305-572A MECHANICS OF ROBOTIC SYSTEMS I
FINAL EXAM

DATE: DECEMBER 21, 2000 TIME: 9:00-12:00

OPEN BOOK

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NOTE:

- The total mark is 60. The weights are indicated in each problem.
- Although the Faculty of Engineering Standard Calculator is allowed, all problems can be solved without using a calculator

PROBLEM 1 (15/60)

A two-revolute orthogonal manipulator (Z_1 perpendicular to Z_2) is shown in **Figure 1** under global coordinate system F_0 (X_0-Y_0-Z_0), The axes for joint frames F_1 (X_1-Y_1-Z_1) and F_2 (X_2-Y_2-Z_2) are defined according to Denavit-Hartenberg notation. F_0 and F_1 have common origin. X_1 is coincident with X_0 and Z_1 makes 45° angle with Y_0. $|O_1O_2| = l$. Moreover, a cylinder is also defined in F_0. The axis the cylinder is parallel to X_0 and intersects with Y_0 at C. $|O_0C| = b$. The radius of its cross section is r.

(1) Compute the rotation matrix that transforms coordinates from F_2 to F_0

(2) A reference point A is defined for the collision avoidance purpose, its coordinates in F_2 being (0, 0, a). Derive the condition under which point A does not collide with surface of the cylinder

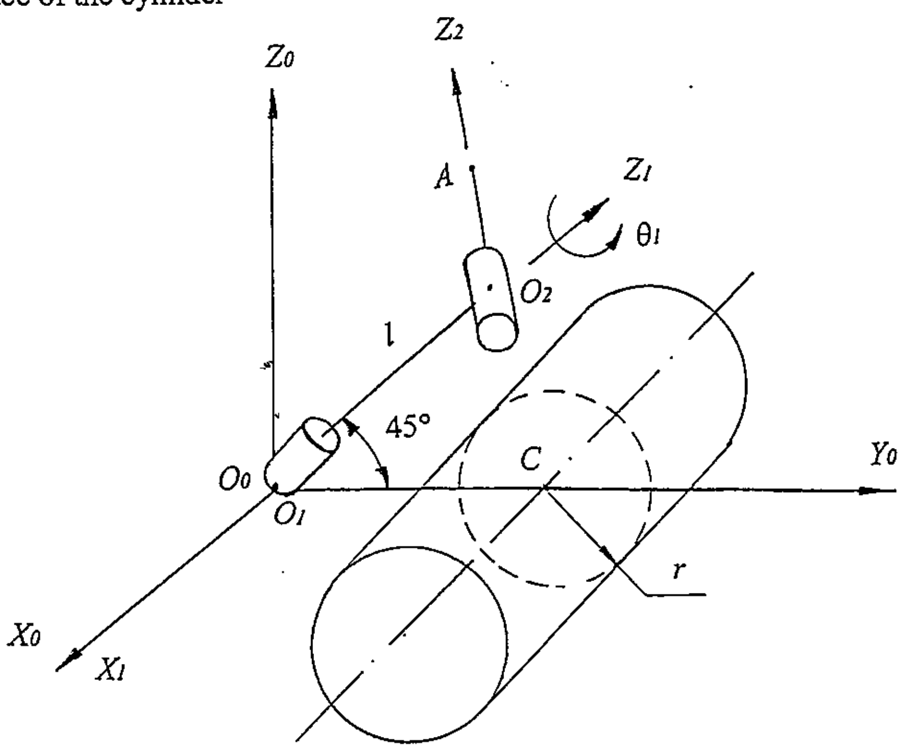

Figure 1

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PROBLEM 2 (10/60)

A three-revolute arm manipulator is shown in **Figure 2**.

(1) Find the Denavit-Hartenberg parameters of the manipulator
(2) Determine the maximum number of inverse kinematic solutions that can be obtained for the current manipulator posture.

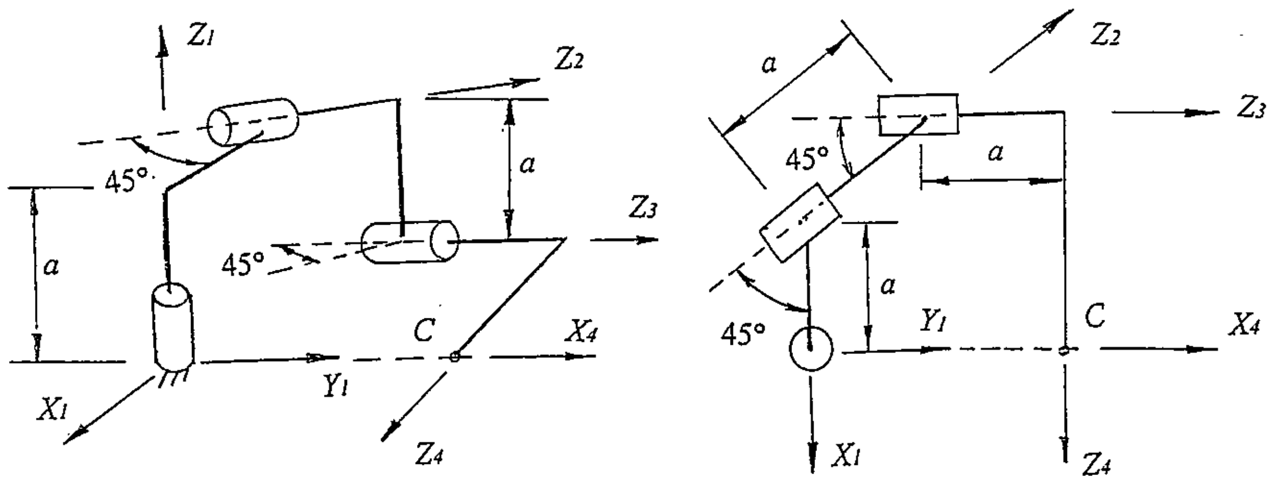

Figure 2

PROBLEM 3 (15/60)

Shown in **Figure 3** is a three-revolute wrist manipulator. Assume that all links are statically balanced and the mass centres of all links are coincident at the wrist centre C. Moreover, the moment of inertia of all links about the mass centre are isotropic such that their inertial matrix can be expressed as $\mathbf{I}_i = \alpha\mathbf{1}$ for $i = 4, 5, 6$, where α is a scalar and $\mathbf{1}$ is the 3×3 identity matrix.

(1) It is proposed during a task planning that the X_6-Y_6-Z_6 frame attain a specific orientation with respect to X_4-Y_4-Z_4 frame as defined by the following rotation matrix

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

that transforms coordinates from X_6-Y_6-Z_6 frame to X_4-Y_4-Z_4 frame. Determine if the proposed orientation is feasible.

(2) Derive the expression for kinetic energy of the manipulator.

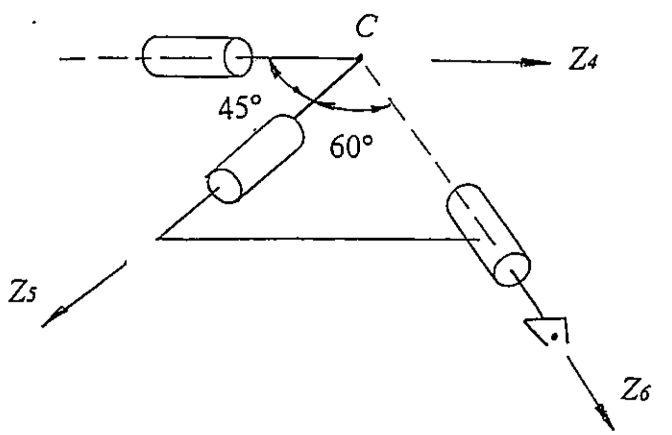

Figure 3

PROBLEM 4 (20/60)

A decoupled manipulator has five revolute joints and one prismatic joint as shown in **Figure 4.** The Denavit-Hartenberg Parameters of the manipulator are given in the following table:

i	a_i	b_i	α_i
1	0	0	90°
2	0	0	90°
3	0	b_3 (Variable)	0
4	0	b_4	90°
5	0	0	90°
6	0	b_6	0

Under the current posture, X_1 axis is vertical and pointing downwards while Z_3 and Z_4 are vertical pointing upwards. Moreover, both Z_6 and Z_7 are parallel to Z_2 that makes 180° with Y_1.

(1) Find the Jacobian matrix
(2) Compute the joint rate that will produce the twist at P as

$$[\omega]_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad [v_P]_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

(3) Compute the joint torque and moments required to balance the following wrench acting at P

$$[\mathbf{n}]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad [\mathbf{f}]_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

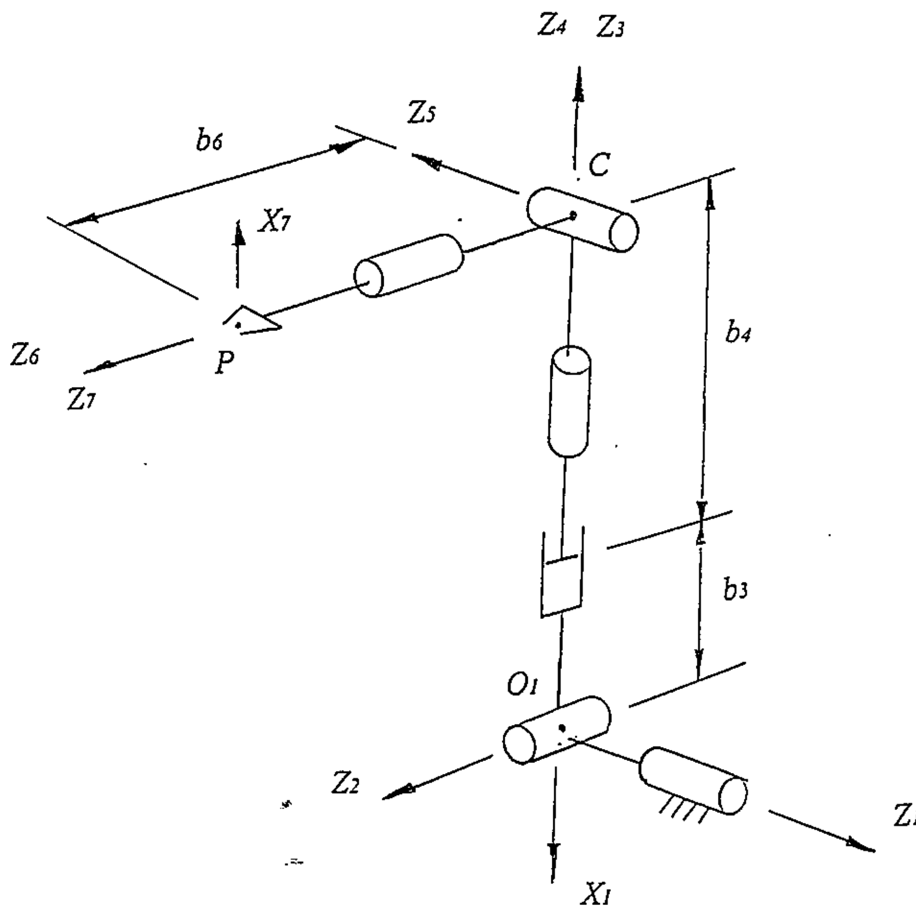

Figure 4

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Problem 1

(1) $Q = Q_0 Q_1$

$Q_0 : \mathcal{F}_0 \to \mathcal{F}_1 \qquad \alpha_0 = 45^\circ \quad \theta_0 = 0 \quad \sin\theta_0 = 0$

$\lambda_0 = \frac{\sqrt{2}}{2} \quad \mu_0 = \frac{\sqrt{2}}{2} \quad \cos\theta_0 = 1$

$$Q_0 = \begin{bmatrix} \cos\theta_0 & -\lambda_0 \sin\theta_0 & \mu_0 \sin\theta_0 \\ \sin\theta_0 & \lambda_0 \cos\theta_0 & -\mu_0 \cos\theta_0 \\ 0 & \mu_0 & \lambda_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$Q_1 : \mathcal{F}_1 \to \mathcal{F}_2$

$\alpha_1 = 90^\circ \quad \lambda_1 = 0 \quad \mu_1 = 1$

$$Q_1 = \begin{bmatrix} \cos\theta_1 & -\lambda_1 \sin\theta_1 & \mu_1 \sin\theta_1 \\ \sin\theta_1 & \lambda_1 \cos\theta_1 & -\mu_1 \cos\theta_1 \\ 0 & \mu_1 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q = Q_0 Q_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \frac{\sqrt{2}}{2}\sin\theta_1 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cos\theta_1 \\ \frac{\sqrt{2}}{2}\sin\theta_1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cos\theta_1 \end{bmatrix}$$

(2) $[\vec{a}]_1 = [\vec{a_1}]_0 + Q[\vec{a}]_2$

$$= \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} l \\ \frac{\sqrt{2}}{2} l \end{bmatrix} + \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \frac{\sqrt{2}}{2}\sin\theta_1 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cos\theta_1 \\ \frac{\sqrt{2}}{2}\sin\theta_1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\cos\theta_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

$$= \begin{bmatrix} a\sin\theta_1 \\ \frac{\sqrt{2}}{2}l - \frac{\sqrt{2}}{2}a\cos\theta_1 \\ \frac{\sqrt{2}}{2}l - \frac{\sqrt{2}}{2}a\cos\theta_1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If A does not collide with the cylinder

$$(y-b)^2 + z^2 > r^2$$

must hold

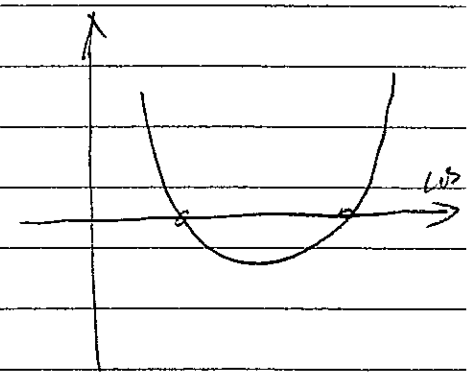
$$\Rightarrow \left(\frac{\sqrt{2}}{2}l - \frac{\sqrt{2}}{2}a\cos\theta_1 - b\right)^2 + \left(\frac{\sqrt{2}}{2}l - \frac{\sqrt{2}}{2}a\cos\theta_1\right)^2 > r^2$$

$$\Rightarrow a^2\cos^2\theta_1 + (\sqrt{2}ab - 2al)\cos\theta_1 + (l^2 + b^2 - r^2 - \sqrt{2}bl) > 0$$

Let $\alpha = a^2$

$\beta = \sqrt{2}ab - 2al$

$r = l^2 + b^2 - r^2 - \sqrt{2}bl$



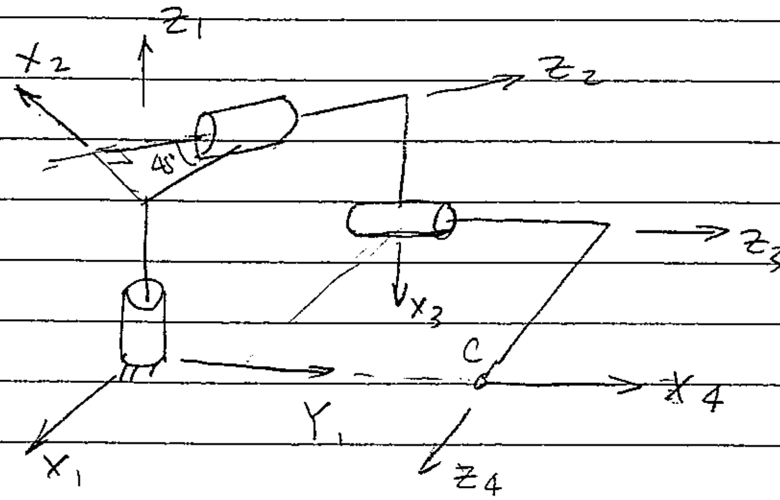
$$\alpha\cos^2\theta_1 + \beta\cos\theta_1 + r > 0$$

$$\begin{cases} \cos\theta_1 < \dfrac{-\beta - \sqrt{\beta^2 - 4\alpha r}}{2\alpha} \\ \cos\theta_1 > \dfrac{-\beta + \sqrt{\beta^2 - 4\alpha r}}{2\alpha} \end{cases}$$

Problem 2.

(1)

i	a_i	b_i	α_i	λ_i	μ_i
1	$\sqrt{2}/2\, a$	a	$90°$	0	1
2	a	$(1+\sqrt{2})a$	$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$
3	0	$(1+\sqrt{2})a$	$90°$	0	1
4	0	0	0		



(2) From the geometry:

$$x_c = 0 \qquad y_c = (1+\sqrt{2})a \qquad z_c = 0$$

eq (4.21c) of the text

$$\Delta_1 = -2a_1\mu_1(x_c^2 + y_c^2) = -\sqrt{2}a(1+\sqrt{2})^2a^2$$
$$= -\sqrt{2}(1+\sqrt{2})^2a^3 \neq 0$$

eq (4.29c)

$$\Delta_2 = a_2^2 + a_3^2(\cos^2\theta_3 + \lambda_2^2\sin^2\theta_3) + b_4^4\mu_3^2(\sin^2\theta_3 + \lambda_2^2\cos^2\theta_3)$$
$$+ 2a_2a_3\cos\theta_3 + 2a_2b_4\mu_3\sin\theta_3 + 2\lambda_2\mu_2(b_3+b_4\lambda_3)(b_4\mu_3\cos\theta_3 -$$
$$a_3\sin\theta_3) + 2a_3b_4(1-\lambda_2^2)\mu_3\sin\theta_3\cos\theta_3 + (b_3+\lambda_3b_4)^2\mu_2^2$$

$$= a_2^2 + 2\lambda_2 u_2 b_3 + b_3 u_2^2$$

$$= a^2 + \frac{3}{2}(1+\sqrt{2})a \neq 0$$

We have both Δ_1 and Δ_2 none zero

~~Thus the max number of inverse kinematic Solution is 4.~~

Problem 3

(1) The last column of Q is $[\vec{e}_6]_4$. The proposed orientation requires

$$[\vec{e}_6]_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

~~Refer to (4.33)~~

$$\xi = 0 \qquad \eta = 0 \qquad \zeta = 1.$$

~~The Radical in (4.38) is~~

$$\Delta = (\xi^2 + \eta^2) u_4^2 - (\lambda_5 - \zeta\lambda_4)^2$$

$$= 0 - \left(\frac{1}{2} - 1\cdot\frac{\sqrt{2}}{2}\right)^2 < 0$$

Thus equ. (4.37) has no Solution

$\Rightarrow$ ~~The proposed orientation is infeasible.~~

(2). $T = T_4 + T_5 + T_6$

$$T_i = \frac{1}{2}\vec{\omega_i}^T I_i \vec{\omega_i} = \frac{1}{2}\alpha \omega_i^T 1 \omega_i = \frac{1}{2}\alpha \|\omega_i\|^2 \quad i=4.5.6$$

$$\vec{\omega_4} = \dot{\theta}_4 \vec{e_4}$$

$$\vec{\omega_5} = \dot{\theta}_4 \vec{e_4} + \dot{\theta}_5 \vec{e_5}$$

$$\vec{\omega_6} = \dot{\theta}_4 \vec{e_4} + \dot{\theta}_5 \vec{e_5} + \dot{\theta}_6 \vec{e_6}$$

$$\|\vec{\omega_4}\|^2 = \dot{\theta}_4^2$$

$$\|\vec{\omega_5}\|^2 = \|\dot{\theta}_4 \vec{e_4}\|^2 + 2\dot{\theta}_4\dot{\theta}_5 \overbrace{\vec{e_4}\cdot\vec{e_5}}^{\cos 45 = \frac{\sqrt{2}}{2}} + \|\dot{\theta}_5 \vec{e_5}\|^2$$

$$= \dot{\theta}_4^2 + \sqrt{2}\dot{\theta}_4\dot{\theta}_5 + \dot{\theta}_5^2$$

$$\|\vec{\omega_6}\|^2 = \dot{\theta}_4^2 + \dot{\theta}_5^2 + \dot{\theta}_6^2 + 2\dot{\theta}_4\dot{\theta}_5 \overbrace{\vec{e_4}\cdot\vec{e_5}}^{\sqrt{2}/2} + 2\dot{\theta}_5\dot{\theta}_6 \overbrace{\vec{e_5}\cdot\vec{e_6}}^{1/2}$$

$$+ 2\dot{\theta}_4\dot{\theta}_6 \vec{e_4}\cdot\vec{e_6}.$$

need $\vec{e_4}\cdot\vec{e_6}$

$$\lambda_4 = \mu_4 = \frac{\sqrt{2}}{2}$$

$$Q_4 = \begin{bmatrix} \cos\theta_4 & -\frac{\sqrt{2}}{2}\sin\theta_4 & \frac{\sqrt{2}}{2}\sin\theta_4 \\ \sin\theta_4 & \frac{\sqrt{2}}{2}\cos\theta_4 & -\frac{\sqrt{2}}{2}\cos\theta_4 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\lambda_5 = \frac{1}{2} \qquad \mu_5 = \frac{\sqrt{3}}{2}$$

$$Q_5 = \begin{bmatrix} \cos\theta_5 & -\frac{1}{2}\sin\theta_5 & \frac{\sqrt{3}}{2}\sin\theta_5 \\ \sin\theta_5 & \frac{1}{2}\cos\theta_5 & -\frac{\sqrt{3}}{2}\cos\theta_5 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

express both $\vec{e_4}$ and $\vec{e_6}$ in $\mathcal{F}_5$

$$[\vec{e_4}]_5 = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \quad \text{— transpose of the 3rd row of } Q_4$$

$$[\vec{e_6}]_5 = \begin{bmatrix} \frac{\sqrt{3}}{2}\sin\theta_5 \\ -\frac{\sqrt{3}}{2}\cos\theta_5 \\ 1/2 \end{bmatrix} \quad \text{— the 3rd column of } Q_5$$

$$\vec{e_4}\cdot\vec{e_6} = -\frac{\sqrt{6}}{4}\cos\theta_5 + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{4}(1-\sqrt{3}\cos\theta_5)$$

Thus.

$$\|\vec{\omega}_6\|^2 = \dot{\theta}_4^2 + \dot{\theta}_5^2 + \dot{\theta}_6^2 + \sqrt{2}\dot{\theta}_4\dot{\theta}_5 + \dot{\theta}_5\dot{\theta}_6 + \frac{\sqrt{2}}{2}(1-\sqrt{3}\cos\theta_5)\dot{\theta}_4\dot{\theta}_6$$

$$T = \frac{1}{2}\alpha\left[3\dot{\theta}_4^2 + 2\dot{\theta}_5^2 + \dot{\theta}_6^2 + 2\sqrt{2}\dot{\theta}_4\dot{\theta}_5 + \dot{\theta}_5\dot{\theta}_6 + \frac{\sqrt{2}}{2}(1-\sqrt{3}\cos\theta_5)\dot{\theta}_4\dot{\theta}_6\right]$$

Problem 4

(1) Decoupled manipulator

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}$$

Where $J_{11} = [\vec{e_1}\ \vec{e_2}\ 0]$ $\quad J_{12} = [\vec{e_4}\ \vec{e_5}\ \vec{e_6}]$

$$J_{21} = [\vec{e_1}\times\vec{r_1}\ \ \vec{e_2}\times\vec{r_2}\ \ \vec{e_3}]$$

$$\vec{e_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{e_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e_4} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e_5} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{e_6} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{r_1} = \vec{r_2} = \begin{bmatrix} -(b_3+b_4) \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e_1} \times \vec{r_1} = \begin{bmatrix} 0 \\ -(b_3+b_4) \\ 0 \end{bmatrix} \qquad \vec{e_2} \times \vec{r_2} = \begin{bmatrix} 0 \\ 0 \\ -(b_3+b_4 \end{bmatrix}$$

$$\Rightarrow \quad J_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad J_{12} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$J_{21} = \begin{bmatrix} 0 & 0 & -1 \\ -(b_3+b_4) & 0 & 0 \\ 0 & -(b_3+b_4) & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -(b_3+b_4) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(b_3+b_4) & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) $$\vec{t_c} = \begin{bmatrix} I & 0 \\ P-C & I \end{bmatrix} \vec{t_p}$$

P, C — cross product matrices of $\vec{p}$ and $\vec{c}$

$$\vec{p} = \begin{bmatrix} -(b_3+b_4) \\ -b_6 \\ 0 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} -(b_3+b_4) \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & -b_6 \\ 0 & 0 & b_3+b_4 \\ b_6 & -(b_3+b_4) & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (b_3+b_4) \\ 0 & -(b_3+b_4) & 0 \end{bmatrix}$$

$$P - C = \begin{bmatrix} 0 & 0 & -b_6 \\ 0 & 0 & 0 \\ b_6 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \dot{\vec{c}} = (P - C)\vec{\omega} + \dot{\vec{p}} = \begin{bmatrix} -b_6 \\ 0 \\ b_6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-b_6 \\ 1 \\ 1+b_6 \end{bmatrix}$$

$$\begin{cases} J_{11}\dot{\vec{\theta}}_a + J_{12}\dot{\vec{\theta}}_w = \vec{\omega} \\ J_{21}\dot{\vec{\theta}}_a = \dot{\vec{c}} \end{cases}$$

$$\dot{\vec{\theta}}_a = J_{12}^{-1}\dot{\vec{c}} = \begin{bmatrix} 0 & -1/(b_3+b_4) & 0 \\ 0 & 0 & -1/(b_3+b_4) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1-b_6 \\ 1 \\ 1+b_6 \end{bmatrix}$$

$$= -\frac{1}{b_3+b_4} \begin{bmatrix} 1 \\ 1+b_6 \\ (b_3+b_4)(1-b_6) \end{bmatrix}$$

$$\dot{\vec{\theta}}_w = J_{12}^{-1}(\vec{\omega} - J_{11}\dot{\vec{\theta}}_a)$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{b_3+b_4} \begin{bmatrix} 0 \\ 1+b_6 \\ 1 \end{bmatrix} \right)$$

$$= \frac{-1}{b_3+b_4} \begin{bmatrix} b_3+b_4 \\ 1+b_3+b_4 \\ -1+b_3+b_4-b_6 \end{bmatrix}$$

(3) $$J_{11}^T \vec{n}_w + J_{21}^T \vec{f} = \vec{\tau}_a$$

$$J_{12}^T \vec{n}_w = \vec{\tau}_w$$

$$\vec{n}_w = n + (\vec{p} - \vec{c}) \times \vec{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -b_6 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-b_6 \\ 1 \\ 1+b_6 \end{bmatrix}$$

$$\vec{\tau}_w = J_{12}^T \vec{n}_w = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1-b_6 \\ 1 \\ 1+b_6 \end{bmatrix} = - \begin{bmatrix} 1-b_6 \\ 1+b_6 \\ 1 \end{bmatrix}$$

$$\tau_a = J_{21}^T \vec{f} + J_{11}^T \vec{n}_w$$

$$= \begin{bmatrix} 0 & -(b_3+b_4) & 0 \\ 0 & 0 & -(b_3+b_4) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1-b_6 \\ 1 \\ 1+b_6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-b_3-b_4+b_6 \\ -1-b_3-b_4 \\ -1 \end{bmatrix}$$