The Dual Generalized Inverses and Their Applications in Kinematic Synthesis

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Abstract The left and right dual Moore-Penrose generalized inverses are the subject of this paper. It is shown that, contrary to the real case, these inverses are not unique, those with *minimum Frobenius norm* being obtained. Their application in kinematic synthesis is discussed. It is shown that, in the case of function-generating RCCC linkages, the left dual generalized inverse leads to a linkage that meets the prescribed input-output relations with both a least-square error and a minimum size. The study concludes with the synthesis of a linkage that approximates a *homokinetic* transmission between shafts with skew, orthogonal axes.

Key words: dual generalized Moore-Penrose inverses, least-square approximation, minimum-Frobenius-norm, homokinetic joint, skew axes.

1 Introduction

Dual numbers are well documented in the literature, an extensive bibliography being available in [1], with 73 entries. The literature is extensive for the scalar case, for vectors and matrices much less so, but some references can be cited, besides the previous one, namely, [2] and [3]. Moreover, dual numbers can be defined over both the real and the complex fields [4]; for the purposes of this paper, real numbers will suffice. The set of dual numbers itself, however, is not a field, but a *ring* [5].

The reason why dual numbers are relevant to kinematics can best be summarized in *The Principle of Transference* [6]:

> *The kinematics and statics relations of spatial linkages and cam mechanisms can be derived upon replacing the real variables occurring in the corresponding relations for spherical linkages by dual numbers.*

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The theory behind dual numbers is well established, but there are still some applications domains that haven't been fully exploited. This paper is a contribution in this direction. One objective of the paper is to shed light on the handling of overdetermined systems of dual linear equations (DLE), as arising in the *approximate synthesis of linkages*, when the number of prescribed conditions to meet exceeds that of linkage parameters available. In this context, the well-known results of linear least squares are revisited in the realm of dual numbers. It is shown that the least-square approximation of an overdetermined system of DLE admits a solution that can be expressed in the form of the dual-equivalent of the *Moore-Penrose generalized inverse,* often referred to a the *left pseudoinverse*. The author does not subscribe to this terminology because it is misleading: the prefix "pseudo" denotes something "false", which is not the case here. One novel contribution is the result that, contrary to the real case, the left dual generalized inverse is not unique, which allows for minimizing the Frobenius norm of the said inverse, thereby obtaining a unique solution that shows a striking similarity with the dual inverse of a square matrix [2]. The same result is shown to apply to the right counterpart of the left generalized inverse. The concepts discussed in Section 2 are then applied to the approximate synthesis of function-generating RCCC linkages.

As an example, the synthesis of a linkage of this type to approximate a *homokinetic transmission* between two shafts of skew axes and lying at right angles is fully discussed. By virtue of the minimum-norm property of the unique left Moore-Penrose generalized inverse—for conciseness, henceforth the foregoing matrix will be referred to as the "left generalized inverse," with a similar denomination for its right counterpart—the linkage thus obtained is one that not only approximates the prescribed number of conditions with a least-square error, but also does so with a minimum size. Moreover, the slight errors present in the optimum solution can be compensated for by means of computer control, upon resorting to an inverse-kinematics approach that guarantees that the linkage output will follow the prescribed input signal upon modulating the linkage input accordingly.

2 Back to Basics: Algebra of Dual Numbers

While dual algebra is a classic subject, and its bases are well established, there is still room for research contributions in the area of applications. One such area is the approximate synthesis of linkages, which often leads to linear least-square problems, the subject of this paper. Their nonlinear counterparts are manageable once the foundations for linear problems have been established.

An item that has not been duly addressed in the pertinent literature is the definition of the derivative of a dual-valued function of a dual argument, but it was discussed by Kotel'nikov in his original book [7]: given the dual function

$$\hat{f}(\hat{x}) \equiv f(\hat{x}) + \varepsilon f_o(\hat{x}), \quad \hat{x} = x + \varepsilon x_o$$

its derivative with respect to its dual argument can be readily obtained as the limit of a ratio of increments, which yields the relation

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}f}{\mathrm{d}x} + \varepsilon\frac{\mathrm{d}f_o}{\mathrm{d}x} \tag{1}$$

consistent with Kotel'nikov [7]. This relation will be needed below.

The extension of the foregoing definitions to vectors and matrices follows as a combination of these definitions and the rules for the counterpart operations for vectors and matrices. The inverse of a dual matrix is given in [1] and [2]. The former also includes a formula for the *dual left generalized inverse*[1]. As the formulas are displayed in that paper without derivation, the paper misses an important point: the generalized inverse in question is not unique. This issue is made apparent below.

For starters the expression for the dual inverse matrix derived in [2] is recalled: let $\hat{\mathbf{A}} = \mathbf{A} + \varepsilon\mathbf{A}_o$ be a dual matrix, with $\mathbf{A},\ \mathbf{A}_o \in \mathbf{R}^{n\times n}$, its inverse being defined as long as $\mathbf{A}$ is invertible, although $\mathbf{A}_o$ need not be so. The inverse of $\hat{\mathbf{A}}$ is given by

$$\hat{\mathbf{A}}^{-1} = \mathbf{A}^{-1} - \varepsilon\mathbf{A}^{-1}\mathbf{A}_o\mathbf{A}^{-1} \tag{2}$$

Paraphrasing the derivation of the expression (2) for the dual inverse, not included here for the sake of conciseness, let $\hat{\mathbf{B}} = \mathbf{B} + \varepsilon\mathbf{B}_o$ be the left generalized inverse of a $m\times n$ dual matrix $\hat{\mathbf{A}}$, with $m > n$. As a consequence, $\hat{\mathbf{B}}$ is bound to be of $n\times m$. In the sequel, it will be made apparent that only $\mathbf{A}$ need be of full rank for the desired generalized inverse to exist, but $\mathbf{A}_o$ can be rank-deficient. Then, $\hat{\mathbf{B}}$ verifies $\hat{\mathbf{B}}\hat{\mathbf{A}} = \mathbf{1}_n$, with $\mathbf{1}_n$ denoting the $n\times n$ identity matrix. Upon expansion of the foregoing left-hand side, two real equations are obtained, one for the primal, one for the dual part:

$$\mathbf{B}\mathbf{A} = \mathbf{1}_n, \quad \mathbf{B}_o\mathbf{A} + \mathbf{B}\mathbf{A}_o = \mathbf{O}_n \tag{3}$$

where $\mathbf{O}_n$ denotes the $n\times n$ zero matrix, the first equation leading to the not so unexpected result $\mathbf{B} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T \equiv \mathbf{A}^I$, i.e., the left generalized inverse of $\mathbf{A}$. When the foregoing expression is substituted into the second of the two above equations, a matrix equation for $\mathbf{B}_o$ is derived:

$$\mathbf{B}_o\mathbf{A} = -\mathbf{A}^I\mathbf{A}_o \ \Rightarrow\ \mathbf{A}^T\mathbf{B}_o^T = -\mathbf{A}_o^T(\mathbf{A}^I)^T \equiv -\mathbf{A}_o^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}$$

which is a system of n^2 equations in $mn > n^2$ unknowns, the real components of $\mathbf{B}_o$. The system is, thus, underdetermined, thereby admitting *infinitely many solutions*. The conclusion is, then, that the *dual left generalized inverse is not unique*. Among all that many solutions, one of minimum *Frobenius* norm, $\mathrm{tr}(\mathbf{B}\mathbf{B}^T)$, can be obtained if one resorts to the right generalized inverse of $\mathbf{A}^T$, denoted $(\mathbf{A}^T)^\dagger$ [8]:

$$(\mathbf{A}^T)^\dagger = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1} \tag{4}$$

whence, after some obvious manipulations,

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[1] Actually, the authors do not stress the difference between the right and the left generalized inverses; they represent both with the same symbol, $(\,\cdot\,)^+$.

$$\mathbf{B}_o = -\mathbf{A}^I\mathbf{A}_o\mathbf{A}^I \tag{5}$$

Therefore, the minimum-Frobenius-norm $\hat{\mathbf{A}}^I$ is

$$\hat{\mathbf{A}}^I = \mathbf{A}^I - \varepsilon\mathbf{A}^I\mathbf{A}_o\mathbf{A}^I \tag{6}$$

which bears a striking similarity with the dual inverse, an expression also displayed in [1].

The right Moore-Penrose generalized inverse of a dual matrix $\hat{\mathbf{C}} = \mathbf{C} + \varepsilon\mathbf{C}_o$, with $\mathbf{C}, \mathbf{C}_o \in \mathbf{R}^{m\times n}$ and $m < n$, is defined as the dual matrix $\hat{\mathbf{C}}^\dagger \equiv \hat{\mathbf{D}}$ such that $\hat{\mathbf{C}}\hat{\mathbf{D}} = \mathbf{1}_m$, with $\hat{\mathbf{D}} = \mathbf{D} + \varepsilon\mathbf{D}_o$ and $\mathbf{D}, \mathbf{D}_o \in \mathbf{R}^{n\times m}$. The computation of $\mathbf{D}$ and $\mathbf{D}_o$ follows the same pattern as that of $\mathbf{B}$ and $\mathbf{B}_o$ above. The details are not included here for conciseness, but the results are displayed below:

$$\hat{\mathbf{C}}^\dagger = \mathbf{C}^\dagger - \varepsilon\mathbf{C}^\dagger\mathbf{C}_o\mathbf{C}^\dagger \tag{7}$$

a formula that is also displayed in [1], but without a proof. Again, as in the case of $\mathbf{A}^I$, $\hat{\mathbf{C}}^\dagger$ is not unique, the formula displayed above being the one with a minimum Frobenious norm.

Now the left dual generalized inverse is applied to the solution of an overdetermined system of m dual linear equations in $n < m$ dual unknowns, grouped in vector $\hat{\mathbf{x}}$, of the form

$$\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}} \tag{8}$$

where $\hat{\mathbf{A}}$ is assumed as above, to be a dual $m \times n$ matrix, with $m > n$ and with a full-rank primal part, $\hat{\mathbf{x}}$ and $\hat{\mathbf{b}}$ being, respectively, n- and m-dimensional dual vectors. As the system is overdetermined, it is not possible to find a vector $\hat{\mathbf{x}}$ that will verify all m dual equations (8), but it will be shown that it is possible to find the vector $\hat{\mathbf{x}}$ that will render the Euclidean norm of the dual error $\hat{\mathbf{e}}$ a minimum, with $\hat{\mathbf{e}}$ defined as

$$\hat{\mathbf{e}} = \hat{\mathbf{b}} - \hat{\mathbf{A}}\hat{\mathbf{x}} \tag{9}$$

whose Euclidean norm[2] $\|\hat{\mathbf{e}}\|$ is the square root of the scalar product $\hat{\mathbf{e}}^T\hat{\mathbf{e}}$, i.e.,

$$\|\hat{\mathbf{e}}\|^2 = \|\hat{\mathbf{b}}\|^2 - 2\hat{\mathbf{b}}^T\hat{\mathbf{A}}\hat{\mathbf{x}} + \|\hat{\mathbf{A}}\hat{\mathbf{x}}\|^2 \tag{10}$$

The error Euclidean norm is minimized upon zeroing the derivative of $\|\hat{\mathbf{e}}\|^2$ with respect to $\hat{\mathbf{x}}$, which readily leads to the *dual normal equations* (DNE):

$$\hat{\mathbf{A}}^T\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{A}}^T\hat{\mathbf{b}} \quad\Rightarrow\quad \hat{\mathbf{A}}^T\hat{\mathbf{e}}_0 = \mathbf{0} \tag{11}$$

thereby stating an important theoretical result: the minimum-norm error—i.e., the error $\hat{\mathbf{e}}_0$ of minimum Euclidean norm—lies in the null space of $\hat{\mathbf{A}}^T$, a restatement of the classical *Projection Theorem*, but now in dual space. Another theoretical result

[2] If $\mathbf{e}$ and $\mathbf{e}_o$ denote the primal and dual parts of $\hat{\mathbf{e}}$, then $\|\hat{\mathbf{e}}\|^2 = \|\mathbf{e}\|^2 + \varepsilon 2\mathbf{e}^T\mathbf{e}_o$.

is the expression for the least-square solution $\hat{\mathbf{x}}_0$, obtained directly from the normal equations (11):

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{A}}^I \hat{\mathbf{b}} \tag{12}$$

Expression (12) is a representation of the *unique* minimum-norm least-square solution of system (8), but should not be used verbatim to compute $\hat{\mathbf{x}}_0$, because of the frequent ill-conditioning of the product $\hat{\mathbf{A}}\hat{\mathbf{A}}^T$. Instead, the QR decomposition [9] should be applied.

Interestingly, having chosen the dual part $\mathbf{B}_o$ of $\hat{\mathbf{A}}^I$ with minimum norm guarantees that the dual part $\mathbf{x}_{o0}$ of the least-square solution $\hat{\mathbf{x}}_0$ is of minimum Euclidean norm. This property will be exploited in the synthesis of a RCCC linkage intended to approximate a homokinetic transmission between two shafts of skew axes, lying at right angles.

3 Synthesis of a RCCC Linkage

The foregoing results will now be applied to the synthesis of the RCCC linkage shown in Fig. 1, with geometric parameters defined using the original Denavit-Hartenberg notation [10].

The input-output (IO) equation of the RCCC linkage was derived by Yang and Freudentstein [11]. The same equation was more recently cast in a framework that allows its analysis in a unified form applicable to planar, spherical and spatial four-bar linkages [12]. For the sake of brevity, the IO equation is not derived here. It is displayed below, as taken from the foregoing reference:

$$\hat{F}(\hat{\psi}, \hat{\phi}) \equiv \hat{k}_1 + \hat{k}_2 \cos\hat{\psi} + \hat{k}_3 \cos\hat{\psi}\cos\hat{\phi} - \hat{k}_4 \cos\hat{\phi} + \sin\hat{\psi}\sin\hat{\phi} = 0 \tag{13}$$

where $\hat{\psi}$, the input angle, has been "hatted", even though this angle is associated with a R joint, which undergoes pure rotations about its axis. In fact, $\hat{\psi} = \psi + \varepsilon b_2$, where b_2 accounts for the location of the common normal between this axis (Z_2) and Z_3. The primal parts of the *dual Freudenstein parameters* (DFP) are given below:

$$k_1 \equiv \frac{\lambda_1\lambda_2\lambda_4 - \lambda_3}{\mu_2\mu_4}, \quad k_2 = \frac{\lambda_4\mu_1}{\mu_4}, \quad k_3 = \lambda_1, \quad k_4 = \frac{\lambda_2\mu_1}{\mu_2} \tag{14}$$

with the definitions $\lambda_i \equiv \cos\alpha_i$ and $\mu_i \equiv \sin\alpha_i \neq 0$, while α_i is displayed in Fig. 1, their dual counterparts being defined as

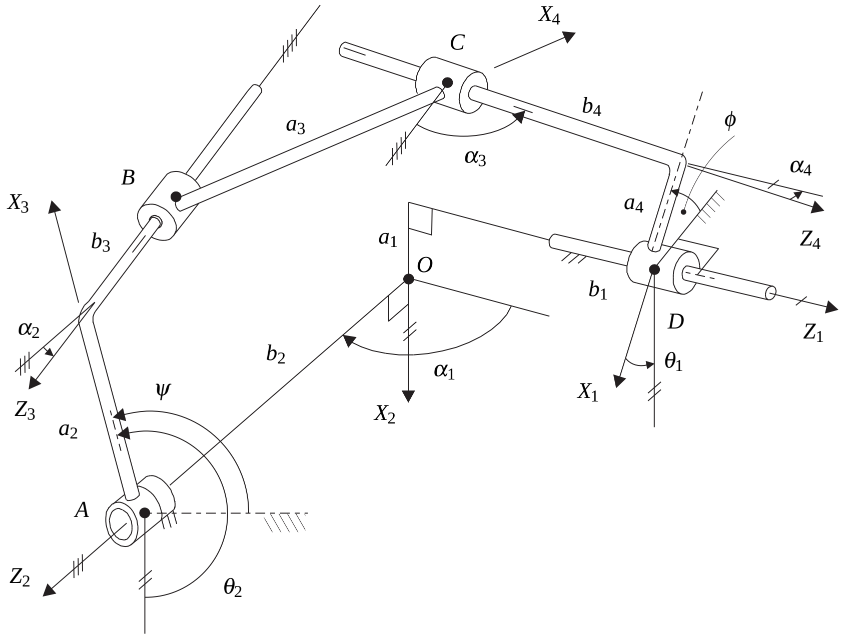

Fig. 1 A generic RCCC linkage

$$
\begin{aligned}
k_{o1} &= -\frac{a_1\lambda_2\lambda_4\mu_1\mu_2\mu_4 + a_2(\lambda_1\lambda_4 - \lambda_2\lambda_3)\mu_4 - a_3\mu_2\mu_3\mu_4 + a_4(\lambda_1\lambda_2 - \lambda_3\lambda_4)\mu_2}{\mu_2^2\mu_4^2} \\
k_{o2} &= \frac{a_1\lambda_1\lambda_4\mu_4 - a_4\mu_1}{\mu_4^2}, \quad k_{o3} = -a_1\mu_1, \quad k_{o4} = \frac{a_1\lambda_1\lambda_2\mu_2 - a_2\mu_1}{\mu_2^2}
\end{aligned}
\tag{15}
$$

The synthesis problem can now be formulated as: given a set of input-angle values $\{\psi_i\}_1^m$ and a set of corresponding output values $\{\phi_i, u_i\}_1^m$, where u_i denotes the ith prescribed value of the output variable[3] b_1, *find the linkage parameters* $\{a_i, \alpha_i\}_1^4$ that will produce a RCCC linkage that meets the prescribed IO relations. Since we have m IO conditions to meet, in the form of the dual equations (13), and four dual linkage parameters, when $m = 4$ the prescribed IO values can be met exactly, which corresponds to *exact synthesis*. For $m > 4$, no linkage will possibly meet all m prescribed IO values. However, it is possible to find the linkage that will meet these values with the minimum error, which is known as *approximate synthesis*. Nevertheless, a word of caution is in order: although the error vector defined in eq.(9) has components with two different units, radians and m, its norm is well defined, as per footnote 2. Hence, a linkage can be found that meets the synthesis equations with an error of minimum Euclidean norm, *independent of the units chosen*. The said equations are obtained upon substitution of the input and output variables by their m prescribed values in the IO equation:

[3] The new variable u_i is introduced with the purpose of avoiding double subscripts.

$$\hat{F}_i(\psi_i, \hat{\phi}_i) \equiv \hat{k}_1 + \hat{k}_2 \cos\hat{\psi}_i + \hat{k}_3 \cos\hat{\psi}_i \cos\hat{\phi}_i - \hat{k}_4 \cos\hat{\phi}_i + \sin\hat{\psi}_i \sin\hat{\phi}_i = 0,\ i = 1, \ldots, m \tag{16}$$

which are linear in the dual Freudenstein parameters $\{\hat{k}_i\}_1^4$. Hence, upon assembling the m foregoing equations, a system of m dual linear equations in the four DFP is obtained:

$$\hat{\mathbf{S}}\hat{\mathbf{k}} = \hat{\mathbf{b}} \tag{17}$$

with

$$\hat{\mathbf{S}} = \underbrace{\begin{bmatrix} 1 & c\psi_1 & c\psi_1 c\phi_1 & -c\phi_1 \\ 1 & c\psi_2 & c\psi_2 c\phi_2 & -c\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c\psi_m & c\psi_m c\phi_m & -c\phi_m \end{bmatrix}}_{\mathbf{S}} + \varepsilon \underbrace{\begin{bmatrix} 0 & -b_2 s\psi_1 & -u_1 c\psi_1 s\phi_1 - b_2 s\psi_1 c\phi_1 & u_1 s\phi_1 \\ 0 & -b_2 s\psi_2 & -u_2 c\psi_2 s\phi_2 - b_2 s\psi_2 c\phi_2 & u_2 s\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -b_2 s\psi_m & -u_m c\psi_m s\phi_m - b_2 s\psi_m c\phi_m & u_m s\phi_m \end{bmatrix}}_{\mathbf{S}_o} \tag{18}$$

with $c(\cdot)$ and $s(\cdot)$ denoting $\cos(\cdot)$ and $\sin(\cdot)$, respectively, while

$$\hat{\mathbf{b}} = -\underbrace{\begin{bmatrix} s\psi_1 s\phi_1 \\ s\psi_2 s\phi_2 \\ \vdots \\ s\psi_m s\phi_m \end{bmatrix}}_{\mathbf{b}} - \varepsilon \underbrace{\begin{bmatrix} u_1 s\psi_1 c\phi_1 + b_2 c\psi_1 s\phi_1 \\ u_2 s\psi_2 c\phi_2 + b_2 c\psi_2 s\phi_2 \\ \vdots \\ u_m s\psi_m c\phi_m + b_2 c\psi_m s\phi_m \end{bmatrix}}_{\mathbf{b}_o} \tag{19}$$

Now, upon equating the primal and the dual parts of eq.(17), two real vector equations are obtained, namely,

$$\mathbf{S}\mathbf{k} = \mathbf{b}, \quad \mathbf{S}\mathbf{k}_o + \mathbf{S}_o\mathbf{k} = \mathbf{b}_o \quad \Rightarrow \quad \mathbf{S}\mathbf{k}_o = \mathbf{b}_o - \mathbf{S}_o\mathbf{k} \tag{20}$$

which amount to two overdetermined linear systems of equations, both with the same matrix coefficient $\mathbf{S}$, one for $\mathbf{k}$, one for $\mathbf{k}_o$. The computation of the least-square solution proceeds in two steps: first the primal equation is solved for $\mathbf{k}$; with the least-square solution thus obtained, $\mathbf{k}_0$, substituted into the dual equation, the least square solution of this equation, $\mathbf{k}_{o0}$, is obtained. Notice that these calculations being done using the QR decomposition, the primal synthesis matrix needs factoring only once. This feature is important if the foregoing procedure is a part of a second, external optimization procedure, that calls for many iterations. It is noteworthy that the DH parameter b_2 is not included in either the primal or the dual part of the DFP, eqs.(14) and (15), respectively, and hence, this parameter has been taken to the right-hand side of the dual synthesis equations (20); b_2 has to be treated not as an unknown, but as a parameter, that can be used to either fine-tune a solution or to optimize an objective function.

Now the RCCC linkage is synthesized so as to approximate a homokinetic transmission for values of the input and the output variables that sweep angles of 120°. Moreover, the primal synthesis equation leading to a spherical linkage, the associated synthesis procedure is identical to that reported in [13]. In that paper, a search is included on the optimum values of the location of the zeros of the input and output dials, which amount to a translation of the data points $\{\psi_i, \phi_i\}_1^m$ *en masse*, i.e., under a rigid-body translation, in the ϕ-vs.-ψ plane. A continuum of values for the optimum shifts were reported in that paper. The values adopted here are $\xi = 146°$ and $\eta = 34°$, for ψ and ϕ, respectively.

The values ψ_i, for $i = 1, \ldots m$, with $m = 501$ prescribed data *triads*[4], are uniformly distributed in the interval $86°(= -60° + 146°) \leq \psi \leq 206°$, while their ϕ_i counterparts are distributed likewise in the interval $-26°(= -60° + 34°) \leq \phi \leq 94°$. The shafts to be coupled lying at right angles, $\alpha_1 = 90°$, whence $\lambda_1 = 0$ and $\mu_1 = 1$. Furthermore, given its desired homokinetic behavior, the linkage is assumed symmetric, as the input and output links play the same role, whence $\alpha_4 = \alpha_2$. In this light, the number of unknown primal Freudenstein parameters reduces to only two, for $k_3 = 0$ and $k_4 = k_2$, a consequence of the foregoing assumptions and relations (14). The number of prescribed points led to an overdetermined linear system of 501 equations in two unknowns, whose least-square solution is

$$k_1 = 1.217,\ k_2 = 0.9439 \ \Rightarrow\ k_4 = 0.9439,\ \alpha_2 = \alpha_4 = 46.65°,\ \alpha_3 = 132.4° \quad (21)$$

with a rms value of the minimum-norm error equal to 0.01942, or 1.942%.

Next, a_1 is set at 240 mm, as imposed by the design conditions, with $b_2 = a_1$ for symmetry. Further, the values u_i of b_1 at the prescribed values of ϕ_i, which complete the ith triad, were distributed symmetrically around $b_1 = 0$, with $u_1 = -a_1/10$, $u_m = a_1/10$, and following a cycloidal motion program:

$$u_i = -\frac{a_1}{10} + U\left(\frac{i-1}{m-1} - \frac{1}{2\pi}\sin\frac{2\pi(i-1)}{m-1}\right) \quad (22)$$

with amplitude U given as $a_1/5$ in order to limit the output sliding b_1. This program was chosen because it starts smoothly with zero velocity and acceleration, and stops likewise. The second system of eqs.(20), of 501 equations for two unknowns, k_{o1} and k_{o2}, led to the least-square solution $\mathbf{k}_{o0}$, with $k_{o3} = -a_1 = -240$ mm not being part of the unknowns, for its value is fixed from the prescribed values for α_1 and a_1, as per eqs.(15). The optimum values were found to be, for the above-mentioned values of $\mathbf{k}_0$,

$$k_{o1} = 319.0\ \text{mm}, \quad k_{o2} = 154.6\ \text{mm} \quad k_{o4} = k_{o2} \quad (23)$$

with a rms value of the minimum-norm dual error of 1.119 mm, or 2.33% of the amplitude U. For the record, the *normalized* dual part of the Euclidean norm of the dual error, $\mathbf{e}^T\mathbf{e}_o/\sqrt{m}$, is 0.3260 mm or 0.07% of the amplitude U.

Computing the DH linkage parameters now is straightforward, as eqs.(15) involve these parameters linearly. The results are displayed below:

[4] This high number was used with the purpose of bringing the optimum *design error* $\mathbf{e}_0$ as close as possible to the *structural error*, which measures the actual deviation of the synthesized output angle from its prescribed value, as per the results reported in [14].

$$a_2 = -76.46\ \text{mm}, \quad a_3 = 209.4\ \text{mm}, \quad a_4 = a_2 \tag{24}$$

where, interestingly, a_i being defined as a *length* in the framework of the DH notation, it must be non-negative. However, a negative value for a_2, and hence, for a_4, was obtained above. The interpretation of the negative sign here is well known within the methodology set forth by Freudenstein [15]: should a_2 (a_4) turn out to be negative as a result of the linkage synthesis for function generation, then measure angle ψ (ϕ) not as indicated in Fig. 1, but from its extension, i.e., add 180° to the prescribed input (output) angles. This completes the solution to the synthesis problem, a CAD model thereof being shown in Fig. 2. In this figure, the output motion of the quasi-homokinetic mechanism is the rotation of the splined shaft, which is mounted on the machine frame by means of standard bearings.

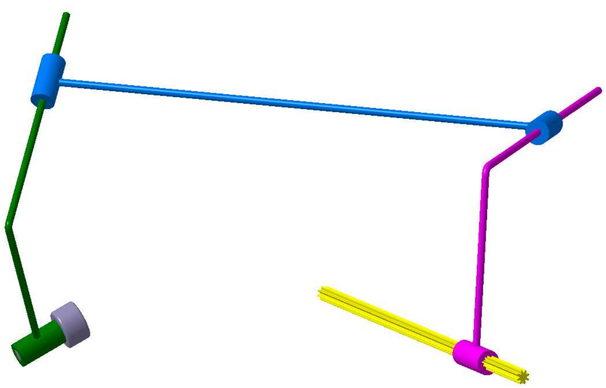

Fig. 2 A CAD model of the synthesized RCCC linkage

4 Conclusions

Some novel results in the realm of the algebra of dual numbers, in connection with dual linear least-square problems were introduced here, then applied to the synthesis of the RCCC function generating linkage. The methodology thus established was then illustrated with the solution of a problem of current interest, the approximate synthesis of a RCCC linkage for homokinetic transmission between shafts with skew axes, lying at right angles.

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