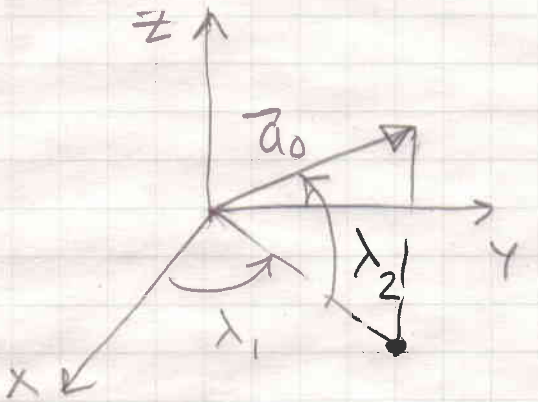
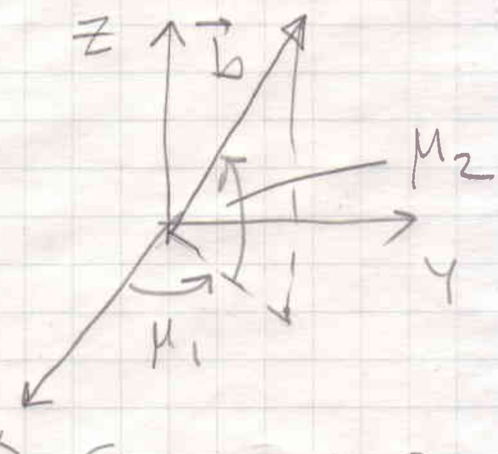
Approximate Synthesis of Spherical Four-bar Linkages for Rigid-body Guidance

Problem similar to planar case (Yao and Angeles, 2000)*. Major difference: $\vec{a}_0$ & $\vec{b}$ no longer independent, as they must obey

$$\|\vec{a}_0\| = 1, \quad \|b\| = 1 \qquad (1a), (1b)$$

Problem can still be formulated with independent unknowns, if spherical coordinates are introduced, as reported in (Angeles and Bai, 2010):





$$\Rightarrow \vec{a}_0 = \begin{bmatrix} \cos\lambda_1 \cos\lambda_2 \\ \sin\lambda_1 \cos\lambda_2 \\ \sin\lambda_2 \end{bmatrix} \quad (2a), \qquad \vec{b} = \begin{bmatrix} \cos\mu_1 \cos\mu_2 \\ \sin\mu_1 \cos\mu_2 \\ \sin\mu_2 \end{bmatrix} \quad (2b)$$

New unknowns: $\vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \mathbb{R}^2$ (3a), $\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \in \mathbb{R}^2$ (3b)

$\Rightarrow$ 4 independent scalar unknowns, just as in planar case.

Follow formulation in (Yao and Angeles, 2000):

$$z = \frac{1}{2}\sum_{1}^{m} f_j^2(\vec{\lambda}, \vec{\mu}) \rightarrow \min_{\vec{\lambda}, \vec{\mu}} \qquad (4)$$

$$f_j = \vec{a}_0^T(\underset{\sim}{Q}_j^T - \underset{\sim}{1})\vec{b} \quad (\neq 0), \; j = 1, \ldots, m > 4 \qquad (5)$$

$$\vec{a}_0 = \vec{a}_0(\vec{\lambda}) \quad (6a) \qquad \vec{b} = \vec{b}(\vec{\mu}) \quad (6b)$$

* Cited references found in Lecture Notes.

Normality conditions: $$\vec{x} = \begin{bmatrix}\vec{\lambda} \\ \vec{\mu}\end{bmatrix} \begin{matrix}\}\,2\\ \}\,2\end{matrix} \in \mathbb{R}^4 \qquad (7)$$

$$\nabla z = \frac{\partial z}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial z}{\partial \vec{\lambda}} \\ \dfrac{\partial z}{\partial \vec{\mu}} \end{bmatrix} = \begin{bmatrix} \vec{0}_2 \\ \vec{0}_2 \end{bmatrix} \qquad (8)$$

$$\frac{\partial z}{\partial \vec{\lambda}} = \sum_1^m f_j \frac{\partial f_j}{\partial \vec{\lambda}} = \vec{0}_2 \qquad (9a)$$

$$\frac{\partial z}{\partial \vec{\mu}} = \sum_1^m f_j \frac{\partial f_j}{\partial \vec{\mu}} = \vec{0}_2 \qquad (9b)$$

Chain rule (See Section 1.4.5 of LN):

$$\frac{\partial f_j}{\partial \vec{\lambda}} = \underbrace{\left(\frac{\partial \vec{a}_0}{\partial \vec{\lambda}}\right)^T}_{\equiv\, \mathbf{A} \in \mathbb{R}^{3\times 2}} \underbrace{\frac{\partial f_j}{\partial \vec{a}_0}}_{\in \mathbb{R}^3} \quad (10a), \qquad \frac{\partial f_j}{\partial \vec{\mu}} = \underbrace{\left(\frac{\partial \vec{b}}{\partial \vec{\mu}}\right)^T}_{\equiv\, \mathbf{B} \in \mathbb{R}^{3\times 2}} \underbrace{\frac{\partial f_j}{\partial \vec{b}}}_{\in \mathbb{R}^3} \quad (10b)$$

$$(5) \Rightarrow \frac{\partial f_j}{\partial \vec{a}_0} = (\mathbf{Q}_j^T - \mathbf{1})\vec{b} \quad (11a), \qquad \frac{\partial f_j}{\partial \vec{b}} = (\mathbf{Q}_j - \mathbf{1})\vec{a}_0 \quad (11b)$$

$$(10a) \,\&\, (11a) \text{ in } (9a) \Rightarrow \mathbf{A}^T \sum_1^m f_j (\mathbf{Q}_j^T - \mathbf{1})\vec{b} = \vec{0}_2 \qquad (12a)$$

$$(10b) \,\&\, (11b) \text{ in } (9b) \Rightarrow \mathbf{B}^T \sum_1^m f_j (\mathbf{Q}_j - \mathbf{1})\vec{a}_0 = \vec{0}_2 \qquad (12b)$$

Let $\vec{u}_1 \equiv \begin{bmatrix}\cos\lambda_1 \\ \sin\lambda_1\end{bmatrix}$, $\vec{u}_2 \equiv \begin{bmatrix}\cos\lambda_2 \\ \sin\lambda_2\end{bmatrix}$, $\vec{v}_1 \equiv \begin{bmatrix}\cos\mu_1 \\ \sin\mu_1\end{bmatrix}$, $\vec{v}_2 \equiv \begin{bmatrix}\cos\mu_2 \\ \sin\mu_2\end{bmatrix}$

$\vec{a}_0$ linear in $\vec{u}_1$ & $\vec{u}_2$, $\vec{b}$ linear in $\vec{v}_1$ & $\vec{v}_2$. Moreover,

$$\mathbf{A} = \left[\begin{array}{c|c} -\sin\lambda_1\cos\lambda_2 & -\cos\lambda_1\sin\lambda_2 \\ \cos\lambda_1\cos\lambda_2 & -\sin\lambda_1\sin\lambda_2 \\ 0 & \cos\lambda_2 \end{array}\right] \quad (13a), \qquad \mathbf{B} = \left[\begin{array}{c|c} -\sin\mu_1\cos\mu_2 & -\cos\mu_1\sin\mu_2 \\ \cos\mu_1\cos\mu_2 & -\sin\mu_1\sin\mu_2 \\ 0 & \cos\mu_2 \end{array}\right] \quad (13b)$$

$\Rightarrow \dfrac{\partial \vec{a}_0}{\partial \vec{\lambda}}$ linear in $\vec{u}_1$ & $\vec{u}_2$, $\dfrac{\partial \vec{b}}{\partial \vec{\mu}}$ linear in $\vec{v}_1$ & $\vec{v}_2$

$\underset{\sim}{A}$ ✓ ✓ ✓ ✓ , $\underset{\sim}{B}$ ✓ ✓ ✓ ✓

f_j bilinear in $\vec{a}_0$ & $\vec{b}$ $\Rightarrow$ bilinear in $\vec{u}_1, \vec{v}_1, \vec{u}_2, \vec{v}_2$

$\underset{\sim}{A}^T(\underline{Q}_j^T - \underline{1})\vec{b}$ bilinear in $\vec{u}_1, \vec{v}_1, \vec{u}_2, \vec{v}_2$

$\underset{\sim}{B}^T(\underline{Q}_j - \underline{1})\vec{a}_0$ ✓ ✓ ✓ ✓ ✓ ✓

$\dfrac{\partial f_j}{\partial \vec{a}_0}$ linear in $\vec{b}$ $\Rightarrow$ bilinear in $\vec{v}_1, \vec{v}_2$

$\dfrac{\partial f_j}{\partial \vec{b}}$ ✓ ✓ $\vec{a}$ $\Rightarrow$ ✓ ✓ $\vec{u}_1, \vec{u}_2$

$\Rightarrow$ RHS of (12a) quadratic in $\vec{u}_1, \vec{u}_2$, cubic in $\vec{v}_1, \vec{v}_2$

✓ ✓ (12b) ✓ $\vec{v}_1, \vec{v}_2$ ✓ $\vec{u}_1, \vec{u}_2$

$\Rightarrow$ 4 equations are quintic in $\{\vec{u}_i, \vec{v}_i\}_1^2$. Upon introduction of the tan-half identities

$$\cos\lambda_i = \frac{1-T_i^2}{1+T_i^2}, \quad \sin\lambda_i = \frac{2T_i}{1+T_i^2}, \quad i = 1, 2$$

$$\cos\mu_i = \frac{1-U_i^2}{1+U_i^2}, \quad \sin\mu_i = \frac{2U_i}{1+U_i^2}, \quad i = 1, 2$$

the foregoing equations become of 10th degree in $\{T_i, U_i\}_1^2$. As we have four such equations, their Bezout number is $10^4 = 10\,000$, or huge. This means that all four equations can be reduced to one single polynomial equation of one single variable, of degree $\leq 10\,000$. Not the way to go. Use contour intersection.