 derived from eq. (4.43):

$$\frac{\sqrt{2}}{2} s_5 = r_{12} c_4 + r_{22} s_4 \Rightarrow s_5 = \sqrt{2}(r_{12} c_4 + r_{22} s_4)$$

$$-\frac{\sqrt{2}}{2} c_5 = -\frac{\sqrt{2}}{2} r_{12} s_4 + \frac{\sqrt{2}}{2} r_{22} c_4 + \frac{\sqrt{2}}{2} r_{32}$$

$$\Rightarrow c_5 = r_{12} s_4 - r_{22} c_4 - r_{32}$$

$\Rightarrow$ one single value of θ_5 for each value of θ_4

Finally, θ_6 is obtained from the two eqns in c_6 & s_6 derived from eq. (4.45):

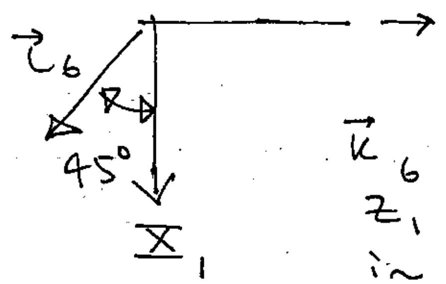
$$c_6 = w_1 c_5 + w_2 s_5$$
$$s_6 = \frac{\sqrt{2}}{2}(-w_1 s_5 + w_2 c_5 + w_3)$$

$\Rightarrow$ one single value of θ_6 for each pair of (θ_4, θ_5) values

$\Rightarrow$ wrist admits two postures for each EE attitude

$\Rightarrow$ Manipulator admits $4 \times 2 = 8$ postures for a given EE pose.

2) a) (5%) Let $\vec{i}_\nu, \vec{j}_\nu, \vec{k}_\nu$ be unit vectors associated with axes X_ν, Y_ν, Z_ν, respectively, for $\nu = 1, 2, \ldots, 6$. Then, apparently, $\vec{i}_6$ lies in the X_1-Y_1 plane:



$$\Rightarrow [\vec{i}_6]_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\vec{k}_6$ makes an angle of $45°$ with Z_1 and has identical components in X_1 & Y_1:

$$[\vec{k}_6]_1 = \begin{bmatrix} \alpha \\ \alpha \\ \sqrt{2}/2 \end{bmatrix} \Rightarrow 2\alpha^2 + \frac{1}{2} = 1 \Rightarrow \alpha^2 = \frac{1}{4}$$

$$\Rightarrow \alpha = \frac{1}{2} \Rightarrow [\vec{k}_6]_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

$$\Rightarrow \vec{j}_6 = \vec{k}_6 \times \vec{i}_6 = \frac{\sqrt{2}}{4}\begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 1 & 1 & \sqrt{2} \\ 1 & -1 & 0 \end{vmatrix}$$

$$[\vec{j}_6]_1 = \frac{\sqrt{2}}{4}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

Using Definition 2.2.1 of text:

$$[\underset{\sim}{Q}]_1 = [\vec{i}_6 \;\; \vec{j}_6 \;\; \vec{k}_6]_1 = \frac{1}{2}\begin{bmatrix} \sqrt{2} & 1 & 1 \\ -\sqrt{2} & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

b) (5%) $\text{vect}(\underset{\sim}{Q}) = \vec{e}\sin\phi$

$$[\text{vect}(\underset{\sim}{Q})]_1 = \frac{1}{4}\begin{bmatrix} -\sqrt{2}-1 \\ 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

$$\Rightarrow \|\text{vect}(\underset{\sim}{Q})\|^2 = \frac{1}{16}\Big[(1+\sqrt{2})^2 + 1 + \overbrace{(1+\sqrt{2})^2}^{3+2\sqrt{2}}\Big] = \frac{7+4\sqrt{2}}{16}$$

$$\Rightarrow \|\mathrm{vect}(\underset{\sim}{Q})\| = \frac{\sqrt{7+4\sqrt{2}}}{4}$$

$$[\vec{e}]_1 = \frac{\sqrt{7+4\sqrt{2}}}{7+4\sqrt{2}} \begin{bmatrix} -1-\sqrt{2} \\ 1 \\ -1-\sqrt{2} \end{bmatrix}$$

(c) (5%) $\cos\phi = \dfrac{\mathrm{tr}(\underset{\sim}{Q})-1}{2} = \dfrac{\frac{1+2\sqrt{2}}{2} - 1}{2}$

$$\cos\phi = \frac{-1+2\sqrt{2}}{4}$$

Since $\cos\phi = \|\mathrm{vect}(\underset{\sim}{Q})\| = \dfrac{\sqrt{7+4\sqrt{2}}}{4}$,

$$\phi = \tan^{-1}\left(\frac{\sqrt{7+4\sqrt{2}}}{-1+2\sqrt{2}}\right)$$

Since both numerator & denominator of $\tan^{-1}(\cdot)$ are positive, ϕ lies in first quadrant.

3. a) (15%) We apply the relation $\underset{\sim}{J}_c^T \vec{w}_c = \vec{\tau}$ (1)

where

$$\underset{\sim}{J}_c = \begin{array}{c} \left[\begin{array}{cc} \underset{\sim}{J}_{11} & \underset{\sim}{J}_{12} \\ \underset{\sim}{J}_{21} & \underset{\sim}{0} \end{array}\right] \begin{array}{l} 3 \\ 3 \end{array} \\ \begin{array}{cc} 3 & 3 \end{array} \end{array}, \quad \vec{w}_c = -\begin{bmatrix} \vec{n} \\ \vec{f} \end{bmatrix}, \quad \vec{\tau} = \begin{bmatrix} \vec{\tau}_a \\ \vec{\tau}_w \end{bmatrix}$$

wrench exerted by the motors!

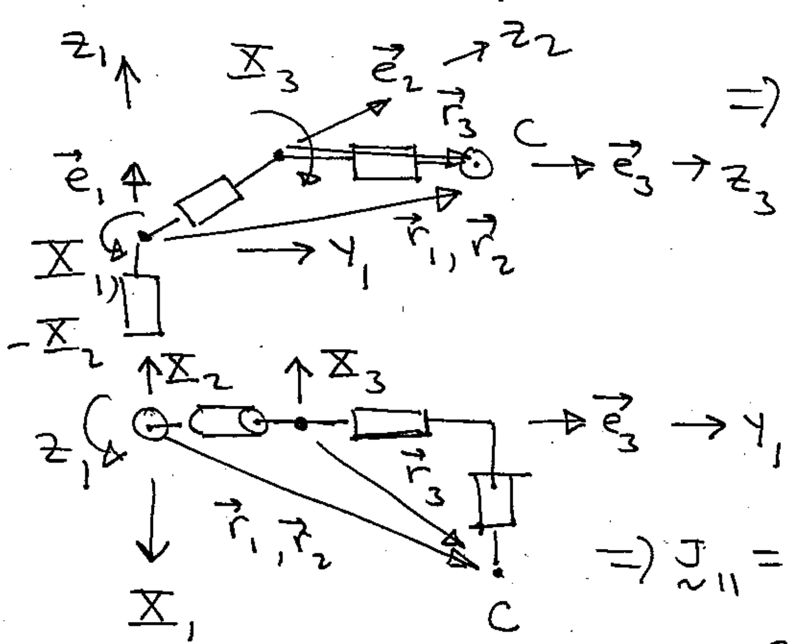
$$\underset{\sim}{J}_{11} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3], \quad \underset{\sim}{J}_{12} = [\vec{e}_4 \ \vec{e}_5 \ \vec{e}_6] \qquad (2a)$$

$$\underset{\sim}{J}_{21} = [\vec{e}_1 \times \vec{r}_1 \quad \vec{e}_2 \times \vec{r}_2 \quad \vec{e}_3 \times \vec{r}_3] \qquad (2b)$$

and $\vec{n}$ is the moment transferred by the force from P to C, i.e.,

$$\vec{n} = (\vec{p} - \vec{c}) \times \vec{f} = (\underset{\sim}{P} - \underset{\sim}{C})\vec{f} \qquad (3a)$$

$$\underset{\sim}{P} \equiv \text{CPM of } \vec{p}; \quad \underset{\sim}{C} \equiv \text{CPM of } \vec{c} \qquad (3b)$$



$$\Rightarrow [\vec{e}_1]_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[\vec{e}_2]_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$[\vec{e}_3]_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{J}_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 1 \\ 1 & \sqrt{2}/2 & 0 \end{bmatrix} \qquad (4)$$

$$[\vec{r}_1]_1 = [\vec{r}_2]_1 = l\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad [\vec{r}_3]_1 = l\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_1 \times \vec{r}_1 = l\begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} \Rightarrow [\vec{e}_1 \times \vec{r}_1]_1 = l\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$[\vec{e}_2 \times \vec{r}_2]_1 = \frac{\sqrt{2}\,\ell}{2}\begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \Rightarrow [\vec{e}_2 \times \vec{r}_2]_1 = \frac{\sqrt{2}\,\ell}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$[\vec{e}_3 \times \vec{r}_3]_1 = \ell\begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} \Rightarrow [\vec{e}_3 \times \vec{r}_3]_1 = \ell\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{J}_{21} = \ell\begin{bmatrix} -2 & -\sqrt{2}/2 & 0 \\ 1 & \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & -1 \end{bmatrix} \qquad (5)$$

By inspection,

$$[\vec{e}_4]_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\vec{e}_5]_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

while $[\vec{e}_6]_1$ was found in Problem 2 as $\vec{k}_6$:

$$[\vec{e}_6]_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} \Rightarrow \underset{\sim}{J}_{12} = \frac{1}{2}\begin{bmatrix} 2 & \sqrt{2} & 1 \\ 0 & \sqrt{2} & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \qquad (6)$$

By inspection,

$$[\vec{p} - \vec{c}]_1 = \ell\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{[\underset{\sim}{P} - \underset{\sim}{C}]_1 = \ell\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}}_{\text{from eq. (2.36c) of text}} \qquad (7)$$

(2a) – (7) into (1):

$$-\begin{bmatrix} \underset{\sim}{J}_{11}^T & \underset{\sim}{J}_{21}^T \\ \underset{\sim}{J}_{12}^T & \underset{\sim}{0} \end{bmatrix}\begin{bmatrix} \underset{\sim}{P} - \underset{\sim}{C} \\ \underset{\sim}{1} \end{bmatrix}\vec{f} = \begin{bmatrix} \vec{\tau}_a \\ \vec{\tau}_w \end{bmatrix}$$

or $$\left[\underset{\sim}{J}_{11}^T(\underset{\sim}{P} - \underset{\sim}{C}) + \underset{\sim}{J}_{21}^T\right]\vec{f} = -\vec{\tau}_a \qquad (8)$$

& $$\underset{\sim}{J}_{12}^T(\underset{\sim}{P} - \underset{\sim}{C})\vec{f} = \vec{\tau}_w \qquad (9)$$

$$\underset{\sim 11}{J}^T (\underset{\sim}{p} - \underset{\sim}{c}) + \underset{\sim 21}{J}^T = l \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$+ l \begin{bmatrix} -2 & 1 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= l \begin{bmatrix} -1 & 1 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1 & 0 & -1 \end{bmatrix} + l \begin{bmatrix} -2 & 1 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & -1 \end{bmatrix} = l \begin{bmatrix} -3 & 2 & 0 \\ -\sqrt{2}/2 & \sqrt{2} & -\sqrt{2} \\ 1 & 0 & -2 \end{bmatrix}$$

$$\Rightarrow \cancel{l'} \begin{bmatrix} -3 & 2 & 0 \\ -\sqrt{2}/2 & \sqrt{2} & -\sqrt{2} \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = -5 \cancel{l'} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix}$$

$(a) + 3(c) \Rightarrow \quad 2f_y - 6f_z = -20 \Rightarrow f_y - 3f_z = -10 \qquad (d)$

$(b) + \frac{\sqrt{2}}{2}(c) \Rightarrow \quad \sqrt{2} f_y - 2\sqrt{2} f_z = -5\left(1 + \frac{\sqrt{2}}{2}\right)$

or $\qquad f_y - 2f_z = -5\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) \qquad (e)$

$(d) \Rightarrow f_y = -10 + 3f_z;\ (e) \Rightarrow f_y = 2f_z - \frac{5}{2}(1+\sqrt{2}) \qquad (f)$

$\Rightarrow -10 + 3f_z = 2f_z - \frac{5}{2}(1+\sqrt{2}) \Rightarrow f_z = -\frac{5}{2}(-3+\sqrt{2}) \qquad (g)$

(g) into $(f) \Rightarrow f_y = -5(-3+\sqrt{2}) - \frac{5}{2}(1+\sqrt{2}) = -\frac{5}{2}(-5+3\sqrt{2}) \qquad (h)$

$(g)\&(h)$ into $(c) \Rightarrow f_x + 5(-3+\sqrt{2}) = -5 \Rightarrow f_x = -5(-2+\sqrt{2}) \qquad (i)$

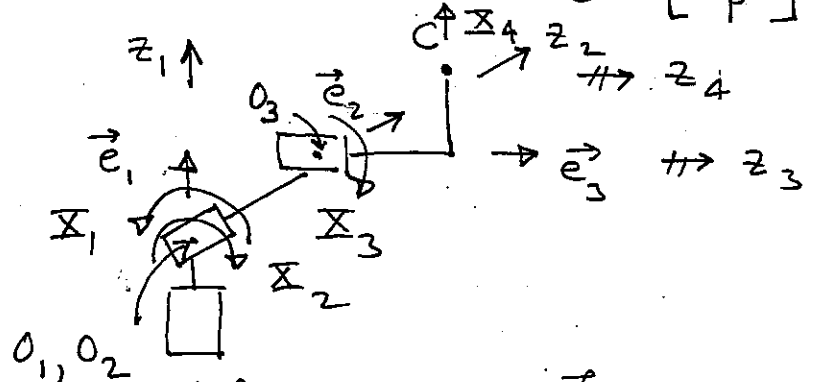
$\Rightarrow \vec{f} = \frac{5}{2}\left[4 - 2\sqrt{2},\ 5 - 3\sqrt{2},\ 3 - \sqrt{2}\right]^T$ Ans. $\qquad (j)$

b) (5%) (6), (7) & (j) into (9):

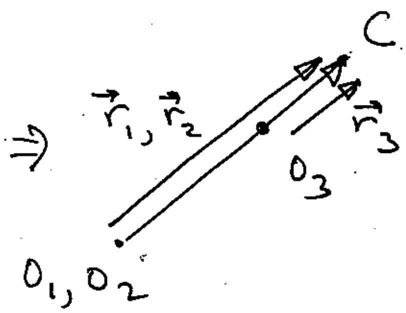
$$\frac{1}{2} l \begin{bmatrix} 2 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \frac{5}{2} \begin{bmatrix} 4 - \sqrt{2} \\ 5 - 3\sqrt{2} \\ 3 - \sqrt{2} \end{bmatrix}}_{\frac{5}{2}\begin{bmatrix} -2 + 2\sqrt{2} \\ 1 \\ 1 - 2\sqrt{2} \end{bmatrix}} = \begin{bmatrix} -\tau_4 \\ -\tau_5 \\ -\tau_6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \frac{5l}{4} \begin{bmatrix} 4(1 - \sqrt{2}) \\ -4 + \sqrt{2} \\ 5 - 3\sqrt{2} \end{bmatrix}$$

4. a) (5%) $\underset{\sim}{J}\,\dot{\vec{\theta}} = \vec{t}_3$, $\vec{t}_3 = \begin{bmatrix} \vec{\omega}_3 \\ \dot{\vec{p}} \end{bmatrix}$



$\Rightarrow$



All vectors in $\mathcal{F}_1$:

$$\vec{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{e}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{r}_1 = \vec{r}_2 = \frac{3}{2}l \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{r}_3 = \frac{l}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{e}_1 \times \vec{r}_1 = \frac{3l}{2}\begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \Rightarrow \vec{e}_1 \times \vec{r}_1 = \frac{3l}{2}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}_2 \times \vec{r}_2 = \frac{3l}{2}\frac{\sqrt{2}}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \Rightarrow \vec{e}_2 \times \vec{r}_2 = \vec{0}$$

$$\vec{e}_3 \times \vec{r}_3 = \frac{l}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \Rightarrow \vec{e}_3 \times \vec{r}_3 = \frac{l}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{J} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \vec{e}_1 \times \vec{r}_1 & \vec{e}_2 \times \vec{r}_2 & \vec{e}_3 \times \vec{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 1 \\ 1 & \sqrt{2}/2 & 0 \\ -3l/2 & 0 & l/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{\vec{\theta}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{t}_3 = \begin{bmatrix} 0 \\ 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \\ -l \\ 0 \end{bmatrix} \Rightarrow \vec{\omega}_3 = \begin{bmatrix} 0 \\ 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \end{bmatrix}, \quad \dot{\vec{p}} = \begin{bmatrix} -l \\ 0 \\ 0 \end{bmatrix} \quad \underline{\underline{\text{Ans.}}}$$

b) (10%) $\vec{t}_3 = \underset{\sim}{J}\,\ddot{\vec{\theta}} + \dot{\underset{\sim}{J}}\,\dot{\vec{\theta}}$

$$\dot{\underset{\sim}{J}} = \begin{bmatrix} \vec{0} & \dot{\vec{e}}_2 & \dot{\vec{e}}_3 \\ \vec{e}_1 \times \dot{\vec{r}}_1 & \dot{\vec{e}}_2 \times \vec{r}_2 + \vec{e}_2 \times \dot{\vec{r}}_2 & \dot{\vec{e}}_3 \times \vec{r}_3 + \vec{e}_3 \times \dot{\vec{r}}_3 \end{bmatrix}$$

$$\dot{\vec{e}}_2 = \vec{\omega}_1 \times \vec{e}_2 = \underbrace{\dot{\theta}_1}_{1} \vec{e}_1 \times \vec{e}_2 = \frac{\sqrt{2}}{2} \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \Rightarrow \dot{\vec{e}}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\vec{e}}_3 = \vec{\omega}_2 \times \vec{e}_3 = (\underbrace{\dot{\theta}_1 \vec{e}_1}_{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} + \underbrace{\dot{\theta}_2 \vec{e}_2}_{\begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}}) \times \underbrace{\vec{e}_3}_{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & \sqrt{2}/2 & 1+\sqrt{2}/2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\dot{\vec{e}}_3 = \begin{bmatrix} -1-\sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{r}_1 = \vec{a}_1 + \vec{a}_2 + \vec{a}_3$$

$$\vec{a}_1 = 0 \Rightarrow \dot{\vec{a}}_1 = \vec{0}$$

$$\vec{a}_2 = l \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \dot{\vec{a}}_2 = \vec{\omega}_2 \times \vec{a}_2 = l \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & \sqrt{2}/2 & 1+\sqrt{2}/2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \dot{\vec{a}}_2 = l \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{a}_3 = l \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \dot{\vec{a}}_3 = \vec{\omega}_3 \times \vec{a}_3 = (\overbrace{\dot{\theta}_1 \vec{e}_1 + \dot{\theta}_2 \vec{e}_2}^{\begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 1+\sqrt{2}/2 \end{bmatrix}} + \overbrace{\dot{\theta}_3 \vec{e}_3}^{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}) \times \overbrace{\vec{a}_3}^{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} l}$$

$$\dot{\vec{a}}_3 = l \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 1+\sqrt{2}/2 & 1+\sqrt{2}/2 \\ 0 & 1/2 & 1/2 \end{vmatrix} \Rightarrow \dot{\vec{a}}_3 = \vec{0}$$

$$\Rightarrow \dot{\vec{r}}_1 = \begin{bmatrix} -l \\ 0 \\ 0 \end{bmatrix}; \quad \dot{\vec{r}}_2 = \dot{\vec{a}}_2 + \dot{\vec{a}}_3 = \begin{bmatrix} -l \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\vec{r}}_3 = \dot{\vec{a}}_3 = \vec{0}$$

$$\vec{e}_1 \times \dot{\vec{r}}_1 = l \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix} \Rightarrow \vec{e}_1 \times \dot{\vec{r}}_1 = l \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\dot{\vec{e}}_2 \times \vec{r}_2 + \vec{e}_2 \times \dot{\vec{r}}_2 = \underbrace{\frac{3}{2}l \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ -\sqrt{2}/2 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}}_{\begin{bmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}} \quad \underbrace{\frac{\sqrt{2}}{2}l \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix}}_{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

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$$\Rightarrow \dot{\vec{e}}_2 \times \vec{r}_2 + \vec{e}_2 \times \dot{\vec{r}}_2 = \frac{\sqrt{2}}{4} l \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\dot{\vec{e}}_3 \times \vec{r}_3 + \vec{e}_3 \times \cancel{\dot{\vec{r}}_3}^{\,0} = \frac{l}{2} \begin{vmatrix} \vec{i}_1 & \vec{j}_1 & \vec{k}_1 \\ -1-\sqrt{2}/2 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} \Rightarrow \dot{\vec{e}}_3 \times \vec{r}_3 + \vec{e}_3 \times \dot{\vec{r}}_3 = \frac{l}{2} \begin{bmatrix} 0 \\ 1+\sqrt{2}/2 \\ -(1+\sqrt{2}/2) \end{bmatrix}$$

$$\Rightarrow \dot{\underset{\sim}{J}} = \begin{bmatrix} 0 & -\sqrt{2}/2 & -1-\sqrt{2}/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -l & \sqrt{2}l/4 & (-1/2+\sqrt{2}/4)l \\ 0 & -\sqrt{2}l/4 & -(1/2+\sqrt{2}/4)l \end{bmatrix}, \quad \dot{\vec{\theta}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \ddot{\vec{\theta}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \dot{\vec{t}}_3 = \begin{bmatrix} 0 \\ 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \\ -l \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1-\sqrt{2} \\ 0 \\ 0 \\ 0 \\ (-1+\sqrt{2})l/2 \\ -(1+\sqrt{2})l/2 \end{bmatrix} = \begin{bmatrix} -1-\sqrt{2} \\ 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \\ -l \\ (-1+\sqrt{2})l/2 \\ -(1+\sqrt{2})l/2 \end{bmatrix}$$

i.e., $\dot{\vec{\omega}}_3 = \begin{bmatrix} -1-\sqrt{2} \\ 1+\sqrt{2}/2 \\ 1+\sqrt{2}/2 \end{bmatrix}, \quad \ddot{\vec{p}} = \begin{bmatrix} -1 \\ (-1+\sqrt{2})/2 \\ -(1+\sqrt{2})/2 \end{bmatrix} l$ <u>Ans.</u>

c) (10%) Let $\underset{\sim}{Q} : \mathcal{F}_1 \to \mathcal{F}_4$

$\vec{i}_4 = \vec{k}_1, \ \vec{j}_4 = \vec{i}_1, \ \vec{k}_4 = \vec{j}_1$

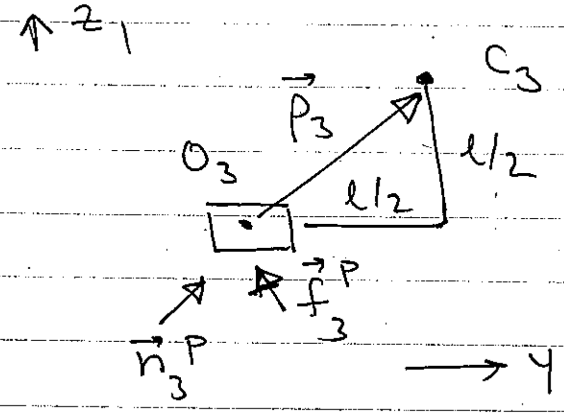
$$\Rightarrow [\underset{\sim}{Q}]_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow [\underset{\sim}{I}]_1 = [\underset{\sim}{Q}]_1 [\underset{\sim}{I}_3]_4 [\underset{\sim}{Q}^T]_1$$

$$\text{i.e., } [\underset{\sim}{I}_3]_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \frac{ml^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[\underset{\sim}{I}_3]_1 = \frac{m\ell^2}{4}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\2&0&0\\0&1&0\end{bmatrix} = \frac{m\ell^2}{4}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

13 **Ans.**

d) (10%) FBD of link 3:



$$\vec{f}_3^{\,P} = m_3\ddot{\vec{c}} = m_3\ddot{\vec{p}}$$

$$= m\ell\begin{bmatrix}-1\\(-1+\sqrt{2})/2\\-(1+\sqrt{2})/2\end{bmatrix}$$

$$\vec{n}_3^{\,P} = \underset{\sim}{I}_3\dot{\vec{\omega}}_3 + \vec{\omega}_3\times\underset{\sim}{I}_3\vec{\omega}_3 + \vec{p}_3\times\vec{f}_3^{\,P}$$

$$[\underset{\sim}{I}_3\dot{\vec{\omega}}_3]_1 = \frac{m\ell^2}{4}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1-\sqrt{2}\\1+\sqrt{2}/2\\1+\sqrt{2}/2\end{bmatrix} = \frac{m\ell^2}{4}\begin{bmatrix}-2(1+\sqrt{2})\\1+\sqrt{2}/2\\1+\sqrt{2}/2\end{bmatrix}$$

$$[\underset{\sim}{I}_3\vec{\omega}_3]_1 = \frac{m\ell^2}{4}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\1+\sqrt{2}/2\\1+\sqrt{2}/2\end{bmatrix} = \frac{m\ell^2}{4}\begin{bmatrix}0\\1+\sqrt{2}/2\\1+\sqrt{2}/2\end{bmatrix}$$

$$[\vec{\omega}_3\times\underset{\sim}{I}_3\vec{\omega}_3]_1 = \frac{m\ell^2}{4}\begin{vmatrix}\vec{i}_1&\vec{j}_1&\vec{k}_1\\0&1+\sqrt{2}/2&1+\sqrt{2}/2\\0&1+\sqrt{2}/2&1+\sqrt{2}/2\end{vmatrix} = \vec{0}$$

$$[\vec{p}_3\times\vec{f}_3^{\,P}]_1 = m\ell\frac{\ell}{2}\begin{vmatrix}\vec{i}_1&\vec{j}_1&\vec{k}_1\\0&1&1\\-1&(-1+\sqrt{2})/2&(1+\sqrt{2})/2\end{vmatrix} \Rightarrow [\vec{p}_3\times\vec{f}_3^{\,P}]_1 = \frac{m\ell^2}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$[\vec{n}_3^{\,P}]_1 = \frac{m\ell^2}{4}\begin{bmatrix}-2\sqrt{2}\\-1+\sqrt{2}/2\\3+\sqrt{2}/2\end{bmatrix} \Rightarrow \tau_3 = [\vec{n}_3^{\,P}\cdot\vec{e}_3]_1 = \frac{m\ell^2}{4}\left[-2\sqrt{2},\,-1+\frac{\sqrt{2}}{2},\,3+\frac{\sqrt{2}}{2}\right]\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\Rightarrow \tau_3 = -\frac{m\ell^2}{4}\left(1-\frac{\sqrt{2}}{2}\right)$$

Ans.