305-572A Mechanics of Robotic Systems I **OPEN BOOK**

All calculators are allowed

Examiner: *Prof. Jorge Angeles* (Local: 6315) J. Angeles Date: December 16, 1997

Associate Examiner: *Dr. Leonid Slutski* (Local: 1041) Time: 2:00–5:00

1. (5%) In testing a piece of software for trajectory-planning, a rotation matrix $\mathbf{Q}$ is computed numerically as

$$\mathbf{Q} = \begin{bmatrix} \star & 0.450 & 0.765 \\ -0.750 & \star & -0.435 \\ -0.415 & 0.725 & \star \end{bmatrix}$$

Due to bugs in the code, the diagonal entries of $\mathbf{Q}$ appeared to be greater than unity in absolute value and were, hence, rejected. However, the off-diagonal entries are dubious. Can you decide if they are also unacceptable? Explain briefly the rationale behind your decision.

2. An orthogonal architecture has been proposed for the *arm* of an experimental robot. This architecture comprises three revolutes with axes that intersect pairwise at 90°, as shown in Fig. 1, in which C denotes the centre of the spherical wrist to be mounted on the arm.

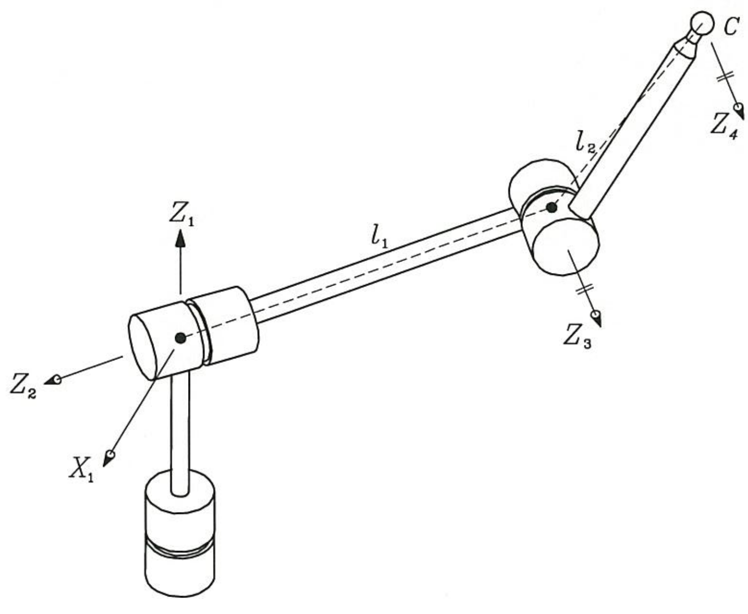

Figure 1: An orthogonal, three-revolute manipulator

(a) (4%) Produce a table showing all Denavit-Hartenberg parameters of the manipulator (architecture parameters and joint variables) at an arbitrary posture.

(b) (4%) Derive expressions for matrices $\mathbf{Q}_i$ and vectors $\mathbf{a}_i$ of the Denavit-Hartenberg notation.

(c) (6%) Find, but do not necessarily solve, two equations in θ_1 and θ_3 that lead to the solution of the associated inverse-kinematics problem.

(d) (10%) Using the above relations, expressions defining the boundary of the workspace can be found, by noticing that, at that boundary, at least two inverse-kinematics solutions merge. Can you tell what shape the workspace has from these expressions? If you cannot, go to next question, since you will not be penalized for not being able to identify the surfaces represented by the above relations.

(e) (8%) Even if you cannot tell the shape of the workspace from the answer to (d) above, you can gain insight into the shape of the workspace by noticing that the manipulator finds itself in a singular posture whenever the third joint axis is perpendicular to the first one. Explain why this posture is singular, for any arbitrary value of θ_3, provided that the second joint remains locked. Note that the locus of point C under these conditions is a circle. Sketch this circle.

(f) (10%) With joint 2 locked, rotate the above circle around the Z_1 axis to generate the workspace (the third joint now becomes irrelevant.) Describe now the shape of the workspace. Moreover, find its volume, under the assumption that $l_2 < l_1$. *Hint: The Pappus-Guldinus Theorem may be useful here (A part of the question is what the P-G Theorem is about!).*

3. The kinematic chain of a three-roll wrist is shown in Fig. 2 in an arbitrary posture. In particular, when $\theta_2 = 90°$, the Jacobian matrix of this wrist takes the form

$$[\mathbf{J}]_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

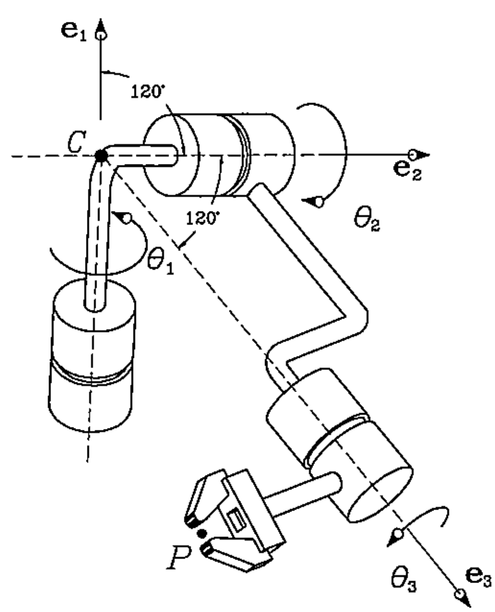

Figure 2: The kinematic chain of a three-roll wrist

(a) (5%) Show that this matrix is a genuine Jacobian for the wrist at hand.

(b) (7%) A force $\mathbf{f}$ applied at the operation point P, a distance r from C, produces a moment $\mathbf{n}$ about C that is estimated by means of torque sensors collocated at the joints. For a certain test, the readouts are

$$\tau_1 = 20 \text{ Nm}, \quad \tau_2 = -10 \text{ Nm}, \quad \tau_3 = 10 \text{ Nm}$$

Find the moment $\mathbf{n}$ in $\mathcal{F}_2$. *Hint: In order to ease the checking of your calculations, you are advised to use* exact arithmetic, *i.e., by longhand calculations, rather than with the aid of a calculator.*

(c) (10%) In another test, we estimate the moment $\mathbf{n}$ and vector $\overrightarrow{CP}$ as

$$[\mathbf{n}]_2 = \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \text{ Nm}, \quad [\overrightarrow{CP}]_2 = \begin{bmatrix} 0.2887 \\ 0.8165 \\ -0.500 \end{bmatrix} r.$$

If we know that $r = 100$ mm and a force sensor collocated at C records a radial component of the applied force $\mathbf{f}$ of 20 N, directed from P to C, find all three components of $\mathbf{f}$ in $\mathcal{F}_2$.

4. (8%) A six-revolute Fanuc robot is to be used for a pick-and-place operation using a cycloidal motion for all joints. The operation requires that the joints sweep the angles given below:

$$\begin{aligned} \Delta\theta_1 &= 30°, \quad \Delta\theta_2 = -9°, \quad \Delta\theta_3 = 11.5°, \\ \Delta\theta_4 &= 36°, \quad \Delta\theta_5 = -24°, \quad \Delta\theta_6 = -36° \end{aligned}$$

Moreover, the manufacturer specifies the maximum joint rates $\{r_i\}_1^6$ for all joints, based in motor performance, as

$$\begin{aligned} r_1 &= 90°/\text{s}, \quad r_2 = 90°/\text{s}, \quad r_3 = 90°/\text{s} \\ r_4 &= 120°/\text{s}, \quad r_5 = 120°/\text{s}, \quad r_6 = 180°/\text{s} \end{aligned} \tag{1}$$

What is the minimum time in which the operation can be performed without exceeding the above joint limits?

5. With regard to the ~~three-roll~~ wrist of Fig. 2, a design has been proposed according to which the links are all statically balanced, with their mass centres all coinciding at the wrist centre. Moreover, the links are designed with an isotropic moment of inertia about their mass centres $\mathbf{I}_i = I\mathbf{1}$ for $i = 1, 2, 3$, where I is a positive scalar and $\mathbf{1}$ is the 3×3 identity matrix.

(a) (8%) Find an expression for the kinetic energy of the manipulator.

(b) (4%) Under the action of gravity which acts in the direction of $-\mathbf{e}_1$, find the potential energy of the wrist.

(c) (6%) Derive an expression for the Coriolis and centrifugal forces of the wrist.

(d) (5%) Derive the mathematical model describing the dynamics of the wrist, when acted upon by torques $\{\tau_i\}_1^3$.

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INSTRUCTIONS

1. Fill in the above carefully.
2. Write your answers on the **ruled side only**. Use the unruled side for rough work or calculations.
3. Do not write in the margin. If a page is accidentally left blank, write "P.T.O." on it.
4. Do not tear pages from this book. All your writing must be handed in.
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C

Problem 1.

Let's take

$$\text{vect}(\underset{\sim}{Q}) = \frac{1}{2}\begin{bmatrix} 0.725+0.435 \\ 0.765+0.415 \\ -0.750-0.450 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.59 \\ -0.60 \end{bmatrix} \stackrel{!}{\doteq} \vec{e}\sin\phi$$

where $\vec{e}$ denotes the unit real eigenvector of $\underset{\sim}{Q}$, parallel to the axis of rotation, while ϕ the angle of rotation. Now,

$\|\text{vect}(\underset{\sim}{Q})\|^2 = 1.045 > 1$ and hence, unacceptable

Ans.

Problem 2

(a)

i	a_i	b_i	α_i	θ_i		λ_i	μ_i
1	0	0	90°	θ_1		0	1
2	0	$-\ell_1$	90°	θ_2	$\Rightarrow$	0	1
3	ℓ_2	0	0°	θ_3		1	0

Under the general architecture of a decoupled manipulator, shown in Fig. 4.8, point C is the intersection of Z_4, X_5, Z_5, X_5 and X_6, and hence, according to the H-D notation, $b_4 = 0$.

b) From the above table,

$$\underset{\sim}{Q}_1 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \underset{\sim}{Q}_2 = \begin{bmatrix} c_2 & 0 & s_2 \\ s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix}, \quad \underset{\sim}{Q}_3 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ -l_1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} l_2 c_3 \\ l_2 s_3 \\ 0 \end{bmatrix}$$

c) From the D-H parameters of the above table, the A, B, C, D, E coefficients of eq. (4.19a) are

$$A = 0, \quad B = 0, \quad C = 0, \quad D = -2 l_1 l_2$$
$$E = l_2^2 + l_1^2 - x_c^2 - y_c^2 - z_c^2 = l_1^2 + l_2^2 + r^2$$

$$r^2 \equiv x_c^2 + y_c^2 + z_c^2$$

$$\Rightarrow -2 l_1 l_2 s_3 + l_1^2 + l_2^2 - r^2 = 0 \qquad (1)$$

Likewise, coefficients F, G, H, I, J of eq. (4.20 a) are

$$F = y_c, \quad G = -x_c, \quad H = 0, \quad I = l_2, \quad J = -l_1$$

$$\Rightarrow y_c c_1 - x_c s_1 + l_2 s_3 - l_1 = 0 \qquad (2)$$

Equations (1) & (2) are the answer to this item.

(d) Equation (1) yields s_3 in the form

$$s_3 = \frac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2} \qquad (3)$$

and hence, two distinct values of θ_3 as long as $|l_1^2 + l_2^2 - r^2| < 2 l_1 l_2$. These two values merge into one when $|l_1^2 + l_2^2 - r^2| = 2 l_1 l_2$

$$\Rightarrow l_1^2 + l_2^2 - r^2 = 2 l_1 l_2 \quad \text{or} \quad l_1^2 + l_2^2 - r^2 = -2 l_1 l_2$$

or

$$x_c^2 + y_c^2 + z_c^2 = (l_1 - l_2)^2 \quad \text{or} \quad x_c^2 + y_c^2 + z_c^2 = (l_1 + l_2)^2$$

thereby indicating two spheres centred at the intersection of the first two axes, of radii $l_1 - l_2$ and $l_1 + l_2$, which correspond to the folded and the extended postures of the arm, respectively.

Furthermore, we introduce the usual trigonometric identies

$$c_1 \equiv \frac{1-z_1^2}{1+z_1^2}, \quad s_1 \equiv \frac{2z_1}{1+z_1^2}, \quad z_1 \equiv \tan\left(\frac{\theta_1}{2}\right)$$

into eq. (2), along with the value of s_3 of eq. (3):

$$2l_1 y_c(1-z_1^2) - 4l x_c z_1 + (-l_1^2 + l_2^2 - r^2)(1+z_1^2) = 0$$

or

$$(2l_1 y_c + l_1^2 - l_2^2 + r^2) z_1^2 + 4 l_1 x_c z_1 - 2l_1 y_c + l_1^2 - l_2^2 + r^2 = 0 \quad (4)$$

thus obtaining a quadratic eqn in z_1, from which two distinct solns can be obtained, as long as the discriminant Δ is positive. When Δ vanishes, the two solutions merge, i.e., a portion of the workspace boundary is obtained upon making $\Delta = 0$:

$$\Delta \equiv 4l_1^2 x_c^2 - (2l_1 y_c + l_1^2 - l_2^2 + r^2)(-2l_1 y_c + l_1^2 - l_2^2 + r^2) =$$

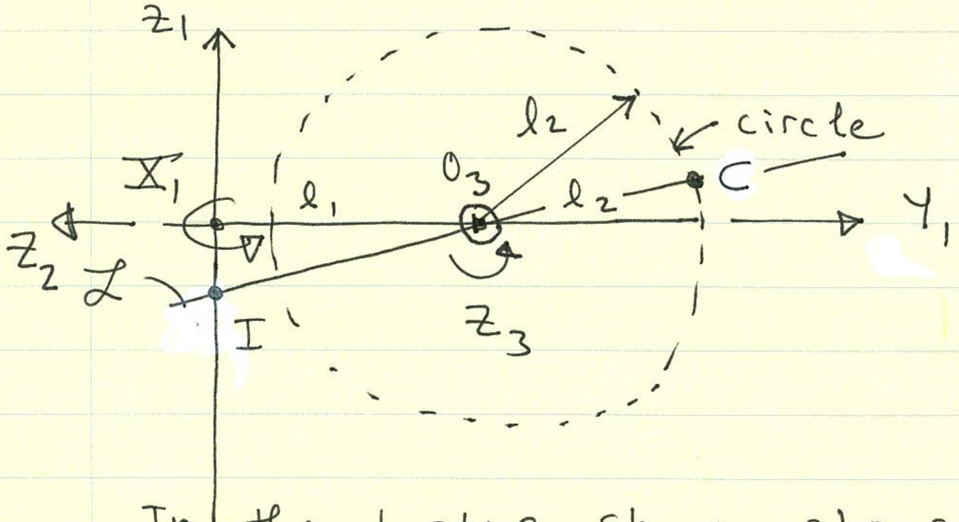
$$= 4l_1^2 x_c^2 - [(l_1^2 - l_2^2 + r^2)^2 - 4l_1^2 y_c^2] =$$

$$= 4l_1^2(x_c^2 + y_c^2) - (l_1^2 - l_2^2 + r^2)^2$$

$$\Delta = 0 \Rightarrow 4l_1^2(x_c^2 + y_c^2) = [l_1^2 - l_2^2 + x_c^2 + y_c^2 + z_c^2]^2 \quad (5)$$

Equation (5) being quartic, it looks like a torus, but its parameters cannot be readily identified, unless one has a handbook with the equations of quartic surfaces. So, let us continue.

(e) With $Z_3 \perp Z_1$, we have



In the posture shown above the line $\mathcal{L}$ passing through C and O_3 intersects Z_1 at I, while Z_2 and Z_3 at O_3. The manipulator is therefore in a singular posture, in which a velocity of C in the direction of $\mathcal{L}$ cannot be produced by any combination of joint rates. By the same token, a force with $\mathcal{L}$ as line of action, applied at C, can be resisted by the manipulator without applying any extra load on the three motors.

(f) The surface generated by the circle of the above figure when the circle turns about the Z_1 axis is a torus. By application of the P-G Theorem, its volume V is

$$V = 2\pi l_1 (\pi l_2^2) = 2\pi^2 l_1 l_2^2$$

Problem 3.

(a) The columns of $\underset{\sim}{J}$ must be unit vectors, which they are because

$$\|\vec{j}_1\|^2 = \frac{1}{4}(\sqrt{3}^2 + 1) = 1$$

$$\|\vec{j}_2\|^2 = \frac{1}{4} 4 = 1$$

$$\|\vec{j}_3\|^2 = \frac{1}{4}(\sqrt{3}^2 + 1) = 1$$

(b) $\underset{\sim}{J}^T \vec{n} = \vec{c} \Rightarrow \frac{1}{2}\begin{bmatrix} 0 & \sqrt{3} & -1 \\ 0 & 0 & 2 \\ \sqrt{3} & 0 & -1 \end{bmatrix}\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$

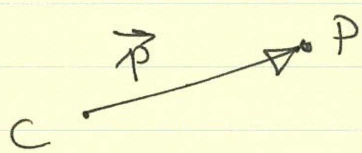
or $\left\{ \begin{array}{ll} \sqrt{3} n_y - n_z = 40 & \text{(a)} \\ 2n_z = -20 \Rightarrow n_z = -10 & \text{(b)} \\ \sqrt{3} n_x - n_z = 20 & \text{(c)} \end{array} \right.$

(b) into (c) $\Rightarrow n_x = (20 - 10)\frac{\sqrt{3}}{3} = 10\frac{\sqrt{3}}{3}$

(b) into (a) $\Rightarrow n_y = (40 - 10)\frac{\sqrt{3}}{3} = 10\sqrt{3}$

$$\Rightarrow [\vec{n}]_2 = \begin{bmatrix} 10\frac{\sqrt{3}}{3} \\ 10\sqrt{3} \\ -10 \end{bmatrix}$$

(c) $[\vec{n}]_2 = \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$ Nm



$\vec{n} = \vec{p} \times \vec{f}$ & $\vec{p} \cdot \vec{f} = 20r$

These two eqns can be cast in the form

$$\underset{\sim}{A}\vec{f} = \vec{b} \qquad \text{(d)}$$

$\underset{\sim}{A} \equiv \begin{bmatrix} \underset{\sim}{P} \\ \vec{p}^T \end{bmatrix}$, $\vec{b} = \begin{bmatrix} \vec{n} \\ 20r \end{bmatrix}$, $\underset{\sim}{P}$ = CPM of $\vec{p}$

$$\underset{\sim}{A}^T(d) \Rightarrow \quad \underset{\sim}{A}^T \underset{\sim}{A} \vec{f} = \underset{\sim}{A}^T \vec{b}$$

$$\underset{\sim}{A}^T\underset{\sim}{A} = \begin{bmatrix} -\underset{\sim}{P} & \vec{p} \end{bmatrix} \begin{bmatrix} \underset{\sim}{P} \\ \vec{p}^T \end{bmatrix} = \underbrace{-\underset{\sim}{P}^2}_{\|\vec{p}\|^2 \underset{\sim}{1} - \vec{p}\vec{p}^T} + \vec{p}\vec{p}^T = \|\vec{p}\|^2 \underset{\sim}{1}$$

$$\|\vec{p}\|^2 = r^2$$

$$\underset{\sim}{A}^T \vec{b} = \begin{bmatrix} -\underset{\sim}{P} & \vec{p} \end{bmatrix} \begin{bmatrix} \vec{n} \\ 20r \end{bmatrix} = \underbrace{-\underset{\sim}{P}\vec{n}}_{-\vec{p}\times\vec{n} = \vec{n}\times\vec{p}} + 20r\vec{p} = \vec{n}\times\vec{p} + 20r\vec{p}$$

$$\left[\vec{n}\times\vec{p}\right]_2 = \begin{vmatrix} \vec{i}_2 & \vec{j}_2 & \vec{k}_2 \\ 20 & -10 & 10 \\ 0.2887 & 0.8165 & -0.500 \end{vmatrix} r = -3.165r\,\vec{i}_2 + 12.887\,r\,\vec{k}_2 + 19.217\,\vec{k}_2$$

$$\Rightarrow \underset{\sim}{A}^T\vec{b} = \begin{bmatrix} -3.165 \\ 12.887 \\ 19.217 \end{bmatrix} r + \begin{bmatrix} 5.774 \\ 16.330 \\ -10.000 \end{bmatrix} r = \underbrace{\begin{bmatrix} 2.609 \\ 29.217 \\ 9.217 \end{bmatrix}}_{\vec{c}} r$$

$$\Rightarrow r^2 \underset{\sim}{1} \vec{f} = r\vec{c}$$

$$\Rightarrow \vec{f} = \frac{1}{r}\vec{c} = \frac{1}{0.10}\begin{bmatrix} 2.609 \\ 29.217 \\ 9.217 \end{bmatrix} = \begin{bmatrix} 26.09 \\ 292.17 \\ 92.17 \end{bmatrix} \text{ N}$$

Problem 4 We have, when using a cycloidal motion, that the minimum time in which the jth joint can perform the manoeuvre is

$$T_j = \frac{|\Delta\theta_j|}{r_j}$$

Hence,

$$T_1 = \frac{30}{90} = \frac{1}{3}\,s, \quad T_2 = \frac{9}{90} = \frac{1}{10}\,s, \quad T_3 = \frac{11.5}{90} = 0.128\,s$$

$$T_4 = \frac{36}{120} = \frac{3}{10}\,s, \quad T_5 = \frac{24}{120} = \frac{1}{5}\,s, \quad T_6 = \frac{36}{180} = \frac{1}{5}\,s$$

Then, $T_{min} = 2\max\{0.333, 0.10, 0.128, 0.3, 0.2, 0.2\}$

$$= 0.667\,s$$

Problem 5. (a) $T = \sum_1^3 T_i$

$$T_i = \frac{1}{2}\vec{\omega}_i^T \underset{\sim}{I}_i \vec{\omega}_i = \frac{1}{2} I \omega_i^T \underset{\sim}{1}\, \vec{\omega}_i = \frac{1}{2} I \|\vec{\omega}_i\|^2, \quad i = 1,2,3$$

$$\vec{\omega}_1 = \dot{\theta}_1 \vec{e}_1 \Rightarrow \|\vec{\omega}_1\|^2 = \dot{\theta}_1^2$$

$$\vec{\omega}_2 = \dot{\theta}_1 \vec{e}_1 + \dot{\theta}_2 \vec{e}_2$$

$$\Rightarrow \|\vec{\omega}_2\|^2 = \|\dot{\theta}_1\vec{e}_1\|^2 + 2\dot{\theta}_1\dot{\theta}_2 \overbrace{\vec{e}_1 \cdot \vec{e}_2}^{\cos 120^\circ = -0.5} + \|\dot{\theta}_2\vec{e}_2\|^2 =$$

$$= \dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2$$

$$\vec{\omega}_3 = \dot{\theta}_1\vec{e}_1 + \dot{\theta}_2\vec{e}_2 + \dot{\theta}_3\vec{e}_3$$

$$\Rightarrow \|\vec{\omega}_3\|^2 = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + 2\dot{\theta}_1\dot{\theta}_2 \overbrace{\vec{e}_1 \cdot \vec{e}_2}^{-0.5} + 2\dot{\theta}_2\dot{\theta}_3 \overbrace{\vec{e}_2 \cdot \vec{e}_3}^{-0.5} + 2\dot{\theta}_1\dot{\theta}_3 \overbrace{\vec{e}_1 \cdot \vec{e}_3}^{?}$$

From eqs. (4.15),

$$[\vec{e}_3]_2 = \vec{u}_2 = \text{third column of } \underset{\sim}{Q}_2$$

Now, to calculate $\underset{\sim}{Q}_2$, we need λ_2 & μ_2:

$\lambda_2 = \cos\alpha_2 = \cos 120° = -0.5$, $\mu_2 = \sin\alpha_2 = \sin 120° = 0.866$

$$\Rightarrow \vec{u}_2 = \begin{bmatrix} 0.866 \\ -0.866 \\ -0.5000 \end{bmatrix} = [\vec{e}_3]_2$$

Likewise,
$[\vec{e}_1]_2 = \vec{o}_1 =$ third row of $\underset{\sim}{Q}_1$, i.e., with

$\lambda_1 = \cos\alpha_1 = \cos 120° = -0.5$ & $\mu_1 = \sin\alpha_1 = \sin 120° = 0.866$

$$\vec{o}_1 = \begin{bmatrix} 0 \\ 0.866 \\ -0.500 \end{bmatrix}$$

$$\Rightarrow \vec{e}_1 \cdot \vec{e}_3 = [\vec{e}_1^T]_2 [\vec{e}_3]_2 = 1.5c_2 + 0.25$$

$$\Rightarrow \|\vec{\omega}_3\|^2 = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 - \dot{\theta}_1\dot{\theta}_2 - \dot{\theta}_2\dot{\theta}_3 + (3c_2 + 0.5)\dot{\theta}_1\dot{\theta}_3$$

$$\Rightarrow T = \frac{1}{2} I \left[3\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + \dot{\theta}_3^2 - \dot{\theta}_2\dot{\theta}_3 + (3c_2 + 0.5)\dot{\theta}_1\dot{\theta}_3\right]$$

(b) Since the mass centres of all links are located at C, and C is stationary, we can then write $V =$ constant, or $V = 0$, for that matter.

(c) Let $\vec{c}(\vec{\theta}, \dot{\vec{\theta}})$ be the vector of Coriolis & centrifugal forces, which is given in eq. (6.21) as the sum of the second & third terms of the LHS:

$$\vec{c}(\vec{\theta}, \dot{\vec{\theta}}) = \dot{\underset{\sim}{I}}(\vec{\theta}, \dot{\vec{\theta}})\dot{\vec{\theta}} - \frac{1}{2}\left[\frac{\partial(\underset{\sim}{I}\dot{\vec{\theta}})}{\partial\vec{\theta}}\right]^T\dot{\vec{\theta}}$$

We thus need $\underset{\sim}{I}(\vec{\theta})$:

$$\underset{\sim}{I}(\vec{\theta}) = \frac{\partial^2 T}{\partial\dot{\vec{\theta}}^2} = \frac{\partial}{\partial\dot{\vec{\theta}}}\left\{\frac{1}{2}I\begin{bmatrix} 6\dot{\theta}_1 + (3c_2+0.5)\dot{\theta}_3 \\ 4\dot{\theta}_2 - \dot{\theta}_3 \\ 2\dot{\theta}_3 - \dot{\theta}_2 + (3c_2+0.5)\dot{\theta}_1 \end{bmatrix}\right\} =$$

$$= \frac{1}{2}I\begin{bmatrix} 6 & 0 & 3c_2+0.5 \\ 0 & 4 & -1 \\ 3c_2+0.5 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \dot{\underset{\sim}{I}}(\vec{\theta}, \dot{\vec{\theta}}) = \frac{1}{2}I\begin{bmatrix} 0 & 0 & -3s_2\dot{\theta}_2 \\ 0 & 0 & 0 \\ -3s_2\dot{\theta}_2 & 0 & 0 \end{bmatrix}$$

$$\dot{\underset{\sim}{I}}(\vec{\theta}, \dot{\vec{\theta}})\dot{\vec{\theta}} = \frac{1}{2}I\begin{bmatrix} -3s_2\dot{\theta}_2\dot{\theta}_3 \\ 0 \\ -3s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}$$

$$\underset{\sim}{I}\dot{\vec{\theta}} = \frac{1}{2}I\begin{bmatrix} 6\dot{\theta}_1 + (3c_2+0.5)\dot{\theta}_3 \\ 4\dot{\theta}_2 - \dot{\theta}_3 \\ (3c_2+0.5)\dot{\theta}_1 - \dot{\theta}_2 + 2\dot{\theta}_3 \end{bmatrix}$$

$$\Rightarrow \frac{\partial(\underset{\sim}{I}\dot{\vec{\theta}})}{\partial\vec{\theta}} = \frac{1}{2}I\begin{bmatrix} 0 & -3s_2\dot{\theta}_3 & 0 \\ 0 & 0 & 0 \\ 0 & -3s_2\dot{\theta}_1 & 0 \end{bmatrix}$$

$$\Rightarrow \left[\frac{\partial \underset{\sim}{I}\dot{\vec{\theta}}}{\partial \vec{\theta}}\right]^T \dot{\vec{\theta}} = \frac{1}{2} I \begin{bmatrix} 0 & 0 & 0 \\ -3s_2\dot{\theta}_3 & 0 & -3s_2\dot{\theta}_1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} =$$

$$= \frac{1}{2} I \begin{bmatrix} 0 \\ -6s_2\dot{\theta}_1\dot{\theta}_3 \\ 0 \end{bmatrix} = I \begin{bmatrix} 0 \\ -3s_2\dot{\theta}_1\dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{c}(\vec{\theta}, \dot{\vec{\theta}}) = \frac{1}{2} I \begin{bmatrix} -3s_2\dot{\theta}_2\dot{\theta}_3 \\ -3s_2\dot{\theta}_1\dot{\theta}_3 \\ -3s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} = -\frac{3}{2} I s_2 \begin{bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_2 \end{bmatrix}$$

(d) We have $V = \text{const} \Rightarrow \frac{\partial V}{\partial \vec{\theta}} = \vec{0}$

Besides,

$$\underset{\sim}{I}\ddot{\vec{\theta}} = \frac{1}{2} I \begin{bmatrix} 6\ddot{\theta}_1 + (3c_2 + 0.5)\ddot{\theta}_3 \\ 4\ddot{\theta}_2 - \ddot{\theta}_3 \\ (3c_2 + 0.5)\ddot{\theta}_1 - \ddot{\theta}_2 + 2\ddot{\theta}_3 \end{bmatrix}, \quad \vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} I \begin{bmatrix} 6\ddot{\theta}_1 + (3c_2 + 0.5)\ddot{\theta}_3 \\ 4\ddot{\theta}_2 - \ddot{\theta}_3 \\ (3c_2 + 0.5)\ddot{\theta}_1 - \ddot{\theta}_2 + 2\ddot{\theta}_3 \end{bmatrix} - \frac{3}{2} I s_2 \begin{bmatrix} \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

i.e.,

$$6\ddot{\theta}_1 + (3c_2 + 0.5)\ddot{\theta}_3 - 3s_2\dot{\theta}_2\dot{\theta}_3 = \frac{2}{I}\tau_1$$

$$4\ddot{\theta}_2 - \ddot{\theta}_3 - 3s_2\dot{\theta}_1\dot{\theta}_3 = \frac{2}{I}\tau_2$$

$$(3c_2 + 0.5)\ddot{\theta}_1 - \ddot{\theta}_2 + 2\ddot{\theta}_3 - 3s_2\dot{\theta}_1\dot{\theta}_2 = \frac{2}{I}\tau_3$$