The Kinematic Synthesis of Steering Mechanisms

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Abstract

We propose a *computational-kinematics* approach based on elimination procedures to synthesize a steering four-bar linkage. In this regard, we aim at minimizing the root-mean square error of the synthesized linkage in meeting the *steering condition* over a number of linkage configurations within the linkage range of motion. A minimization problem is thus formulated, whose *normality conditions* lead to two polynomial equations in two unknown design variables. Upon eliminating one of these two variables, a monovariate polynomial equation is obtained, whose roots yield *all* locally-optimum linkages. From these roots, the global optimum, as well as unfeasible local optima, are readily identified. The global optimum, however, turns out to be impractical because of the large differences in its link lengths, which we refer to as *dimensional unbalance*. To cope with this drawback, we use a kinematically-equivalent *focal mechanism*, i.e., a six-bar linkage with an input-output function identical to that of the four-bar linkage. Given that the synthesized linkage requires a rotational input, as opposed to most existing steering linkages, which require a translational input, we propose a spherical four-bar linkage to drive the steering linkage. The spherical linkage is synthesized so as to yield a speed reduction as close as possible to 2 : 1 and to have a maximum transmission quality.

La synthèse cinématique des mécanismes de direction

Résumé

Nous proposons ici une approche de *cinématique algorithmique* basée sur une méthode d'élimination de variables, qui permet de synthétiser un mécanisme de direction à quatre barres articulées. Dans cette optique, nous nous efforçons de minimiser l'erreur quadratique moyenne du mécanisme obtenu, en vérifiant la *condition de direction* dans un grand nombre de configurations possibles pour le mécanisme, ce qui mène à un problème de minimisation. Les *conditions*

de normalité de ce dernier donnent lieu à deux équations polynômiales à deux variables de conception inconnues. L'élimination de l'une de ces variables permet d'obtenir une seule équation polynômiale à une seule variable, dont les racines réelles donnent tous les minimums locaux. Il est donc possible d'en tirer le minimum global et de distinguer les minimums inadmissibles. Cependant, le minimum global comporte des longueurs de maillons très diverses, ce qui le rend peu utile. Néanmoins, le concept de *mécanisme focal* équivalent, à savoir un mécanisme à six barres articulées ayant la même relation entrée-sortie que le mécanisme à quatre barres, permet de remplacer ce dernier par un mécanisme de proportions plus acceptables. Puisque le mécanisme ainsi synthétisé exige un entraînement par couple rotoïde, contrairement aux mécanismes de direction existants qui exigent un entraînement par couple glissant, nous proposons un mécanisme sphérique pour coupler le mouvement de l'arbre de direction au maillon entraînant du mécanisme de direction. Le mécanisme sphérique en question est synthétisé afin de donner une relation entré-sortie équivalente à une transmission par engrenages à réduction 2:1 et une qualité de transmission optimale.

1 Introduction

The determination of the type and optimum dimensions of steering mechanisms for road vehicles has remained a challenging problem [1–12]. The complexity of the proposed designs varies from a planar four-bar linkage to a very sophisticated spatial mechanism. A planar four-bar linkage seems to have been used as a steering mechanism very early [13]. Since its inception, this simple structure has been extensively applied in different types of road vehicles. In this paper, we focus on the synthesis of a four-bar steering linkage. Wolfe [14] used the input-output function of the linkage and the steering condition to set up an error equation, then expanded it in series and chose coefficients optimally so as to minimize the underlying structural error. Dudiţă and Alexandru [9] expanded the steering-condition equation, took its first three terms and substituted them into the loop equation of the linkage; then, they chose the proper dimensions of the mechanism to verify the steering condition as closely as possible. Both methods are based on a series expansion, which yields linkages with a small error within a rather limited range of motion. Dijksman [1] used exact synthesis to obtain the four-bar linkage, which nevertheless does not reduce the error of the linkage in the whole range of motion. Fahey and Huston [2] used a numerical method to minimize the root-mean square value of the structural error in the whole steering motion range. While these authors reported only one local minimum, we aim in this paper at finding all local minima, and hence, the global minimum. To this end, the root-mean square value of the design error of the steering four-bar linkage is minimized over a rich set of points within the mobility range of interest. As shown in [15], a large cardinality of the set of prescribed input-output values in function-generation problems leads to a minimum structural error when minimizing the design error. We exploit this result here because finding the least-square design error can be done *directly*. On the contrary, finding the least-square structural error is a nonlinear problem, and hence, calls for an *iterative procedure*.

Instead of a numerical method, however, we resort to symbolic elimination in order to reduce the two *normal equations* of the optimization problem at hand to one monovariate polynomial equation whose roots yield all local minima. Nevertheless, it turns out that all minima entail a high *dimensional unbalance*, i.e., a large ratio of the largest to the smallest link-lengths. In order to cope with this unbalance, we resort to the concept of *focal mechanism* of the four-bar linkage [1,16,17]. Indeed, the *focal six-bar equivalent* of the original four-bar linkage produces the same input-output relation, but has links of similar lengths.

Finally, the issue of how the steering linkage is to be driven is worth discussing. Indeed, most steering linkages are driven by means of a sliding joint, which is actuated by means of a worm-gear transmission [18]. In our case, however, the most appropriate mode of driving is via a revolute joint. If we assume that the steering linkage lies in a horizontal plane, then the input revolute has a vertical axis, to be driven by the steering wheel, whose axis is inclined with respect to the vertical. Here, a bevel-gear transmission would be the obvious means of driving; however, bevel gears, like any other gears, entail friction losses, backlash and a low load-carrying capacity. For this reason we propose as a viable alternative a spherical four-bar linkage that produces, within the desired range of motion, a speed reduction as uniform as possible, of $N:1$, for an integer N. The remaining question to answer here is what value to assign to this integer. If we think of the bevel-gear transmission, then a reasonable value of N would be 2, which is what we use in this paper. Tests have to be conducted to validate this choice. Should a larger value of N be needed, this can be accommodated with a spherical four-bar linkage, but it would be extremely difficult to do so with a simple bevel-gear train. We propose in the paper a method to determine the suitable input-output pairs needed to synthesize the spherical four-bar linkage, whose geometry (angles between neighbouring axes) is then determined via a least-square procedure as well.

2 Kinematic Synthesis of the Steering Four-Bar Linkage

Within the kinematic synthesis of a steering four-bar linkage, there is not really an input and an output link, but the two links must move in a coordinated fashion so as to comply with the *steering condition*. Under this condition, the two wheel-carrying links are required to move in such a way that the two axes of the wheels, $\mathcal{L}$ and $\mathcal{M}$, of Fig. 1, intersect the rear axis $\mathcal{N}$ of the vehicle. Therefore, these two links, although remaining close to parallel upon steering, rotate about their vertical axes by differing angles. While this difference is relatively small, it is large enough to warrant careful consideration when designing the mechanism. Moreover, a steering four-bar linkage is required to be symmetric since it turns the vehicle both right and left.

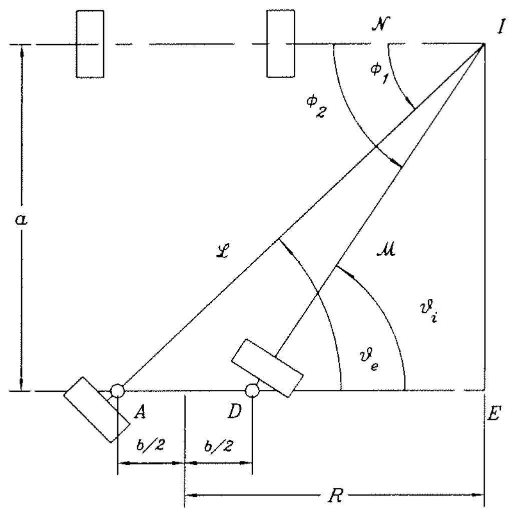


Figure 1: The wheel-steering condition

Figure 1, based on one given in [6], depicts the configuration of the wheels under a left turn. According with the steering condition, the axes of the two wheels must intersect the rear axis at a common point I. From that figure, the steering condition is readily obtained. Indeed, from triangles AIE and DIE, we have

$$\frac{b+\overline{DE}}{a} = \frac{1}{\tan\vartheta_e}, \quad \frac{\overline{DE}}{a} = \frac{1}{\tan\vartheta_i} \tag{1}$$

whence

$$\frac{1}{\tan\vartheta_e} - \frac{1}{\tan\vartheta_i} = \frac{b}{a} \tag{2}$$

which is the *steering condition* sought.

2.1 The Set of Equations for Steering-Linkage Synthesis

The steering condition of eq.(2) is numerically unsuitable because it includes variables that can attain unbounded values. A numerically more convenient form of this condition will be derived. To this end, we redraw Fig. 1 in a more convenient form, in Fig. 2, with the purpose of deriving later the input-output equation of the steering four-bar linkage in a more familiar layout.

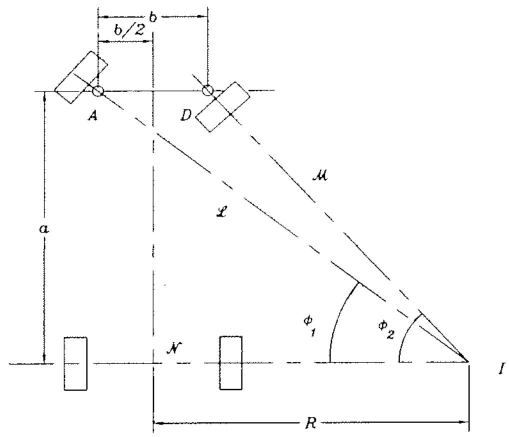


Figure 2: Notation on the steering condition

Now we write the steering condition in terms of angles ϕ_1 and ϕ_2, illustrated in Figs. 1 and 2, thus obtaining

$$\sin(\phi_2 - \phi_1) - \rho \sin\phi_1 \sin\phi_2 = 0 \tag{3a}$$

with parameter ρ defined as

$$\rho \equiv \frac{b}{a} \tag{3b}$$

Equation (3a) is numerically better behaved than eq.(2), for it is expressed in terms of quantities that vary between -1 and $+1$. Moreover, the input-output equation of the four-bar linkage at hand is written in the standard form of the Freudenstein equation [19], in terms of angles σ_1 and σ_2, indicated in Fig. 3. This equation is

$$k_1 + k_2 \cos\sigma_2 - k_3 \cos\sigma_1 = \cos(\sigma_1 - \sigma_2) \tag{4}$$

where k_1, k_2, and k_3 are dimensionless parameters defined as

$$k_1 \equiv \frac{a_1^2 + a_2^2 - a_3^2 + a_4^2}{2a_2a_4}, \quad k_2 \equiv \frac{a_1}{a_2}, \quad k_3 \equiv \frac{a_1}{a_4} \tag{5}$$

Note that we have $a_2 = a_4$, and hence, when the vehicle travels along a straight course, the mid planes of the two front wheels remain parallel. This leads to $\phi_1 = 0$ when $\phi_2 = 0$, while eq.(4) leads to

$$a_3 = a_1 - 2a_2 \sin\beta \tag{6}$$

Further, we use the relation between σ_1 and σ_2 with ϕ_1 and ϕ_2, which is apparent from Fig. 3b:

$$\sigma_1 = \pi/2 - (\beta + \phi_1), \quad \sigma_2 = \pi/2 + (\beta - \phi_2) \tag{7}$$

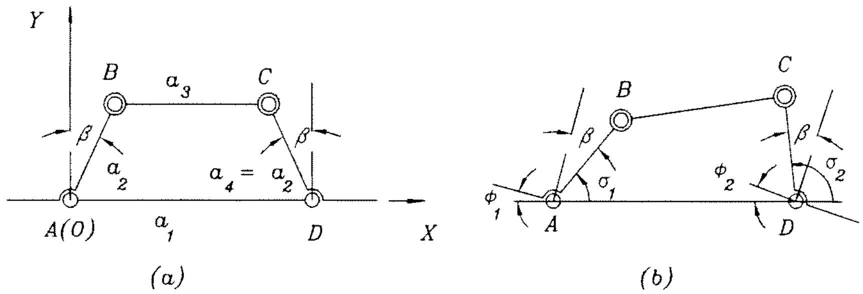


Figure 3: Four-bar linkage with the standard Freudenstein notation in (a) a straight-course configuration; (b) an arbitrary configuration

Now, let x and y be the *nondimensional* coordinates of point B of Fig. 3b, i.e.,

$$x = \frac{a_2}{a_1}\sin\beta \equiv \frac{1}{k_2}\sin\beta \tag{8a}$$

$$y = \frac{a_2}{a_1}\cos\beta \equiv \frac{1}{k_2}\cos\beta \tag{8b}$$

It is apparent from Fig. 3 that neither $x = 0$ nor $y = 0$ can be accepted as viable solutions. Indeed, if $y = 0$, then $\beta = 90°$, which, in the presence of a symmetric linkage, produces a parallelogram, whose input and output links sweep identical angles, and cannot, therefore, generate the steering condition. If, in turn, $x = 0$, then the linkage becomes a structure, with all its joint centres aligned.

In light of definitions (8a) and (8b), the Freudenstein equation becomes

$$f(\phi_1, \phi_2, x, y) \equiv Ax^2 - Ay^2 + Bx + Cy + Dxy = 0 \tag{9a}$$

where

$$\begin{aligned} A &\equiv 1 - \cos(\phi_1 - \phi_2), \quad B \equiv 2 - \cos(\phi_1) - \cos(\phi_2) \\ C &\equiv \sin(\phi_2) - \sin(\phi_1), \quad D \equiv 2\sin(\phi_1 - \phi_2) \end{aligned} \tag{9b}$$

Thus, the new Freudenstein equation includes only two geometric parameters, β and k_2 or, correspondingly, x and y, which are to be determined so as to produce the relation between angles ϕ_1 and ϕ_2 given by the steering condition. By virtue of the number of linkage parameters, however, the symmetric four-bar linkage at hand cannot meet *exactly* the steering condition (3a) at more than two pairs of (ϕ_1, ϕ_2) values. This linkage, however, can *approximate* more than two pairs of those values with a minimum error if β and k_2 or, equivalently x and y, are chosen appropriately. We then formulate the synthesis task under discussion as an optimization problem:

$$z \equiv \frac{1}{2}\sum_{i=1}^{n} f_i^2 \quad \rightarrow \quad \min_{x,y} \tag{10a}$$

where f_i denotes function $f(\phi_1, \phi_2; x, y)$, defined in eq.(9a), when evaluated at the ith pair of (ϕ_1, ϕ_2) values, and n is the number of such pairs, i.e.,

$$f_i = A_i x^2 - A_i y^2 + B_i x + C_i y + D_i xy = 0, \quad i = 1, 2, \ldots, n \tag{10b}$$

with A_i, B_i, and C_i defined as the values attained by A, B, and C, respectively, at the ith pair of (ϕ_1, ϕ_2) values.

In order to minimize z, we obtain the *normality conditions* of the *unconstrained* minimization problem of eq.(10a), i.e.,

$$\frac{\partial z}{\partial x} = \sum_{i=1}^{n} f_i(2A_i x + B_i + D_i y) = 0 \tag{11a}$$

$$\frac{\partial z}{\partial y} = \sum_{i=1}^{n} f_i(-2A_i y + C_i + D_i x) = 0 \tag{11b}$$

It should be noted that the normality conditions are notorious for their ill-conditioned behaviour [20], and hence, unsuitable for a purely numerical approach; however, we will not pursue here such an approach. Instead, we will resort to elimination procedures, whereby the normality conditions offer the advantage of providing all possible solutions. By the same token, a numerical solution of Problem (10a) yields only one local minimum at a time, the *global minimum* never being guaranteed. Also note that f_i is quadratic in x and y, the two foregoing conditions thus being cubic in x and y, and hence, can be expressed as

$$p_1 x^3 + p_2 x^2 + p_3 x + p_4 = 0 \tag{12}$$

$$q_1 x^3 + q_2 x^2 + q_3 x + q_4 = 0 \tag{13}$$

where p_1 and q_1 are constant, all other coefficients being polynomials in y, with p_2 and q_2 linear, p_3 and q_3 quadratic, and p_4 and q_4 cubic.

2.2 Monovariate Polynomial

In order to solve eqs.(12) and (13) for x and y we eliminate one unknown from the above two equations, thus deriving a single monovariate polynomial in y whose roots yield all solutions. We aim here at eliminating x. To this end, we multiply the two normality conditions first by x and then by x^2, thereby producing four additional polynomial equations, which, together with eqs.(12) and (13), yield six linear homogeneous equations in $[x^5,\ x^4,\ x^3,\ x^2,\ x,\ 1]^T$, i.e.,

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 & 0 & 0 \\ 0 & p_1 & p_2 & p_3 & p_4 & 0 \\ 0 & q_1 & q_2 & q_3 & q_4 & 0 \\ 0 & 0 & p_1 & p_2 & p_3 & p_4 \\ 0 & 0 & q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

Obviously, the solution sought cannot be trivial, for the last component of the unknown vector is unity. The condition for a nontrivial solution to exist is that the determinant of the coefficient matrix be zero. Upon expansion, the determinant turns out to be nonic in y, namely,

$$y(y^8 + r_7 y^7 + r_6 y^6 + r_5 y^5 + r_4 y^4 + r_3 y^3 + r_2 y^2 + r_1 y + r_0) = 0 \tag{15}$$

where coefficients $\{r_k\}_0^8$ are constant. Moreover, notice that the solution $y = 0$ was already found unacceptable, and hence, the above nonic equation reduces to an octic equation, i.e.,

$$y^8 + r_7 y^7 + r_6 y^6 + r_5 y^5 + r_4 y^4 + r_3 y^3 + r_2 y^2 + r_1 y + r_0 = 0 \tag{16}$$

2.3 Solution Procedure

After solving for y from the foregoing equation and substituting it into eq.(14), we obtain a linear system of equations in the powers of x, which thus yields one unique value of this variable for each of the roots of eq.(16). With x and y known, we obtain k_2 and β:

$$k_2 = \frac{1}{\sqrt{x^2+y^2}}, \quad \beta = \arctan\left(\frac{y}{x}\right) \tag{17}$$

Moreover, once a_1 has been decided on, based on the space available, a_2, a_3 and a_4 are determined as

$$a_2 = a_4 = \frac{a_1}{k_2}, \quad a_3 = a_1 - 2a_2\cos(\beta) \tag{18}$$

thereby completing the synthesis procedure.

2.4 Numerical Examples

We illustrate the foregoing procedure with two examples taken from the literature, for comparison purposes.

2.4.1 Dudiţă's Linkage [9]

Let ϕ_1 vary from $-40.0°$ to $30.89°$, ϕ_2 varying correspondingly from $-30.89°$ to $40.0°$, with $n = 40$, and all 40 points uniformly spaced along the ϕ_1 axis. Moreover, we set $\rho = 0.46$, as in [9], and set $a_1 = 1$ m. Equations (12) and (13) become now, with six decimals displayed,

$$\begin{aligned}0.153878x^3 + (8.964945 + 7.44537y)x^2 + (-209.385578y + 86.606y^2 + 123.884916)x \\ -2.48179y^3 + 44.78154y - 34.551135y^2 = 0 \\ 2.48179x^3 + (-104.691 + 81.60599y)x^2 + (-69.10227y - 7.44537y^2 + 44.78154)x \\ +0.153878y^3 + 16.2788567y + 3.177492y^2 = 0\end{aligned}$$

In pursuing the monovariate polynomial approach, eq.(16) becomes, with only four decimals displayed,

$$\begin{aligned}-0.0008y^8 + 0.0084y^7 - 0.0393y^6 + 0.1069y^5 - 0.1858y^4 \\ +0.2130y^3 - 0.159y^2 + 0.0716y - 0.0149 = 0\end{aligned}$$

Upon solving the foregoing equation for y, we obtain four real roots:

$$y_1 = 1.9117, \quad y_2 = 1.5131, \quad y_3 = 1.2568, \quad y_4 = 1.166$$

which are all nondimensional. Now, we substitute the foregoing values of y into eq.(14), thereby obtaining, correspondingly, four real solutions for x:

$$x_1 = 1.2373, \quad x_2 = 1.9272, \quad x_3 = 3.5746, \quad x_4 = 0.4992$$

Table 1: Geometrical Parameters of the Linkages

linkage	a_1 (m)	a_2, a_4 (m)	a_3 (m)	β	e_{rms}(degree)
1	1.0	2.2771	−1.4745	32.91°	0.17
2	1.0	2.4502	−2.8543	51.86°	0.08
3	1.0	3.7891	−6.1491	70.63°	0.01
4	1.0	1.2684	0.0016	23.767°	*

We thus find four possible linkages producing the steering condition with locally-minimum error. Shown in Table 1 are the geometric parameters of these linkages, using $a_1 = 1$ m, with the corresponding root-mean square error e_{rms}.

However, only the first three of the above solutions are acceptable, the fourth solution giving an extremely short coupler link, which yields virtually a triangle, rather than a quadrilateral. For this reason, the corresponding root-mean square error has not been computed, the associated entry in Table 1 being filled with an asterisk. Shown in Fig. 4 are the plots of the error distributions of the corresponding linkages. These have better error distributions and smaller maximum errors over the whole steering motion than the linkage obtained by Dudiţă et al., as shown in Fig. 5. It should be noted that a_3 has negative length in this example. From the definition of eq.(6), this means that the two links AB and CD are crossed when the four-bar linkage is in its first position. A discussion on how to deal with negative values of a_2 or a_4 in the context of linkage synthesis for function generation is available in [19].

2.4.2 Fahey and Houston's Linkage [2]

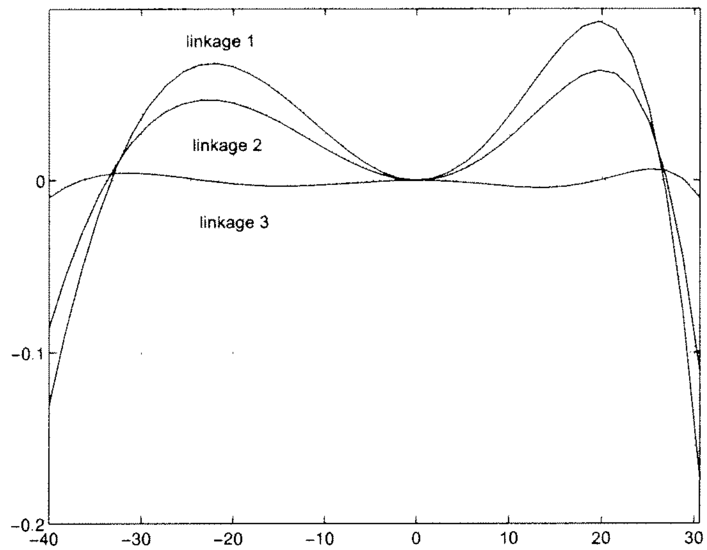
The second example is that studied by Fahey and Huston [2], in which $\rho = 0.6$ and the full motion range is $[-61°, 41°]$. Following the procedure proposed here, we obtain four real solutions as well. The error plot of the globally-optimum solution is shown in Fig. 6. The result shows that the maximum structural error is 0.7° over the range $[-60°, 0]$, which is much smaller than that obtained by Fahey and Huston, of 2.93°.

3 Replacement of the Four-Bar Linkage with Its Focal Six-Bar Equivalent

Although the four-bar steering linkage obtained above satisfies the steering condition with high accuracy, its large dimensional unbalance (its largest link is more than six times longer than its shortest link) makes it unsuitable for practical applications. As an alternative, a kinematically-equivalent six-bar linkage is derived below from the four-bar linkage, with a better dimensional balance.

3.1 A Four-Bar Linkage and Its Focal Six-Bar Equivalent

Shown in Fig. 7 are a four-bar linkage $ABCD$ and its focal six-bar equivalent, $APFSDQF$, in which σ_1 and σ_2 stand for the angular displacements of links AB and CD. In order to let links AP and



ϕ_1 (degree)

Figure 4: The error plots for linkages 1, 2 and 3 of Table 1

DQ of the six-bar linkage undergo the same angular displacements as AB and CD, respectively, the conditions shown below must be observed [17]:

$$\frac{ab}{mn+pq}=\frac{a}{m}=\frac{b}{n}=\frac{a^2+b^2-c^2+d^2}{m^2+n^2-p^2-q^2+d^2}\equiv\lambda \tag{19a}$$

and

$$\frac{s}{(d-s)}=\frac{mp}{nq}\equiv\mu \tag{19b}$$

When the dimensions a, b, c, and d of the four-bar linkage are given and a point S is chosen along line AD, a distance s from point A, we can reduce eqs.(19a) and (19b) to a quadratic equation in λ. With λ known, we can calculate m, n, p, and q by means of eq.(19a) and

$$r^2=\frac{m^2n^2(p^2+q^2)+(m^2+n^2)p^2q^2+mnpq(m^2+n^2+p^2+q^2-d^2)}{(mp+nq)^2} \tag{20}$$

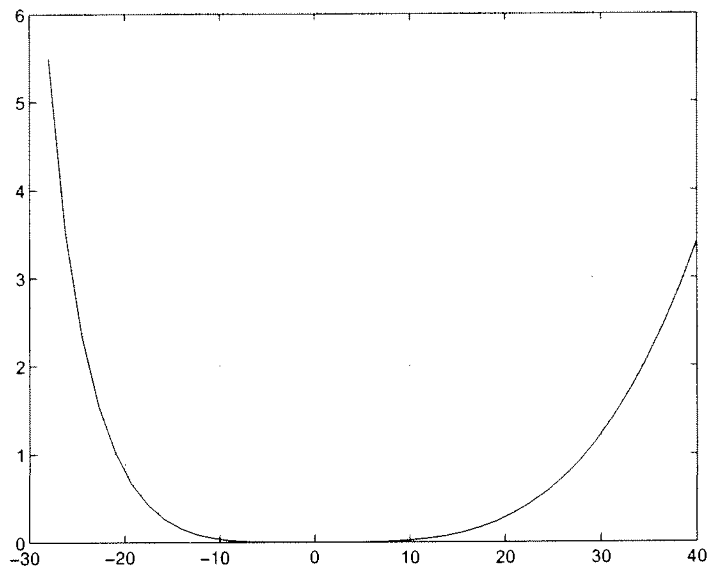
For the first example, $a=b=2.4502$, $c=2.8543$, and $d=1$, with all lengths given in meter; taking $s=0.5$ m, we have, for $\lambda=15.2971, m=n=0.1602$ m, $p=q=0.6056$ m and $r=0.3774$ m for the focal six-bar linkage, as shown in Fig. 8, whereby it is apparent that the focal six-bar linkage is dimensionally better balanced than the original four-bar linkage.

3.2 General Form of the Focal Six-Bar Linkage

With reference to Fig. 9, it is possible to change the position of the fixed joint S of link SF to S' by a combination of stretching and rotation of segment AS. This transformation can be obtained, e.g., as

$$\| \overrightarrow{AS'} \| = \mu \| \overrightarrow{AS} \| \tag{21}$$

$$\mathbf{s}'=\mu\mathbf{R}(\mathbf{s}-\mathbf{a})+\mathbf{a} \tag{22}$$



ϑ_i (degree)

Figure 5: The error plot of the linkage obtained by Dudiţă et al. [9]

where $\mathbf{R}$ is a rotation matrix through an angle φ_1 and μ is a stretching factor. Then, the original six-bar linkage $APFSDQF$ can be transformed into the general $AP'F'F''Q'BS'$ linkage. To this end, the condition below must be satisfied [1,16]:

$$APFS \sim AP'F'S' \quad \text{and} \quad BQFS \sim BQ'F''S' \tag{23}$$

so that the input-output relation between (AP, DQ) and (AP', DQ') remains unchanged. When μ and φ_1 are given, the general six-bar linkage can be obtained from the position vectors $\mathbf{s}'$, $\mathbf{p}'$, $\mathbf{f}'$, $\mathbf{q}'$ and $\mathbf{f}''$, of points S', P', F', Q' and F'', respectively, which are given below:

$$\mathbf{s}' = \mu\mathbf{R}(\mathbf{s} - \mathbf{a}) + \mathbf{a} \tag{24}$$

$$\mathbf{p}' = \mu\mathbf{R}(\mathbf{p} - \mathbf{a}) + \mathbf{a} \tag{25}$$

$$\mathbf{f}' = \mu\mathbf{R}(\mathbf{f} - \mathbf{a}) + \mathbf{a} \tag{26}$$

$$\nu = \frac{\| \overrightarrow{S'D} \|}{\| \overrightarrow{SD} \|} \tag{27}$$

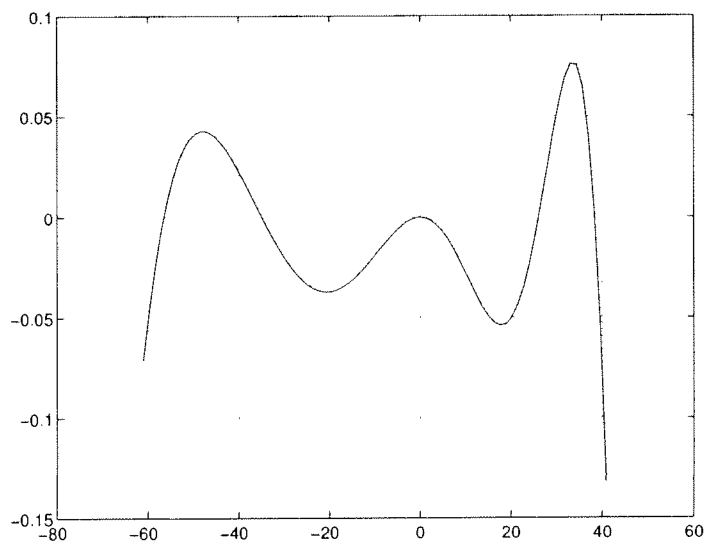
$$\mathbf{q}' = \nu\mathbf{S}(\mathbf{q} - \mathbf{d}) + \mathbf{d} \tag{28}$$

$$\mathbf{f}'' = \nu\mathbf{S}(\mathbf{f} - \mathbf{d}) + \mathbf{d} \tag{29}$$

where $\mathbf{S}$ is a rotation matrix through an angle φ_2. When requiring point S' to be located on the perpendicular bisector of $A'D$ and $S'F' = S'F'' = F'F''$, we can choose $\varphi_1 = \varphi_2 = 30°$ to do so.

3.3 Transmission Quality

In Subsection 3.1, when a different s is chosen, different focal six-bar linkages are obtained, with different *transmission indices*, and SF as driving link. The transmission index was introduced by



ϕ_1 (degree)

Figure 6: The error curve for the global optimum of the second example

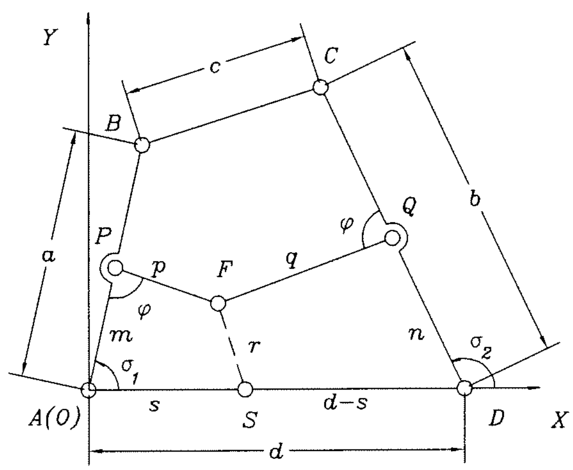


Figure 7: Four-bar linkage and its focal six-bar equivalent

Sutherland and Roth [21] as a generalization of the concept of *transmission angle*. In the case of planar four-bar linkages, the transmission index reduces to the sine of the transmission angle. Just like the transmission angle, which varies with the configuration of the linkage, the transmission index is also configuration-dependent. A *global performance index* measuring the *goodness* of the force-and-motion transmission of the linkage, is the *transmission quality* [22], which is defined in terms of Sutherland and Roth's transmission index.

In Fig. 10, we distinguish two four-bar linkages with a common input link, SF, of output links AP and DQ. In that figure, link SF is shown in its two extreme positions sweeping the whole range

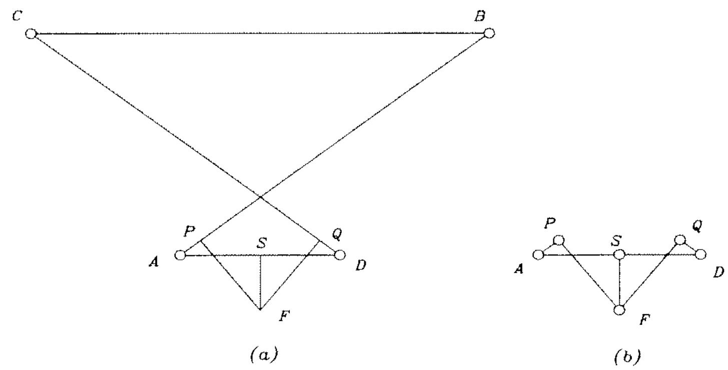


Figure 8: (a) Steering linkage and (b) its focal equivalent

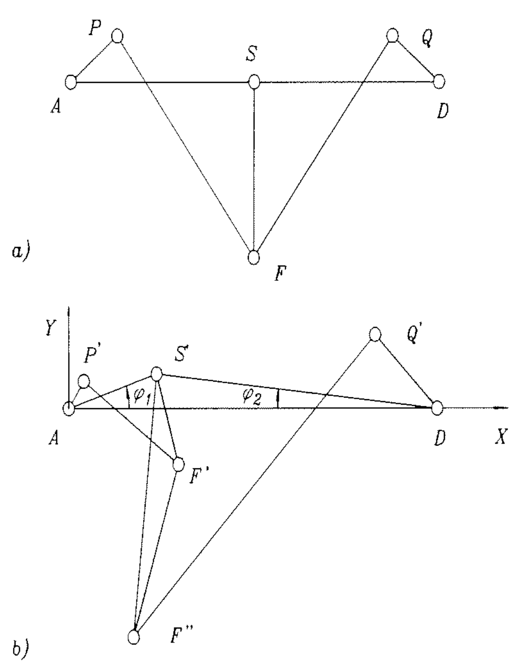


Figure 9: The general six-bar focal linkage

of motion, SF_1 and SF_2. Links AP and DQ are displayed in their corresponding extreme positions, AP_1 and AP_2 for the former, DQ_1 and DQ_2 for the latter. The input angle γ is indicated likewise, for its two extreme values, γ_1 and γ_2. The transmission angle μ_l of the left four-bar linkage of Fig. 10 is known to be $\angle FPA$, while its right counterpart, $\mu_r = \pi - \mu_l$, is $\angle FQD$. Let Q_l and Q_r stand for the transmission quality of the left and right linkages, respectively, i.e.,

$$Q_l = \frac{1}{\gamma_1 - \gamma_2} \int_{\pi-\gamma_2}^{\pi-\gamma_1} \sin^2 \mu_l d\gamma, \quad Q_r = \frac{1}{\gamma_1 - \gamma_2} \int_{-\gamma_2}^{-\gamma_1} \sin^2 \mu_r d\gamma \tag{30}$$

The plot of the transmission quality of any of the two linkages is shown in Fig. 11 vs. parameter s, defined in Fig. 7. It is apparent from this plot that a maximum value of the transmission quality, of $Q_l = Q_r = 0.83$, occurs at $s = 0.5$, a result that should be expected by symmetry. Hence, in order to achieve a maximum transmission quality, link SF should be anchored onto the vehicle chassis halfway between the anchor points of the two output links AP and DQ.

Because of condition (23), the general focal six-bar linkage and its original counterpart have both the same transmission quality.

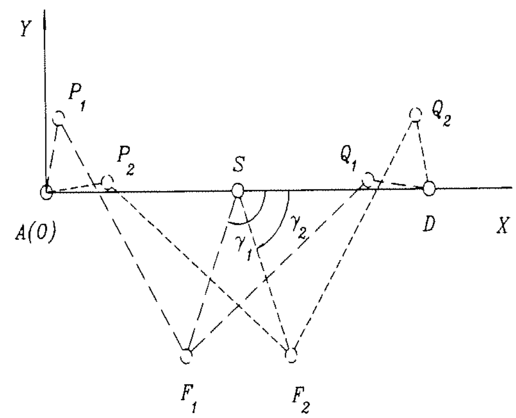


Figure 10: The limit positions of link AP of the focal six-bar linkage

4 The Synthesis of a Spherical Four-Bar Linkage Coupling the Steering Wheel with the Steering Mechanism

Steering linkages are driven by the rotation of the steering wheel using a mechanical transmission that is based, in most cases, on a worm gear [18]. This transmission is suitable to the current design of steering linkages, but would not be suitable to our proposed design. Since the driving link of our design, henceforth referred to as the *driving link* for brevity, is coupled to the vehicle chassis via a revolute joint, the most suitable transmission in this case is one that transmits force and motion from the steering wheel, whose axis is inclined with respect to the vertical by an angle α *directly*, to the

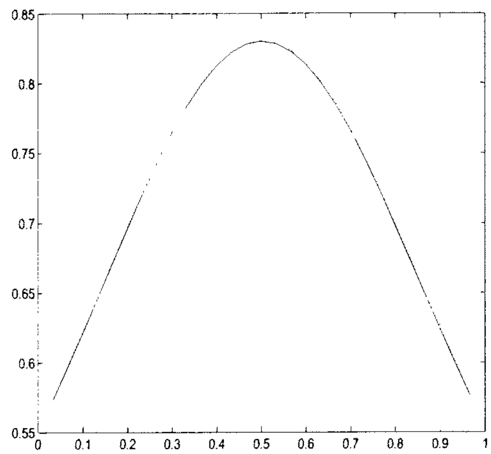


Figure 11: Transmission quality of the steering linkage as a function of parameter s of Fig. 7

driving link of the steering linkage, whose axis can be assumed to be vertical. Furthermore, the axis of the steering wheel is offset from the vehicle midplane, while the axis of the driving link is located in this plane. The offset can be compensated for by means of a double universal joint, and hence, we can safely assume that the input motion to the steering linkage is provided via a shaft of axis intersecting that of the driving link at the above-mentioned angle α. The obvious transmission, then, is apparently a spherical four-bar linkage. The balance of this section is devoted to the synthesis of this linkage.

4.1 Description of the Vehicle Rotation

The spherical linkage of interest will be synthesized as a function generator [19], which requires a set of input-output pairs of values that the linkage is to meet. In order to define the input-output pairs, it will prove convenient to derive a relation between the curvature of the vehicle trajectory and the steering angle ϕ_1.

The radius R of the circular trajectory traced by the vehicle upon turning is often used to indicate the extent of the vehicle rotation. From Fig. 2, R can be found to be

$$R = \frac{a}{\tan\phi_1} - \frac{b}{2} \tag{31}$$

with ϕ_1 being positive when the vehicle turns to the right and negative otherwise, the same holding for R. When ϕ_1 tends to 0, R tends to infinity and, hence $R(\phi_1)$ becomes discontinuous. For this reason, we prefer to use the curvature κ of the above trajectory, which is the reciprocal of R, i.e.,

$$\kappa = \frac{2a - b\tan\phi_1}{2\tan\phi_1} \tag{32}$$

a function of ϕ_1 that remains continuous in the interval $[\kappa_1, \kappa_2]$, where

$$\kappa_1 = \kappa(\phi_1^a), \quad \kappa_2 = \kappa(\phi_1^b)$$

with ϕ_1^a and ϕ_1^b defined as the two extreme values of ϕ_1; in our case, $\phi_1^a = -40°$ and $\phi_1^b = 31.1918°$. Shown in Fig. 12 is a plot of ϕ_1 vs. κ, which was obtained using eq.(32).

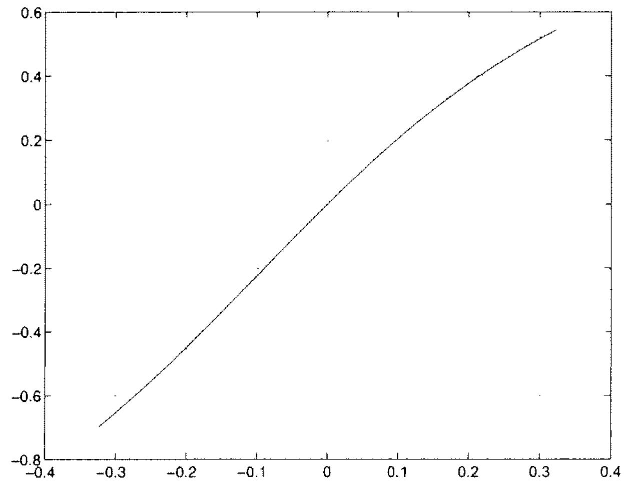


Figure 12: Angle ϕ_1 vs. curvature κ, in m^{-1}

4.2 Input-Output Function of the Spherical Four-Bar Linkage

A spherical four-bar linkage is shown in Fig. 13, in which TW is the fixed link; TU the input link, whose motion is identical to that of the steering wheel; and VW the output link, connected rigidly to FS of the steering six-bar linkage. The positions for the input and output links are defined by $\vartheta_0 + \vartheta$ and $\gamma_0 + \gamma$, respectively. For proper steering, the curvature must be proportional to the input angle, i.e.,

$$\kappa = k\vartheta \tag{33}$$

where k is a scaling factor. Therefore, when κ and k are given, the input displacement ϑ can be obtained from eq.(33). The output displacement γ, in turn, is obtained by imposing an input-output relation. We would like to provide a *uniform* transmission ratio throughout the given motion range. Additionally, a torque amplification is required, which we obtain by requiring that the transmission ratio be of 2:1, but other transmission ratios can be accommodated, as needed. For a value of $k = 0.4313$, the plot of γ vs. ϑ is shown in Fig. 14.

We discuss below the synthesis of the spherical linkage using an optimization approach.

4.3 Linkage Optimization

The input-output function of a spherical four-bar linkage is known to be [23]

$$\begin{aligned} F(\vartheta, \gamma) \equiv k_1 + k_2\cos(\vartheta_0 + \vartheta) + k_3\cos(\vartheta_0 + \vartheta)\cos(\gamma_0 + \gamma) \\ -k_4\cos(\gamma_0 + \gamma) + \sin(\gamma_0 + \gamma)\sin(\vartheta_0 + \vartheta) = 0 \end{aligned} \tag{34}$$

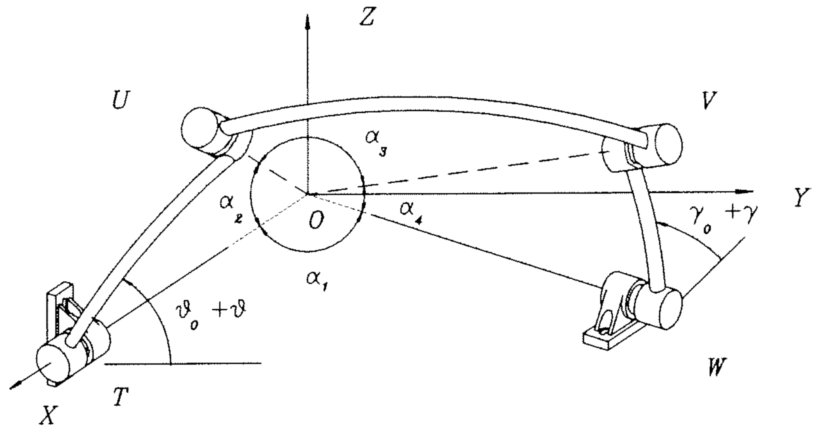


Figure 13: The spherical four-bar linkage

where

$$
\begin{aligned}
k_1 &= \frac{\cos\alpha_1 \cos\alpha_2 \cos\alpha_4 - \cos\alpha_3}{\sin\alpha_2 \sin\alpha_4} \\
k_2 &= \frac{\sin\alpha_1 \cos\alpha_4}{\sin\alpha_4} \\
k_3 &= \cos\alpha_1 \\
k_4 &= \frac{\sin\alpha_1 \cos\alpha_2}{\sin\alpha_2}
\end{aligned}
$$

In synthesizing the spherical linkage, we assume that the steering six-bar linkage lies in a horizontal plane and that the axis of the steering-wheel axle makes 45° with the vertical. Hence $\alpha_1 = 45°$, the optimization procedure thus consisting of determining α_2, α_3 and α_4 that produce a linkage meeting a set of input-output pairs with a minimum error. We aim at the minimization of the Euclidean norm of the design error over a rich set of input-output pairs, namely

$$z \equiv \frac{1}{2}\sum_{i=1}^{n} F_i^2 \quad \rightarrow \quad \min_{\alpha_2,\alpha_3,\alpha_4} \tag{35}$$

where

$$F_i = F(\vartheta_i, \gamma_i) \tag{36}$$

and $\{(\vartheta_i, \gamma_i)\}_1^n$ is a set of n points along the input-output function $F(\vartheta, \gamma)$ of eq.(34) and displayed in Fig. 14. When ϑ_0 and γ_0 are known, the foregoing minimization reduces to solving a linear least-square problem. For example, with $\vartheta_0 = 50°$ and $\gamma_0 = -80°$, we specify 40 pairs of equally-spaced input-output values $\{(\vartheta_i, \gamma_i)\}_1^{40}$. The results of the optimization are shown in Table 2.

In the optimization procedure we impose the condition that the input link be a crank. This can be readily done upon recalling the mobility criteria derived by Zheng and Angeles [23]:

$$(k_2 + k_1)^2 - (k_3 - k_4)^2 \leq 0, \quad (k_2 - k_1)^2 - (k_3 + k_4)^2 \leq 0$$

The transmission quality of the synthesized linkage is 0.9642, which is quite acceptable, the error $\gamma_{des} - \gamma_{gen}$ being plotted in Fig. 15.

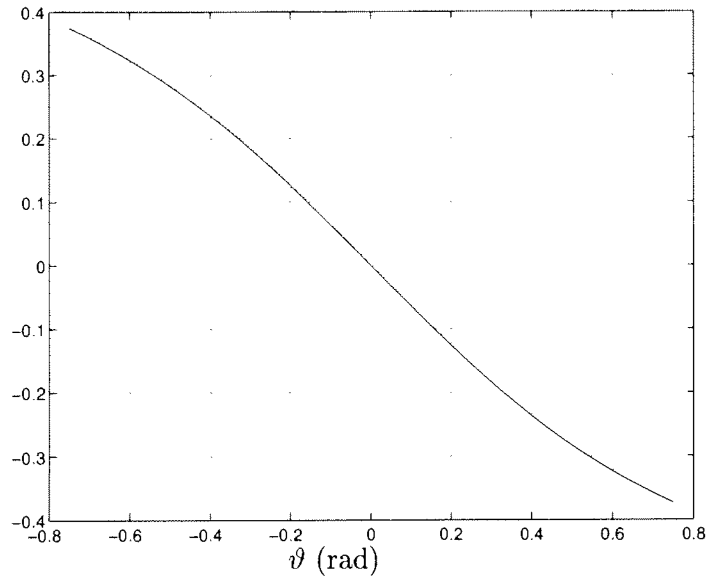


Figure 14: The input-output function of the spherical four-bar linkage

Table 2: Optimum linkage with $\vartheta_0 = 50°$ and $\gamma_0 = -80°$

k_1	k_2	k_3	k_4	α_1	α_2	α_3	α_4
0.1334	2.0196	0.7071	4.4809	45°	8.9675°	49.2807°	19.2961°

Now, we regard ϑ_0 and γ_0 as additional design variables to minimize the sum of the squares of the *structural error* at the given input values. The optimum results are $\vartheta_0 = -1.4825°$, $\gamma_0 = -92.554°$, the other parameters being shown in Table 3. The output link of the second linkage turns out to be a crank as well, with a transmission quality of 0.7689, which is still acceptable, while the error plot is shown in Fig. 16. Note that the second linkage offers an error which is one order of magnitude smaller

Table 3: Spherical linkage with optimum values of ϑ_0 and γ_0

k_1	k_2	k_3	k_4	α_1	α_2	α_3	α_4
-0.0991	0.0037	0.7071	2.2652	45°	17.3366°	88.1041o	89.6981°

than that of the first one. However, this is done at the expense of a lower transmission quality. In practice, such a small error is futile, for the tight tolerances required to implement it increases the production costs. However, the transmission quality impacts on the life span of the mechanism, and hence, becomes a more relevant performance index. In summary, the first linkage is preferred over the second one.

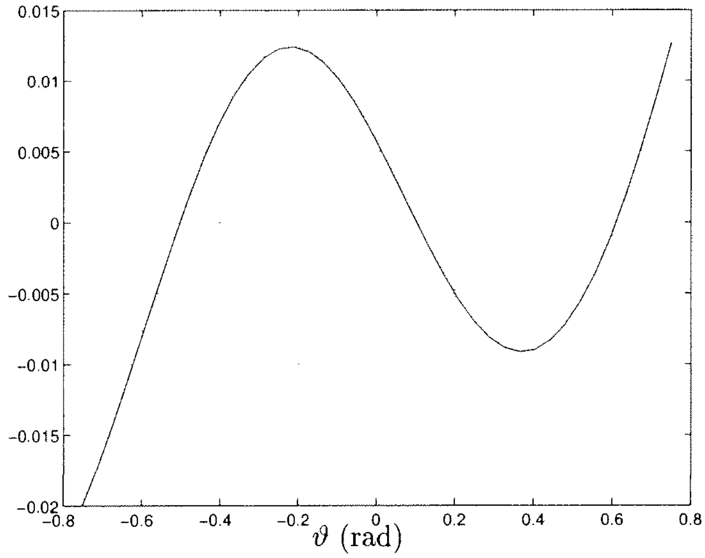


Figure 15: Error plot of the linkage of Table 3

5 Conclusions

A computational-kinematics approach was introduced to synthesize a steering four-bar linkage. The objective of this approach is to minimize the root-mean square value of the design error in the steering condition. We acknowledge that the minimization of the design error is not equivalent to that of the structural error. Nevertheless, we invoke a recent result indicating that the two errors become closer to each other as the cardinality of the data set of prescribed input-output pairs in the synthesis of function generators increases. By means of an elimination procedure, a polynomial equation in one unknown is derived from the associated normality condition, from which all local optima are computed and, hence, the global optimum can be readily found. Moreover, we synthesize a kinematically-equivalent six-bar focal mechanism, to replace the steering four-bar linkage, which turns out to show an unacceptable dimensional unbalance. The six-bar equivalent linkage shows much better-balanced dimensions. Finally, we propose an optimum spherical four-bar linkage to couple the steering-wheel axle with the steering mechanism. In fact, two candidate mechanisms are obtained, with slightly different performances. Based on practical considerations, the mechanism with the higher transmission quality, if with a lower accuracy, is preferred.

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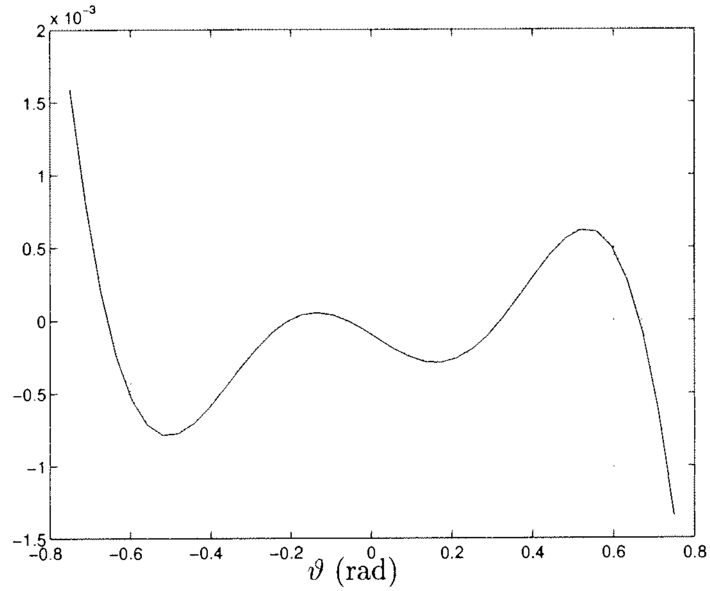

Figure 16: Error plot ($\gamma_{des} - \gamma_{\text{gen}}$)

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