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DATA-CONDITIONING IN THE OPTIMIZATION OF FUNCTION-GENERATING LINKAGES

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Abstract

Rank-deficiencies and ill-conditioning of the synthesis matrix in the optimization of function-generating linkages are often caused by an improper selection of the input-output data points given by the $\{\psi_i,\ \phi_i\}_1^m$ pairs, where ψ and ϕ denote the input and output values, respectively. Ten basic cases of rank-deficiencies in the synthesis matrix are discussed in this paper, the associated curves, termed *singularity curves*, being plotted in the ψ-ϕ plane. Measures to remedy the ill-conditioning that arises in the optimization procedure and means to find the best-conditioned synthesis matrices by minimizing their condition number are also proposed.

1 Introduction

In the optimization of function-generating linkages, we are looking for the optimum dimensions of a linkage, defined by n parameters, which can produce a prescribed set of m input-output (I/O) pairs $\{\psi_i,\ \phi_i\}_1^m$ with a minimum error. The optimization procedure is often formulated as finding the optimum solution to an overdetermined linear system of equations

$$\mathbf{S}\mathbf{x} = \mathbf{b} \qquad (1)$$

where S and $\mathbf{b}$ are an $m \times n$ matrix and an m-dimensional vector containing the prescribed I/O pairs, respectively, whereas $\mathbf{x}$ is the n-dimensional vector of design variables. Here, we assume that $m > n$. In order to find $\mathbf{x}$ from eq.(1), we define an error vector $\mathbf{d}$, termed the *design error*, as

$$\mathbf{d} = \mathbf{b} - \mathbf{S}\mathbf{x} \qquad (2)$$

and minimize the Euclidean norm of $\mathbf{d}$, a procedure often referred to as *design-error minimization*. The linkage can also be optimized by minimizing the same norm of *structural error*, defined in terms of the difference between the desired and the actual output variable, i.e., the vector $\mathbf{s}$ whose ith component is given as

$$s_i = \varphi_i - \phi_i \qquad (3)$$

where φ_i stands for the actual value of the output angle ϕ attained at $\psi = \psi_i$. As the structural error minimization is a nonlinear procedure, the associated Jacobian matrix is often required in finding the optimum solution. It is known that this Jacobian matrix can be expressed in terms S (Tinubu and Gupta, 1984; Angeles, 1989), i.e., as

$$\frac{d\mathbf{s}}{d\mathbf{x}} = \left(\frac{\partial \mathbf{d}}{\partial \boldsymbol{\psi}}\right)^{-1}\mathbf{S} \qquad (4)$$

in which $\boldsymbol{\psi}$ is am m-dimensional vector defined as

$$\boldsymbol{\psi} = [\psi_1, \ldots, \psi_m]^T \qquad (5)$$

It is apparent that S, which we call the *synthesis matrix*, plays an important role in the optimization of function-generating linkages when minimizing a norm of either the design or the structural error. Since the numerical conditioning of this matrix affects the optimum results directly, a well-conditioned synthesis matrix is always desired. An ill-conditioned synthesis matrix, on the other hand, leads to failure of the optimization schemes and should be avoided.

Wilde (1982, 1983) discussed the singularities of the 3×3 matrix $\mathbf{S}^T\mathbf{S}$ in a least-square scheme for the design of planar function generators. In order to solve the problem when singularities occur, Wilde proposed to reduce the number of design variables and use a well-conditioned 2×2 matrix to replace the original matrix. It is apparent that the singularity of $\mathbf{S}^T\mathbf{S}$ is caused by a rank-deficient synthesis matrix $\mathbf{S}$. The rank-deficiency of $\mathbf{S}$, in turn, is often due to the selection of the I/O pairs $\{\psi_i,\ \phi_i\}_1^m$ in the optimization procedure. This means that an improper location of the design data in the ψ-ϕ plane gives rise to ill-conditioning, while a proper choice of the I/O pairs leads to a well-conditioned synthesis matrix. The objective of the research reported here is to investigate this issue and provide guidelines for the design of function generators.

Guidelines along the same lines have only been discussed in the context of precision-point synthesis (Freudenstein, 1959; Hall, 1961). These guidelines were derived mainly from experience and cannot be applied in linkage optimization. In this paper, we will examine the effect of I/O data on the synthesis matrix of different types of linkages and develop a systematic method to analyze the data conditioning in the design of function-generating linkages. Ten possible cases of I/O pairs that cause rank-deficiencies of the synthesis matrix, generally referred to as *singularities*, the worst-conditioned cases, are discussed here. We will study these cases in connection with commonly-used planar, spherical, and spatial linkages and will plot the corresponding curves, called *singularity curves*, in the ψ-ϕ plane. If the original I/O pairs lie along or close to a certain singularity curve, it is still possible to eliminate the singularities thus occurring with simple measures. We will discuss this issue and ways of improving the conditioning of synthesis matrices in this paper.

2 Condition Number

Because the synthesis matrix $\mathbf{S}$ consists of the sines and cosines of the input and output angles ψ and ϕ, respectively, the conditioning of this matrix depends directly upon the I/O pairs used in the optimization procedure. Thus, the question to be answered here is: what class of $\{\psi_i,\ \phi_i\}_1^m$ pairs lead to an ill-conditioned $\mathbf{S}$ matrix and how can we avoid ill-conditioning, if ill-conditioning can be avoided?

To tell whether a matrix is ill-conditioned or not, we recall the concept of *condition number*, which measures the amplification of the relative round-off error in the solution, caused by a perturbation of the data. For matrix $\mathbf{S}$, the condition number is defined here as

$$\kappa \equiv \text{cond}(\mathbf{S}) = \frac{\sigma_M}{\sigma_m} \tag{6}$$

where σ_M and σ_m are the largest and the smallest singular values of $\mathbf{S}$, respectively. Moreover, the condition number is bounded from below but unbound from above, i.e.,

$$1 \leq \kappa < \infty \tag{7}$$

A matrix is rank-deficient if its condition number is infinity; it is ill-conditioned if its condition number is very large. Thus, a well-conditioned matrix has a relatively small condition number, say of the order of 10. Sometimes, it is convenient to use the reciprocal of the condition number to describe the conditioning of a matrix, which we will denote by γ, i.e.,

$$\gamma = \frac{1}{\kappa} \tag{8}$$

Obviously, γ is not only bounded from below but also from above, i.e.,

$$0 \leq \gamma \leq 1 \tag{9}$$

and hence, γ is more convenient to work with than κ. A matrix is rank-deficient if $\gamma = 0$ and well-conditioned when γ is close to 1. Ill-conditioned matrices have a $\gamma << 1$.

3 Rank-Deficiencies in the Synthesis Matrix

Our discussion here focuses on four types of linkages, namely, planar *RRRR*, spherical *RRRR*, spatial *RCCC* and *RSSR* linkages. If m I/O pairs $\{\psi_i,\ \phi_i\}_1^m$ are selected for a problem, we can write the synthesis matrix in the form

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_m^T \end{bmatrix} \tag{10}$$

where the n-dimensional vectors $\mathbf{s}_i$, for $i = 1, \ldots, m$, are defined for different linkages as follows:

$$\begin{aligned}
\text{Planar } RRRR: &\quad \mathbf{s}_i = [1,\ \cos\phi_i,\ -\cos\psi_i]^T \\
\text{Spherical } RRRR: &\quad \mathbf{s}_i = [1,\ -\cos\phi_i,\ \cos\psi_i, \\
&\qquad \cos\phi_i \cos\psi_i]^T \\
\text{Spatial } RSSR: &\quad \mathbf{s}_i = [1,\ \cos\phi_i,\ \sin\phi_i,\ -\cos\psi_i, \\
&\qquad -\sin\psi_i,\ -\cos\phi_i \cos\psi_i]^T \\
\text{Spatial } RCCC: &\quad \mathbf{s}_i = [1,\ \sin\phi_i,\ \sin\psi_i, \\
&\qquad -\sin\phi_i \sin\psi_i]^T
\end{aligned}$$

Matrix **S** becomes rank-deficient if the I/O pairs selected for the optimization procedure verify certain conditions that render the columns of **S** linearly dependent. Summarized below are such conditions on I/O pairs that govern the basic cases of rank-deficiency of their synthesis matrices. As mentioned before, we call such situations *singularities*.

Singularity 1:

$$\sin\phi = c_1 \quad \text{or} \quad \cos\phi = c_1 \tag{11}$$

where c_1 is a constant bounded as $-1 \leq c_1 \leq 1$.

Singularity 2:

$$\sin\psi = c_2 \quad \text{or} \quad \cos\psi = c_2 \tag{12}$$

where c_2 is a constant bounded as $-1 \leq c_2 \leq 1$.

Singularity 3:

$$\cos\phi + c_3\cos\psi = 0 \tag{13}$$

where c_3 is a non-zero scalar.

Singularity 4:

$$\cos\phi\cos\psi = c_4, \tag{14}$$

where c_4 is a constant bounded as $-1 \leq c_4 \leq 1$.

Singularity 5:

$$\cos\phi + c_5\sin\psi = 0 \tag{15}$$

where c_5 is a non-zero scalar.

Singularity 6:

$$\cos\psi + c_6\sin\phi = 0 \tag{16}$$

where c_6 is a non-zero scalar.

Singularity 7:

$$\sin\phi + c_7\cos\phi\cos\psi = 0 \tag{17}$$

where c_7 is a non-zero scalar.

Singularity 8:

$$\sin\psi + c_8\cos\phi\cos\psi = 0 \tag{18}$$

where c_8 is a non-zero scalar.

Singularity 9:

$$\sin\phi + c_9\sin\psi = 0 \tag{19}$$

where c_9 is a non-zero scalar.

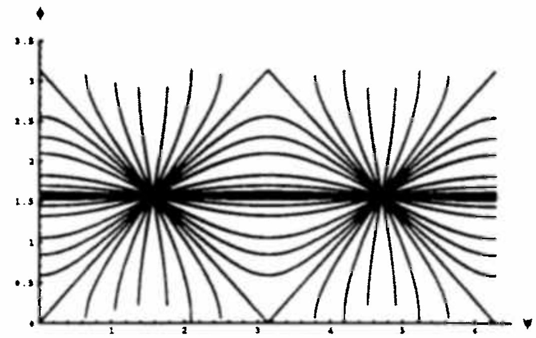

Figure 1: Singularity 3

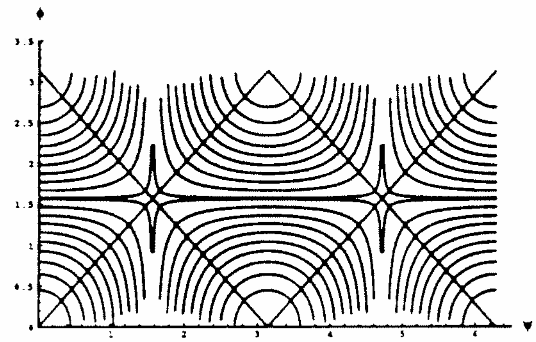

Figure 2: Singularity 4

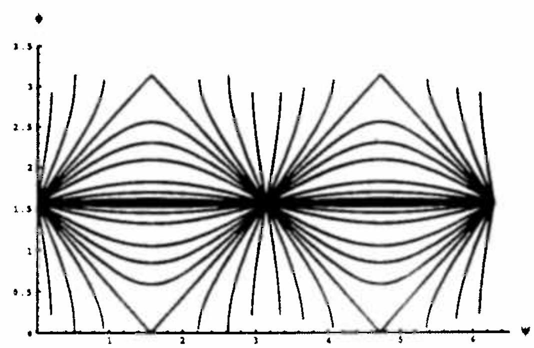

Figure 3: Singularity 5

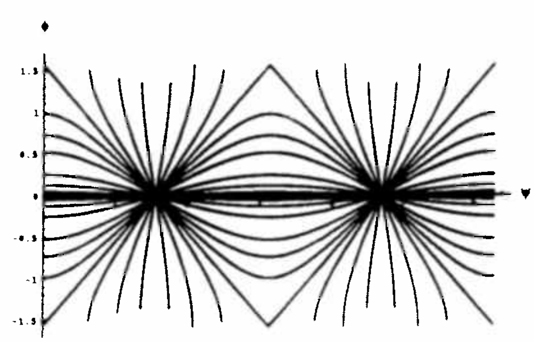

Figure 4: Singularity 6

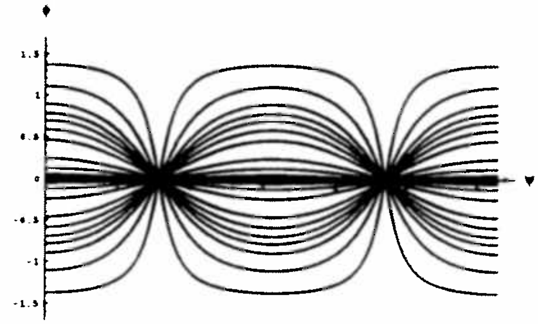

Figure 5: Singularity 7

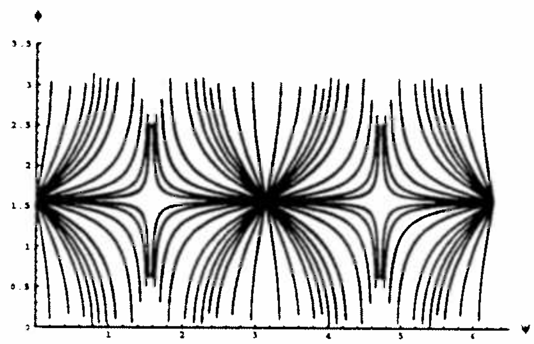

Figure 6: Singularity 8

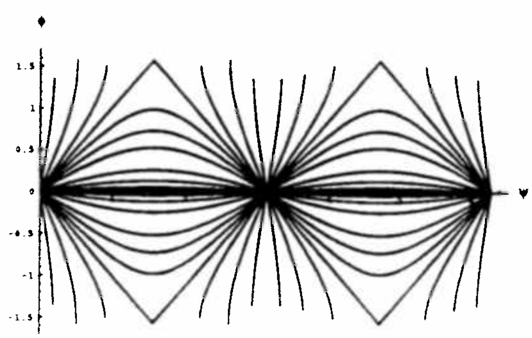

Figure 7: Singularity 9

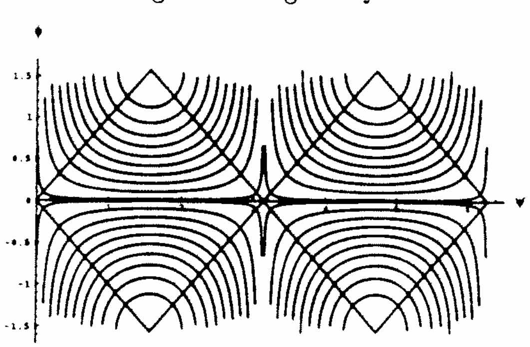

Figure 8: Singularity 10

Linkage Type	Possible Singularities
Planar *RRRR*	1, 2, 3
Spherical *RRRR*	1, 2, 3, 4
Spatial *RSSR*	1, 2, 3, 4, 5, 6, 7, 8, 9
Spatial *RCCC*	1, 2, 9, 10

Table 1: Singularities for different linkages

Singularity 10:

$$\sin\phi \sin\psi = c_{10} \qquad (20)$$

where c_{10} is a constant bounded as $-1 \leq c_{10} \leq 1$.

Geometrically, each condition in eqs.(11)–(20) represents a family of curves, called *singularity curves*, corresponding to different possible values of c_i, in the ψ-ϕ plane. As singularity curves (11) and (12) are readily visualized, we plot here curves (13)–(20), as shown in Figs. 1–8. I/O pairs selected directly from these curves lead to singularities in the synthesis matrices. A synthesis matrix with I/O pairs lying close to one of the relevant singularity curves will be very likely ill-conditioned.

Due to the different structures in the synthesis matrices, singularities associated with each type of linkage are different. Given in Table 1 are the possible singularities for different linkages.

4 Elimination of Singularities

If the I/O pairs happen to lie very close to or fall into one of the singularity curves discussed in the last section, we will encounter an ill-conditioned or rank-deficient synthesis matrix, which will very likely hamper our optimization procedure. In this section, we will discuss two basic measures that have proven to be effective in eliminating singularities in the optimization procedure.

4.1 Adding Supplementary Data Points

By adding supplementary points away from a certain singularity curve, the overall I/O data distribution in the ψ-ϕ plane changes and singularity can thus be eliminated. To show this idea, we use the I/O pairs given in Table 2, which lie very close to a curve of singularity 3. If we add supplementary points, away from the curve of singularity 3, into the data, say two pairs of I/O values (145°, 85°) and (155°, 55°), the condition

ψ (deg.)	ϕ (deg.)
15.512	36.585
29.017	43.220
42.576	52.146
55.813	62.080
68.021	71.827
81.007	82.515
97.891	96.570
112.245	108.389

Table 2: I/O pairs leading to singularity 3

number of the synthesis matrices will decrease significantly. To show this fact, we use planar $RRRR$, spherical $RRRR$ and spatial $RSSR$ linkages whose synthesis matrices are formed using the data of Table 2. The condition numbers of the synthesis matrices of these linkages, before and after adding the aforementioned supplementary points, are given in Table 3.

Linkages	κ (before)	κ (after)
Planar $RRRR$	2.87×10^5	4.14
Spherical $RRRR$	3.05×10^5	8.39
Spatial $RSSR$	4.29×10^5	95.79

Table 3: Examples of adding supplementary points

A scheme for the optimization of function-generating linkages based on I/O curve planning has been proposed earlier (Liu and Angeles, 1992-1,2). In using this method, it is always necessary to add supplementary data points to satisfy certain design requirements while rendering the I/O curves of a desired shape. As we can see here, adding supplementary points can also be used to improve the conditioning of the synthesis matrix at hand, thus deriving a direct and simple way of eliminating singularities.

4.2 Changing the Zeros of the Input and Output Dials

Changing the zeros in the input and output dials can also improve the conditioning of the synthesis matrices. Here, we can regard the original I/O pairs as a set of input and output angle increments $\{\Delta\psi_i,\ \Delta\phi_i\}_1^m$. If α and β represent the zeros of the dials of the input and the output angle, respectively, we then have

$$\psi_i = \alpha + \Delta\psi_i, \qquad \phi_i = \beta + \Delta\phi_i, \qquad i = 1, \ldots, m \quad (21)$$

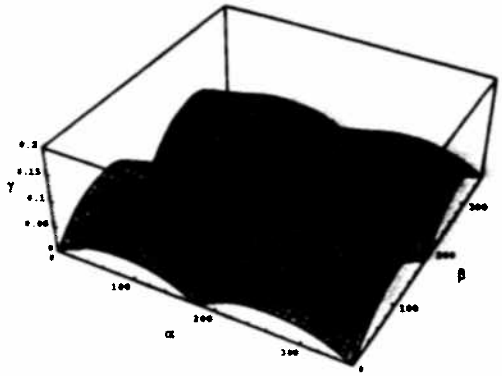

Figure 9: Distribution of γ of planar linkage

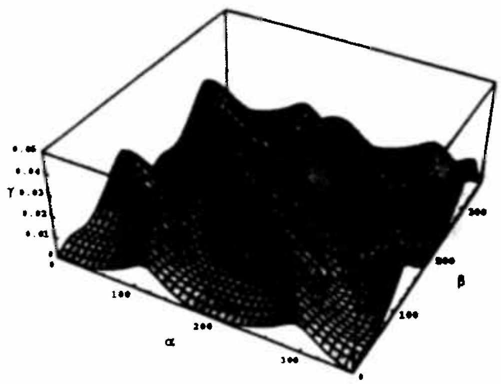

Figure 10: Distribution of γ of spherical linkage

Geometrically, this means that we move all the points in the ψ-ϕ plane under a pure translation given by the values of α and β so that they lie farthest away from the singularity curves. Again, we use the data given in Table 2 to show how this idea works. It is found that, if we choose $\alpha = 25°$ and $\beta = 30°$ for these I/O pairs, the associated condition number κ for a planar linkage changes from 2.87×10^5 to 126.1, or $\gamma = 7.9 \times 10^{-3}$, the improvement thus being significant. As different α and β values lead to different condition numbers of the synthesis matrix, various values of α and β have been tried with the I/O pairs given in Table 2 for planar and spherical linkages and the associated κ and γ values have been obtained. Plotted in Figs. 9 and 10 are the γ values for different values of α and β associated with planar and spherical linkages.

It should be pointed out that, although changing the zeros of the dials of the input and output angles improves the condition number of planar $RRRR$, spherical $RRRR$ and spatial $RCCC$ linkages, this technique cannot be applied to the spatial $RSSR$ linkage when its synthesis matrix encounters singularities 3, 5, 6 and 9. The reason is that, regardless of the locations of the zero on the dials of the input and the output angle, some column vectors in the synthesis matrix are linearly dependent. A proof of this claim is given below.

With the zeros of the dials of the input and the output angles of a spatial *RSSR* linkage set at values α and β, respectively, the synthesis matrix can be written columnwise as

$$\mathbf{S} = [\mathbf{c}_1,\ \mathbf{c}_2\ \ \mathbf{c}_3\ \ \mathbf{c}_4\ \ \mathbf{c}_5\ \ \mathbf{c}_5\ \ \mathbf{c}_6] \qquad (22)$$

where

$$\mathbf{c}_1 = [1,\ \ldots,\ 1]^T \qquad (23)$$
$$\mathbf{c}_2 = [\cos(\beta+\Delta\phi_1),\ \ldots,\ \cos(\beta+\Delta\phi_m)]^T \qquad (24)$$
$$\mathbf{c}_3 = [\sin(\beta+\Delta\phi_1),\ \ldots,\ \sin(\beta+\Delta\phi_m)]^T \qquad (25)$$
$$\mathbf{c}_4 = [\cos(\alpha+\Delta\psi_1),\ \ldots,\ \cos(\alpha+\Delta\psi_m)]^T \qquad (26)$$
$$\mathbf{c}_5 = [\sin(\alpha+\Delta\psi_1),\ \ldots,\ \sin(\alpha+\Delta\psi_m)]^T \qquad (27)$$
$$\mathbf{c}_6 = [\cos(\beta+\Delta\phi_1)\cos(\alpha+\Delta\psi_1),\ \ldots,\ \cos(\beta+\Delta\phi_m)\cos(\alpha+\Delta\psi_m)]^T \qquad (28)$$

Expanding $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$ and $\mathbf{c}_5$, we obtain

$$\mathbf{c}_2 = \begin{bmatrix} \cos\beta\cos\Delta\phi_1 - \sin\beta\sin\Delta\phi_1 \\ \vdots \\ \cos\beta\cos\Delta\phi_m - \sin\beta\sin\Delta\phi_m \end{bmatrix}, \qquad (29)$$

$$\mathbf{c}_3 = \begin{bmatrix} \sin\beta\cos\Delta\phi_1 + \cos\beta\sin\Delta\phi_1 \\ \vdots \\ \sin\beta\cos\Delta\phi_m + \cos\beta\sin\Delta\phi_m \end{bmatrix}, \qquad (30)$$

$$\mathbf{c}_4 = \begin{bmatrix} \cos\alpha\cos\Delta\psi_1 - \sin\alpha\sin\Delta\psi_1 \\ \vdots \\ \cos\alpha\cos\Delta\psi_m - \sin\alpha\sin\Delta\psi_m \end{bmatrix}, \qquad (31)$$

$$\mathbf{c}_5 = \begin{bmatrix} \sin\alpha\cos\Delta\psi_1 - \cos\alpha\sin\Delta\psi_1 \\ \vdots \\ \sin\alpha\cos\Delta\psi_m - \cos\alpha\Delta\sin\psi_m \end{bmatrix} \qquad (32)$$

If a set of I/O pairs is given that leads to singularity 3, i.e., if the set verifies eq.(13), then $\mathbf{c}_2$ and $\mathbf{c}_3$ can be written as

$$\mathbf{c}_2 = \begin{bmatrix} c_3\cos\beta\cos\Delta\psi_1 - \sin\beta\sin\Delta\phi_1 \\ \vdots \\ c_3\cos\beta\cos\Delta\psi_m - \sin\beta\sin\Delta\phi_m \end{bmatrix}, \qquad (33)$$

$$\mathbf{c}_3 = \begin{bmatrix} c_3\sin\beta\cos\Delta\psi_1 + \cos\beta\sin\Delta\phi_1 \\ \vdots \\ c_3\sin\beta\cos\Delta\psi_m + \cos\beta\sin\Delta\phi_m \end{bmatrix} \qquad (34)$$

From eqs.(31), (32), (33), and (34), we can readily obtain the relation

$$\mathbf{c}_2\cos\beta + \mathbf{c}_3\sin\beta = c_3(\mathbf{c}_4\cos\alpha + \mathbf{c}_5\sin\alpha) \qquad (35)$$

which means that the column vectors $\mathbf{c}_2$, $\mathbf{c}_3$ $\mathbf{c}_4$ and $\mathbf{c}_5$ are linearly dependent, regardless of the values of the α and β. Hence, the synthesis matrix is rank-deficient for any values of α and β. Similarly, we can prove that the matrix of eq.(22) is rank-deficient for the I/O pairs leading to singularities 5, 6 and 9. Thus, if these singularities occur in the synthesis matrix of *RSSR* linkages, we have to rely on adding supplementary points to eliminate them.

ψ (deg.)	ϕ (deg.)
15	343
37	324
50	312
80	279
103	257
145	216
197	162
241	119

Table 4: Non-singular I/O pairs

By changing the zeros of the dials of the input and output angles, we can also improve the conditioning of a synthesis matrix, even if it is of full-rank, i.e., minimize its condition number. Given in Table 4 is a set of I/O pairs which, when used for planar function generators, leads to a synthesis matrix with a condition number $\kappa = 114.1$, or $\gamma = 8.76 \times 10^{-3}$. The 3-dimensional plot of γ against different values of α and β is shown in Fig. 11. Moreover, the *isoconditioning contours*, i.e., the contours of constant condition number in the α-β plane, are plotted in Fig. 12, from which values of α and β corresponding to a minimum condition number can be readily selected. For example, if we use $\alpha = 78°$ and $\beta = 0°$ based on the display of Fig. 12, we then end up with a new condition number $\kappa = 2.18$ or $\gamma = 0.459$, the improvement on the conditioning of the synthesis matrix thus being very significant. A scheme is presented in Section 5 to find values of α and β that minimize the condition number of the synthesis matrices.

5 Condition-Number Minimization

From Subsection 4.2, it is apparent that changing the locations of the zeros of the input and output dials can eliminate singularities and improve the conditioning of the synthesis matrix. Here, we will discuss how to find

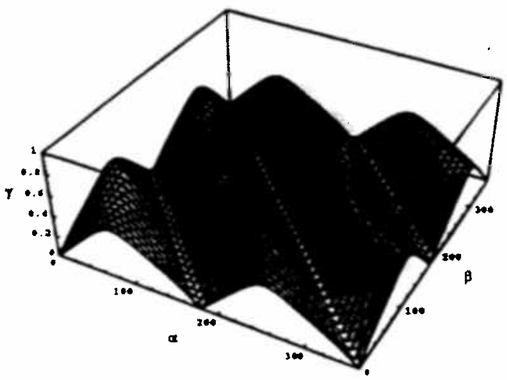

Figure 11: Distribution of γ for the non-singular data

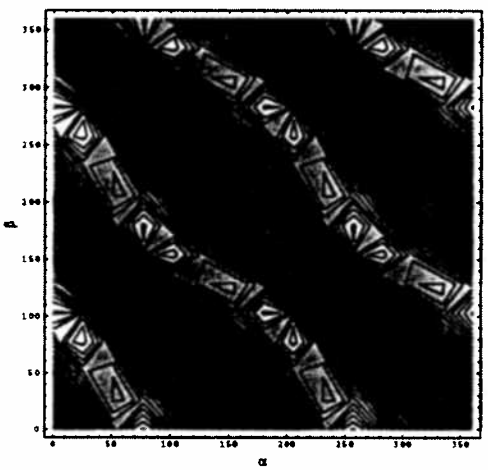

Figure 12: Isoconditioning contours in the α-β plane

the values of α and β that lead to optimally-conditioned synthesis matrices.

If we regard the original I/O pairs as a set of increments, as in eq.(21), the condition number κ can then be written as a function of α and β, i.e.,

$$\kappa = \kappa(\alpha, \ \beta) \qquad (36)$$

from which κ can be minimized over α and β. Since the condition number κ, as defined in eq.(6), is computed using the singular values of the synthesis matrix S, it is very difficult to obtain an explicit expression of κ in terms of α and β as well as of its gradient. Thus, the *polytope algorithm* (Nelder and Mead, 1965), a direct-search method that does not require the gradient of the objective function, is employed here to find the minimum condition number. Using this method, we can locate the minimum condition number κ_{min} of the synthesis matrices of planar and spherical linkages using the I/O data given in Table 2. With the help of the plots in Figs. 9 and 10, proper initial guesses α_0 and β_0 are first located. The optimum values of α_{opt} and β_{opt} are then found using the aforementioned method, results being given in Table 5.

Linkages	Planar *RRRR*	Spherical *RRRR*
$\alpha_0, \ \beta_0$	23°, 0°	195°, 115°
$\alpha_{opt}, \ \beta_{opt}$	102.9°, 16.6°	216.8°, 99.5°
κ_{min}	13.4	62.7

Table 5: Examples of condition-number minimization

6 Conclusions

Rank-deficiencies and ill-conditioning of synthesis matrices in the optimization of function-generating linkages are caused by the distribution of the I/O data. Geometrically, if the points representing the given I/O pairs $\{\psi_i, \ \phi_i\}_1^m$ in the ψ-ϕ plane lie on one of the ten singularity curves reported in this paper, the synthesis matrix will become rank-deficient. Two methods have been proposed to cope with this problem, i.e., changing of zeros of the dials of ψ and ϕ or adding supplementary points to the I/O set. The conditioning of the synthesis matrix can also be improved through minimizing its condition number. Numerical examples are provided in the paper.

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