Notice that, in this case, all the points of the body undergo the same displacement, but the object is neither rotated nor distorted. We thus have

$$\boldsymbol{M} = \mathbf{1}, \quad \boldsymbol{t} = \begin{bmatrix} D & H & L \end{bmatrix}^T \tag{4.33b}$$

The values of D, H, L represent the relative translation of the point in the x, y, z directions, respectively.

In Fig. 4.18, we can see examples of translations.

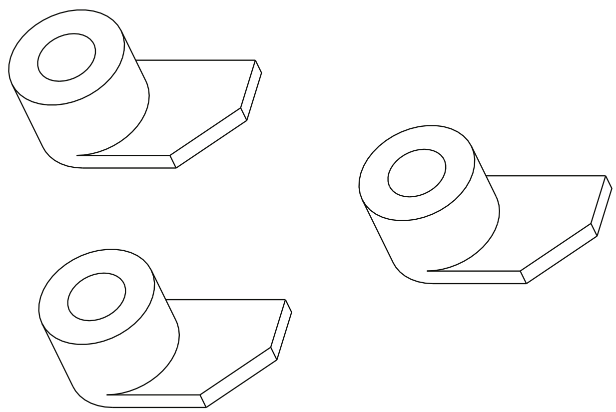

Figure 4.18: Translations in 3D

4.3.3 Rotation

A rotation in 3D is another special case of rigid-body displacement. Under a rotation, the distance between every pair of object points is preserved and one point of the object remains stationary. The object is said to rotate about that point.

Rotations in three dimensions are more complex than their two-dimensional counterparts, because an axis of rotation, rather than a centre of rotation, must be specified. Rotations about an axis passing through the origin are characterized by a *proper orthogonal* matrix $\boldsymbol{M}$ and a zero translation, $\boldsymbol{t} = \mathbf{0}$. In this case, $\boldsymbol{M}$ has the properties below:

$$\boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{M}\boldsymbol{M}^T = \mathbf{1}, \quad \det(\boldsymbol{M}) = +1 \tag{4.34}$$

In particular, the rotation matrix for a Z-axis rotation through an angle θ is:

$$\boldsymbol{M}_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.35}$$

which produces the mapping:

$$x' = x\cos\theta - y\sin\theta,\ y' = x\sin\theta + y\cos\theta,\ z' = z$$

In a similar manner, a rotation of θ about the Y-axis can be obtained by means of

$$x' = x\cos\theta + z\sin\theta,\ y' = y,\ z' = -x\sin\theta + z\cos\theta$$

and is correspondingly represented by

$$\boldsymbol{M}_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{4.36}$$

A rotation about the X-axis is:

$$x' = x,\ y' = y\cos\theta - z\sin\theta,\ z' = y\sin\theta + z\cos\theta$$

which is represented by

$$\boldsymbol{M}_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{4.37}$$

Sometimes, rotations about arbitrary axes are specified as a sequence of rotations about the coordinate axes, as illustrated with the solar panel of Fig. 4.19(a), used in telecommunications satellites to provide energy to their different instruments. In this case we have

$$\boldsymbol{M}_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{M}_Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{M}_Z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4.3.1 With reference to Fig. 4.19, a) find the homogeneous transformation matrix that carries the solar panel from attitude (a) to attitude (c) and takes its point P, that coincides with the origin, to a new location, labelled $Q(1, 2, 3)$; then b) find the homogeneous transformation matrix that will carry the same panel back to its original configuration.

Solution:

a) Let $\boldsymbol{M}_{ac}$ be the matrix representing the rotation from attitude (a) to attitude (c). Thus,

$$\boldsymbol{M}_{ac} = \boldsymbol{M}_Y\boldsymbol{M}_X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

the corresponding homogeneous transformation $\boldsymbol{T}_r$ being readily obtained as

$$\boldsymbol{T}_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we need a homogeneous transformation $\boldsymbol{T}_t$ to translate the panel without rotating it. Given the translation vector $\boldsymbol{t}$ involved, obtaining $\boldsymbol{T}_t$ is straightforward:

$$\boldsymbol{T}_t = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

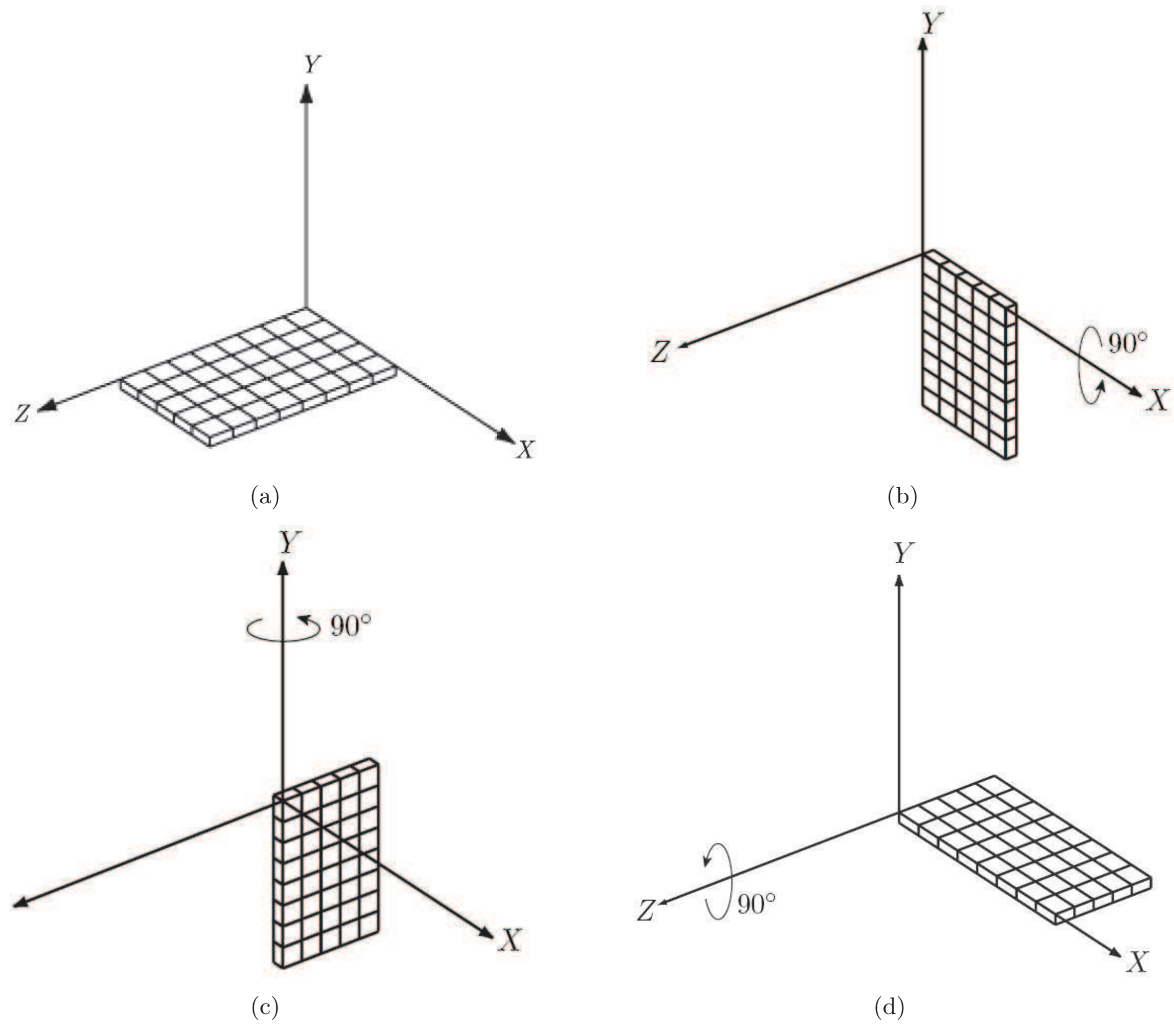

Figure 4.19: A solar panel: (a) in its original configuration; (b) after a rotation through 90° about the X-axis; (c) after a second rotation through 90° about the Y-axis; and (d) about a third rotation through 90° about the Z-axis

whence, the total transformation matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \boldsymbol{T}_t\boldsymbol{T}_r = \begin{bmatrix} \mathbf{1} & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{M}_{ac} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{ac} & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

whence,

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) To compute $\boldsymbol{T}^{-1}$, we recall the general expression (1.99) for the inverse of a homogeneous transformation matrix:

$$\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{M}_{ac}^{-1} & -\boldsymbol{M}_{ac}^{-1}\boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{ac}^{T} & -\boldsymbol{M}_{ac}^{T}\boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where the orthogonality of $\boldsymbol{M}_{ac}$ has been recalled, to simplify the foregoing expression, the final result thus being

$$\boldsymbol{T}^{-1} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

thereby completing the calculations required.

Example 4.3.2 *Matrix $\boldsymbol{M}$ shown below is claimed to represent a rotation of an object $\mathcal{B}$ rotating about the origin*

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

a) Prove that the matrix indeed represents a rotation about the origin; then

b) Find its axis and its angle of rotation

Solution:

a) To represent a rotation about the origin, $\boldsymbol{M}$ must be proper orthogonal. We thus compute

$$\boldsymbol{M}\boldsymbol{M}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M}^T\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$