Least-Square Optimization of Planar and Spherical Four-Bar Function Generator Under Mobility Constraints

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An optimization scheme for four-bar function generators under mobility constraints, which can be applied to both planar and spherical four bar linkages, is presented in this paper. The design error, *defined as the residual in the input-output equation, is minimized over the vector of linkage parameters. The mobility constraints, given as a set of inequalities, are converted into equalities by introducing slack variables. The problem is thus formulated as an equality-constrained minimization problem, which is then solved using the orthogonal-decomposition algorithm, an iterative numerical method introduced elsewhere. To reduce the dimensional unbalance, which often occurs in solving a synthesis problem, a penalty function is combined with the original objective function, whose minimization leads to dimensionally balanced linkages. A numerical example is included.*

1 Introduction

The synthesis of four-bar function-generating linkages is a classical topic in kinematics. The problem consists of finding all the relevant parameters of a four-bar linkage that can produce a prescribed set of m input-output pairs $\{\psi_i, \phi_i\}_1^m$, where ψ and ϕ denote the input and output variables, respectively, as shown in Figs. 1 and 2. Let n be the number of independent parameters required to define the four-bar linkage. For planar linkages we have $n = 3$, while for spherical linkages $n = 4$. The synthesis problem can be termed *exact* or *approximate*, depending on the relationship between m and n. This paper focuses on the approximate-synthesis problem, the exact synthesis being regarded as a particular case of the former. Thus, the approximate error needs to be established first. In this context, two types of error are usually defined, namely *design error* and *structural error* (Tinubu and Gupta, 1984). The former is based on the error residual in the input-output equation, which was first proposed for planar linkages by Freudenstein (1955) and was later extended to the spherical and spatial cases by Hartenberg and Denavait (1964). The latter is defined as the difference between the actual and the desired output angles corresponding to an input angle. The two types of error are related to each other as shown by Tinubu and Gupta (1984) and Angeles (1989).

As far as the approximate synthesis of four-bar function generators is concerned, intensive research work has been done in the past. Freudenstein (1955) first proposed an algebraic formulation for the approximate synthesis equation, which was later used and extended by other researchers (Suh and Rad-

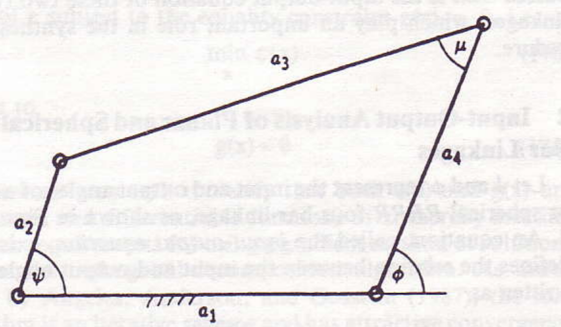

Fig. 1 Planar *RRRR* four-bar linkage

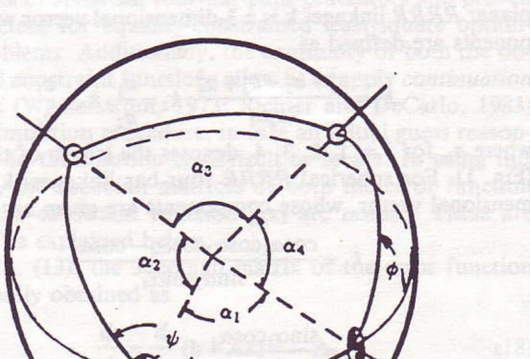

Fig. 2 Spherical *RRRR* four-bar linkage

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cliffe, 1967; Suh, 1968; Mohan Rao et al., 1973; Luck, 1976). Wilde (1982, 1983) applied error linearization techniques for the synthesis problem, while Tinubu and Gupta (1984) solved the problem with elimination of branch defect. However, very few papers have dealt with the mobility condition directly in the synthesis procedure. Although many criteria for mobility analysis of different types of linkages have been developed, including them in the optimization procedure remains a problem. The main difficulty here is the inequalities involved, which increase the complexity of the synthesis procedure. In many practical problems concerning the design of four-bar function generators, mobility requirements are present, and in most of the cases, using a set of inequalities to define the full-mobility region of the input or output link of a four-bar linkage becomes inevitable. The significance for devising a general method for tackling these constraints is hence apparent.

The optimization scheme presented in this paper aims at minimizing the design error in solving the problem of synthesis of a four-bar function generator under mobility conditions. The objective function is written in a quadratic form and the whole problem is formulated as a constrained nonlinear least-square minimization problem. By introducing slack variables, the inequalities representing the mobility conditions are readily transformed into equalities, which allows us to solve the problem in the context of equality-constrained nonlinear least-squares, for whose solution very effective procedures are available. Thus, the arising problem is solved using the orthogonal-decomposition algorithm, as introduced in Angeles, Anderson, and Gosselin (1987). Additionally, penalty terms are added to the objective function in order to eliminate the problems of dimensional unbalance. The optimization procedure is illustrated with a numerical example.

The method presented here can be used for the synthesis of planar and spherical four-bar function generators. Spatial four-bar linkages, however, do not lead to simple equality mobility constraints, and hence, are not considered in this paper. Discussed next is the input-output equation of these two types of linkages, which play an important role in the synthesis procedure.

2 Input-Output Analysis of Planar and Spherical Four-Bar Linkages

Let ψ and ϕ represent the input and output angles of a planar or spherical *RRRR* four-bar linkage, as shown in Figs. 1 and 2. An equation, called the *input-output equation*, exists that defines the relation between the input and output angles. It is written as

$$f(\psi,\phi,\mathbf{k})=0 \tag{1}$$

where $\mathbf{k}$ is known as the vector of linkage parameters. For the planar *RRRR* linkage, $\mathbf{k}$ is a 3-dimensional vector whose components are defined as

$$k_1=\frac{a_1^2+a_2^2-a_3^2+a_4^2}{2a_2a_4},\ k_2=\frac{a_1}{a_2},\ k_3=\frac{a_1}{a_4} \tag{2}$$

where a_i, for $i = 1, 2, 3, 4$, denotes the length of the ith link (Fig. 1). For spherical *RRRR* four-bar linkages, $\mathbf{k}$ is a 4-dimensional vector, whose components are given as

$$\begin{aligned} k_1&=\frac{\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3}{\sin\alpha_2\sin\alpha_4}\\ k_2&=\frac{\sin\alpha_1\cos\alpha_4}{\sin\alpha_4}\\ k_3&=\cos\alpha_1\\ k_4&=\frac{\sin\alpha_1\cos\alpha_2}{\sin\alpha_2} \end{aligned} \tag{3}$$

where α_i, for $i = 1, 2, 3, 4$, is the angle between each two adjacent joint axes, as shown in Fig. 2. From the known value of $\mathbf{k}$, the link lengths in terms of its components are obtained by simply inverting the above equations. In the planar case, the inversion yields

$$\begin{aligned} a_1&=1,\quad a_2=\frac{1}{k_2},\\ a_3&=\frac{\sqrt{k_2^2+k_3^2+k_2^2k_3^2-2k_1k_2k_3}}{|k_2k_3|}\\ a_4&=\frac{1}{k_3} \end{aligned} \tag{4}$$

while in the spherical case, we have the following:

$$\begin{aligned} \sin\alpha_1&=\sqrt{1-k_3^2},\quad \cos\alpha_1=k_3\\ \sin\alpha_2&=\sqrt{\frac{1-k_3^2}{1+k_4^2-k_3^2}},\quad \cos\alpha_2=\frac{k_4}{\sqrt{1+k_4^2-k_3^2}},\\ \sin\alpha_3&=\sqrt{1-\frac{[k_2k_3k_4-k_1(1-k_3^2)]^2}{(1-k_3^2+k_4^2)(1+k_2^2-k_3^2)}},\\ \cos\alpha_3&=\frac{[k_2k_3k_4-k_1(1-k_3^2)]}{\sqrt{(1-k_3^2+k_4^2)(1+k_2^2-k_3^2)}},\\ \sin\alpha_4&=\sqrt{\frac{1-k_3^2}{1+k_2^2-k_3^2}},\quad \cos\alpha_4=\frac{k_2}{\sqrt{1+k_2^2-k_3^2}} \end{aligned} \tag{5}$$

where all the angles are assumed to lie within the range from 0 to π, and $k_3^2 < 1$.

Given the set of the input-output pairs $\{\psi_i,\ \phi_i\}_1^m$, we can obtain a system of m equations from Eq. (1), namely,

$$f(\psi_i,\phi_i,\mathbf{k})=0,\quad i=1,\,2,\,\ldots,\,m \tag{6}$$

which is linear in $\mathbf{k}$, and hence, Eq. (6) can be written in the form of

$$\mathbf{Ak}=\mathbf{b} \tag{7}$$

For planar linkages, $\mathbf{A}$ and $\mathbf{b}$ are given as

$$\mathbf{A}=\begin{bmatrix} 1 & \cos\phi_1 & -\cos\psi_1\\ 1 & \cos\phi_2 & -\cos\psi_2\\ \vdots & \vdots & \vdots\\ 1 & \cos\phi_m & -\cos\psi_m \end{bmatrix}, \tag{8a}$$

Table 1 Mobility conditions of planar and spherical *RRRR* four-bar linkages

Linkage type	Conditions for Full-Mobolity	
	Input link	Output link
Planar *RRRR* four-bar linkage	$(k_1+k_3)^2 \le (1+k_2)^2$ $(k_1-k_3)^2 \le (1-k_2)^2$	$(k_1+k_2)^2 \le (1+k_3)^2$ $(k_1-k_2)^2 \le (1-k_3)^2$
Spherical *RRRR* four-bar linkage	$(k_2+k_1)^2 \le (k_3-k_4)^2$ $(k_2-k_1)^2 \le (k_3+k_4)^2$	$(k_1-k_4)^2 \le (k_2+k_3)^2$ $(k_1+k_4)^2 \le (k_2-k_3)^2$

$$\mathbf{b}=\begin{bmatrix}\cos(\psi_1-\phi_1)\\ \cos(\psi_2-\phi_2)\\ \vdots\\ \cos(\psi_m-\phi_m)\end{bmatrix} \tag{8b}$$

and, in the spherical case, we have

$$\mathbf{A}=\begin{bmatrix}1 & \cos\psi_1 & \cos\psi_1\cos\phi_1 & -\cos\phi_1\\ 1 & \cos\psi_2 & \cos\psi_2\cos\phi_2 & -\cos\phi_2\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \cos\psi_m & \cos\psi_m\cos\phi_m & -\cos\phi_m\end{bmatrix}, \tag{9a}$$

$$\mathbf{b}=-\begin{bmatrix}\sin\psi_1\sin\phi_1\\ \sin\psi_2\sin\phi_2\\ \vdots\\ \sin\psi_m\sin\phi_m\end{bmatrix} \tag{9b}$$

Clearly, **A** and **b** are functions of the given input-output pairs and the ith row of **A** and the ith component of **b** are functions of ψ_i and ϕ_i only. Equation (7) is the basic equation in forming the objective function in the optimization procedure.

3 Treatment of Mobility Conditions

In this section, the constraint relations suitable for our problem are derived. The mobility conditions of four-bar planar and spherical linkages have been extensively studied in the past (Grashof, 1883; Bricard, 1927; Duditza and Dittrich, 1969; Gupta and Radcliffe, 1971; Angeles and Bernier, 1987). For our problem, the mobility criterion for planar four-bar linkages presented in Gosselin and Angeles (1988) and the one for spherical four-bar linkages given in Liu (1988) are employed. They are stated in Table 1.

The conditions of Table 1 are the basic inequalities for deriving our constraint equations. Notice that these constraints are given in terms of quadratic positive functions on both sides of the inequalities listed in Table 1. We can make both sides equal by simply adding a third positive term on the smaller side, thereby replacing the inequalities by equality constraints. This additional term is chosen as the square of an additional variable, which will be taken into account in the optimization procedure. For example, to obtain the expression for the full-mobility condition of the input link of a spherical four-bar linkage, we introduce two slack variables, k_5 and k_6, which leads to the following:

$$g_1\equiv(k_2+k_1)^2-(k_3-k_4)^2+k_5^2=0 \tag{10a}$$

$$g_2\equiv(k_2-k_1)^2-(k_3+k_4)^2+k_6^2=0 \tag{10b}$$

and k_5 and k_6 will be included as additional design parameters in the optimization procedure. This also applies to other inequalities in Table 1. This will ease the problem formulation, and, eventually, its solution.

4 Optimization Scheme

Now the method for the optimum synthesis of four-bar function generators with mobility considerations on the input or output link is outlined. As mentioned before, the optimization scheme focuses on minimizing the design error, which is defined as the error residuals in the input-output equation. If we use **e** to represent this error, we have

$$\mathbf{e}=\mathbf{b}-\mathbf{A}\mathbf{k} \tag{11}$$

where **e** and **b** are m-dimensional vectors, m being the number of prescribed input-output pairs: **A** is an $m \times n$ matrix and **k** is an n-dimensional vector, where $n = 3$ for planar linkages and $n = 4$ for spherical ones. If $m > n$, Eq. (7) is an overdetermined system of equations and the Euclidean norm of **e** can be minimized directly with the value of **k** given as (Golub and Van Loan, 1983):

$$\mathbf{k}=(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{12}$$

However, this direct scheme will not guarantee the optimal linkage to meet any mobility conditions. Here, we propose a least-square optimization scheme subject to equality constraints that will satisfy the mobility conditions. We will use **x** as the design vector and express the design error as

$$\mathbf{e}=\mathbf{b}-\mathbf{A}\mathbf{x} \tag{13}$$

The objective function is written as follows:

$$z(\mathbf{x})=\frac{1}{2}\,\mathbf{e}^T\mathbf{W}\mathbf{e} \tag{14}$$

where **W** is a positive-definite constant weight matrix which accounts for the necessary normalization and scaling.

Now that the inequality constraints have been properly treated and converted into equality form as shown in Eqs. (10a and b), we can use them directly in the formulation. To this end, they are grouped in a vector function in the form of

$$\mathbf{g}(\mathbf{x})=\mathbf{0} \tag{15}$$

where **0** is the zero vector, whose dimension, identical to that of **g**, depends on the number of scalar equations imposed by the mobility conditions. The design vector **x** is defined as:

$$\mathbf{x}=[k_1,\ldots,k_n,\,k_{n+1},\ldots,k_{n+l}]^T \tag{16}$$

where k_i for $i = 1, \ldots, n$ are the linkage parameters as defined in Section 2, while k_i for $i = n+1, \ldots, n+l$ are the slack variables, which are constraint-condition dependent; for example, if the full mobility condition is imposed only on the input or output link, then $l = 2$; if these conditions are imposed on both the input and output links, then $l = 4$. Due to the extra variables, matrix **A** in Eq. (13) will have extra zero columns comparing with those given in Eqs. (8a and 9a). Therefore, the problem is formulated as minimizing the Euclidean norm of **e** subject to the equality constraint $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, i.e.,

$$\min_{\mathbf{x}} z(\mathbf{x}) \tag{17a}$$

subject to

$$\mathbf{g}(\mathbf{x})=\mathbf{0} \tag{17b}$$

It is now an easy task to verify that both $\mathbf{e}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are continuous and differentiable functions of **x**, thereby meeting the basic requirements for using a gradient method of solution, such as the orthogonal-decomposition algorithm. As introduced by Angeles, Anderson, and Gosselin (1987), the said algorithm is an iterative scheme and has attractive convergence properties. It has been used by Liu (1988) to solve the problem of synthesis of spherical four-bar path generators and proved to be efficient for equality-constrained least-square optimization problems. Additionally, the continuity of both the objective and constraint functions allow us to apply *continuation* techniques (Wasserstrom, 1973; Richter and DeCarlo, 1983) in the optimization procedure, in case an initial guess reasonably close to the solution is difficult to locate. In using this algorithm, the Jacobian matrices of both the error function $\mathbf{e}(\mathbf{x})$ and the constraint function $\mathbf{g}(\mathbf{x})$ are needed. These are computed as explained below.

From Eq. (13), the Jacobian matrix of the error function can be readily obtained as

$$\frac{d\mathbf{e}}{d\mathbf{x}}=\frac{d}{d\mathbf{x}}(\mathbf{b}-\mathbf{A}\mathbf{x})=-\mathbf{A} \tag{18}$$

while the Jacobian matrix of the constraint function does not have a general form and is dealt with on a case-by-case basis.

One problem that may arise in the optimization procedure is dimensional unbalance, which sometimes leads to very large link length ratios. A means to cope with this relies on the

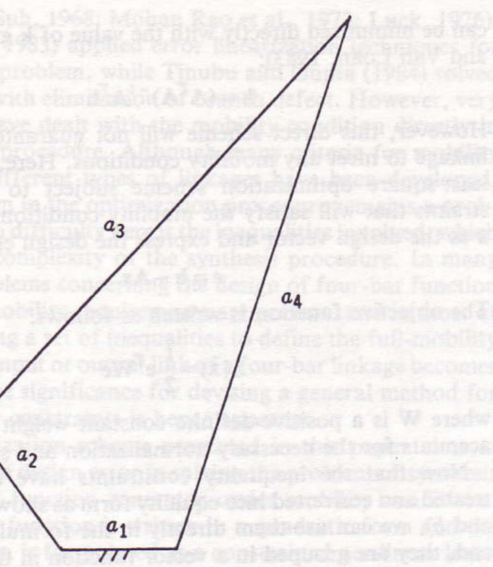

Fig. 3 The optimum linkage

introduction of penalty terms in the objective function, which are defined in such a way that their value increases with dimensional unbalance and decreases when the link lengths become proportionate. To suit the formulation adopted at the outset, based on least-squares, we write the penalty term in a quadratic form. For example, if we introduce a p-dimensional penalty function $\mathbf{p}(\mathbf{x})$, we can write it in the following form

$$\zeta(\mathbf{x}) = \frac{1}{2}\,\mathbf{p}^T\mathbf{R}\mathbf{p} \tag{19}$$

where $\mathbf{R}$ is a diagonal matrix containing penalty factors, i.e.,

$$\mathbf{R} = \mathrm{diag}(r_1, r_2, \ldots, r_p) \tag{20}$$

while $\mathbf{p}$ is suitably defined as explained in the example. Moreover, the penalty factors r_i, for $i = 1, \ldots, p$, will be adjusted in the optimization procedure. Now a new objective function $Z(\mathbf{x})$ can be written as

$$Z(\mathbf{x}) = z(\mathbf{x}) + \zeta(\mathbf{x})$$

$$= \frac{1}{2}\,\mathbf{e}^T\mathbf{W}\mathbf{e} + \frac{1}{2}\,\mathbf{p}^T\mathbf{R}\mathbf{p}$$

$$= \frac{1}{2}\,\mathbf{f}^T\mathbf{S}\mathbf{f} \tag{21}$$

where

$$\mathbf{f} = \begin{bmatrix} \mathbf{e} \\ \mathbf{p} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{W} & \mathbf{0}_{m\times p} \\ \mathbf{0}_{p\times m} & \mathbf{R} \end{bmatrix} \tag{22}$$

$\mathbf{0}_{p\times m}$ and $\mathbf{0}_{m\times p}$ being the $p \times m$ and $m \times p$ zero matrices, respectively. The Jacobian matrix of the new function $\mathbf{f}(\mathbf{x})$ is computed as

$$\frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \dfrac{d\mathbf{e}}{d\mathbf{x}}, & \dfrac{d\mathbf{p}}{d\mathbf{x}} \end{bmatrix}^T \tag{23}$$

where $d\mathbf{e}/d\mathbf{x}$ has already computed in Eq. (18) and $d\mathbf{p}/d\mathbf{x}$ is problem dependent.

5 Example

A design problem is solved here to show the application of the optimization method presented in this paper.

It is required to design a planar *RRRR* four-bar linkage to meet the input-output relations shown in Table 2. Moreover, the input link should be a crank.

The problem consists of seven given pairs of input-output angles, i.e., $m = 7$. For a planar linkage, $n = 3$, and hence

Table 2 Input-output relations

ψ (deg.)	70	80	90	100	110	130	140
ϕ (deg.)	40	45	50	58	64	74	80

we have an overdetermined linear system of equations, which will be solved together with the constraint conditions by using our optimization method.

For comparison purposes, the problem is first solved by using Eq. (12). To this end, we minimize the design error directly, without considering any constraints on the mobility conditions. This leads to the following results:

$$k_1 = 1.3311, \; k_2 = -0.7730, \; k_3 = -0.4421$$

and it is easy to verify, using Table 1, that the input link is a rocker, which does not meet the mobility condition imposed, although the Euclidean norm of the design error is very small. Now, our optimization scheme is applied to include the mobility constraint. From Table 1 we can readily obtain the constraint equations, namely,

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} (k_1+k_3)^2 - (1-k_2)^2 + k_4^2 \\ (k_1-k_3)^2 - (1-k_2)^2 + k_5^2 \end{bmatrix} = \mathbf{0}$$

where $\mathbf{0}$ is the 2-dimensional zero vector and k_4 and k_5 are the slack variables. We then have the design vector written as

$$\mathbf{x} = [k_1, k_2, k_3, k_4, k_5]^T$$

The Jacobian matrix of the constraint function is readily computed as

$$\frac{d\mathbf{g}(\mathbf{x})}{d\mathbf{x}} = 2\begin{bmatrix} k_1+k_3 & -1-k_2 & k_1+k_3 & k_4 & 0 \\ k_1-k_3 & 1-k_2 & k_3-k_1 & 0 & k_5 \end{bmatrix}$$

Since $\mathbf{x}$ has included two extra components, namely, k_4 and k_5, matrix $\mathbf{A}$, given in Eq. (8*a*), has to be modified; that is, two additional zero vectors will appear as its 4th and 5th columns. The ith row of matrix $\mathbf{A}$ then becomes

$$A_i = [1, \cos\phi_i, -\cos\psi_i, 0, 0], \quad i = 1, 2, \ldots, m$$

Now we can use the optimization method presented in this paper. From the initial guess $\mathbf{x}^0 = [0.28, 0.74, 0.12, 1.69, 0.2]^T$ we end up with the following solution:

$$k_1 = 0.3248, \quad k_2 = 0.5875, \quad k_3 = -0.009725,$$
$$k_4 = 1.556, \quad k_5 = 0.2415$$

the Euclidean norm of the error function being 5×10^{-2}, which is larger than the one obtained for the unconstrained least-square problem. The corresponding link lengths are

$$a_1 = 1, \quad a_2 = 1.702, \quad a_3 = 103.4, \quad a_4 = 102.8$$

Although the mobility conditions are satisfied now, this is, unfortunately, still not a good result since the dimensional unbalance is very serious. As we can see, a_3 and a_4 are more than 100 times larger than a_1. To solve this problem, we resort to the penalty-function method to penalize this unbalance. From Eq. (2), it is apparent that, when all the link lengths are identical, we have $k_1 = k_2 = k_3 = k_4 = 1$; when variations in the link lengths occur, the values of k's turn to be either greater or smaller than 1. This gives us the basic idea to form the penalty function. Indeed, we define the following

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} k_1 - \dfrac{1}{k_1}, & k_2 - \dfrac{1}{k_2}, & k_3 - \dfrac{1}{k_3} \end{bmatrix}^T$$

Clearly, when all the link lengths are identical, $\mathbf{p}(\mathbf{x})$ vanishes; otherwise, its norm becomes large as the dimensional unbalance grows. Computing now the penalty function $\zeta(\mathbf{x})$ defined in Eq. (19) and adding it to the objective function, the problem of dimensional unbalance can be controlled. Therefore, we now can write the objective function in the form of Eq. (21) and compute the subjacobian matrix $d\mathbf{p}(\mathbf{x})/d\mathbf{x}$ as follows

$$\frac{d\mathbf{p}(\mathbf{x})}{d\mathbf{x}}=\begin{bmatrix} 1+1/k_1^2 & 0 & 0 & 0 & 0 \\ 0 & 1+1/k_2^2 & 0 & 0 & 0 \\ 0 & 0 & 1+1/k_3^2 & 0 & 0 \end{bmatrix}$$

Now we repeat the optimization procedure with the new objective function and a value of $r_i = 0.002$, for $i = 1, 2, 3$. The following results are obtained

$$k_1=0.215, \quad k_2=0.7533, \quad k_3=0.2015,$$
$$k_4=1.7031, \quad k_5=0.2463$$

with the linkage dimensions calculated as:

$$a_1=1, \quad a_2=1.327, \quad a_3=4.955, \quad a_4=4.962$$

The Euclidean norm in the error function for this solution is 9×10^{-2}. As we can see, with the penalty function formed, the link lengths of a_3 and a_4 decrease about 20 times. The problem of dimensional unbalance is therefore eliminated. Obviously, this solution has a larger design error than the one without the penalty terms, as expected. The optimum linkage is shown in Fig. 3.

6 Conclusions

A method for the optimization of four-bar function generators under mobility constraints was presented, which is applicable to both planar and spherical four-bar linkages. The method aims at solving approximate synthesis problems in which the residual in the input-output equation, termed the design error, is minimized over the linkage parameters. The mobility condition, originally expressed in inequality form, was properly treated and converted into equality form by introducing slack variables. Hence, the synthesis problem is formulated as an equality-constrained nonlinear least-square optimization problem. The orthogonal-decomposition algorithm, an iterative numerical scheme, is employed in solving our problem. The dimensional unbalance that often occurs in solving the synthesis problem is discussed and a penalty-function-based method is proposed in dealing with this problem. While the optimization scheme works fine for planar and spherical linkages, as discussed in this paper, it can possibly be extended to other types of linkages, as long as an input-output equation, along with a set of inequalities defining its mobility region, can be found, that bears a form similar to that in the planar or spherical cases.

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