MECH 541 Kinematic Synthesis
The Spherical Burmester Problem—Fall 2009

1 Problem Formulation

The spherical Burmester Problem is stated below:

Problem 1 *Find a spherical four-bar linkage that will conduct its coupler link through a set $\mathcal{S}$ of m attitudes given by the orthogonal matrices $\{\mathbf{Q}_j\}_1^m$, defined with respect to a reference attitude given by $\mathbf{Q}_0 = \mathbf{1}$, where $\mathbf{1}$ denotes the 3×3 identity matrix.*

The four-bar linkage in question is depicted in Fig. 3.3, with its four linkage dimensions $\{\alpha_j\}_1^4$. To be consistent with the notation used for the planar Burmester Problem, the two grounded revolutes are labelled B, the two moving revolutes A_0. The linkage is thus fully defined by the two dyads BA_0. If a distinction is needed between the two dyads, one will be labelled $B^\star A_0^\star$.

The axes of the revolutes of one dyad are thus given by the segments $\overline{OB}$ and $\overline{OA_0}$, points B and A_0 being the intersections of these axes with the unit sphere; their position vectors are $\mathbf{b}$ and $\mathbf{a}_0$, both of unit magnitude, i.e.,

$$\|\mathbf{b}\| = 1, \quad \|\mathbf{a}_0\| = 1 \tag{1}$$

Such as in the planar Burmester Problem, point B is called *centre point*, while A_0 *circular point*. As the coupler link moves, while visiting the m given attitudes, the circular point, which is common to both the grounded link $\overline{BA_0}$ and the coupler link, attains positions $A_1, \ldots, A_m$, the segments along the axis of the moving revolute of the dyad thus becoming $\overline{OA_1}, \ldots, \overline{OA_m}$. The *synthesis equation* is obtained upon imposing the geometric constraint that the angle between $\overline{OA_j}$ and $\overline{OB}$ remains equal to that between $\overline{OA_0}$ and $\overline{OB}$, i.e.,

$$\mathbf{a}_j^T\mathbf{b} = \mathbf{a}_0^T\mathbf{b} \quad \text{or} \quad (\mathbf{a}_j - \mathbf{a}_0)^T\mathbf{b} = 0, \quad j = 1, \ldots, m \tag{2}$$

where, apparently,

$$\mathbf{a}_j = \mathbf{Q}_j\mathbf{a}_0 \tag{3}$$

whence conditions (2) become

$$\mathbf{a}_0^T(\mathbf{Q}_j^T - \mathbf{1})\mathbf{b} = 0, \quad j = 1, \ldots, m \tag{4}$$

An expression for $\mathbf{Q}_j$ is readily obtained from the general expression for the rotation matrix displayed in eq.(2.1c):

$$\mathbf{Q}_j = \mathbf{1} + s_j\mathbf{E}_j + (1 - c_j)\mathbf{E}_j^2, \quad c_j \equiv \cos\phi_j, \quad s_j \equiv \sin\phi_j \tag{5}$$

where $\mathbf{E}_j = \text{CPM}(\mathbf{e}_j)$, $\mathbf{e}_j$ denoting the unit vector that defines the direction of the axis of rotation of $\mathbf{Q}_j$, and ϕ_j the corresponding angle. Hence,

$$\mathbf{Q}_j - \mathbf{1} = [s_j\mathbf{1} + (1 - c_j)\mathbf{E}_j]\mathbf{E}_j \tag{6}$$

Therefore, eq.(4) becomes

$$\mathbf{a}_0^T\mathbf{E}_j[s_j\mathbf{1} - (1 - c_j)\mathbf{E}_j]\mathbf{b} = 0 \tag{7}$$

In order to ease the ensuing discussion, let

$$\mathbf{g}_j \equiv (\mathbf{Q}_j - \mathbf{1})\mathbf{a}_0 \tag{8}$$

whence eq.(4) takes the form

$$\mathbf{g}_j^T\mathbf{b} = 0, \quad j = 1, \ldots, m \tag{9}$$

2 Three Poses

In this case, $m = 2$, i.e., two constraint equations occur:

$$\mathbf{g}_1^T\mathbf{b} = 0 \quad \text{and} \quad \mathbf{g}_2^T\mathbf{b} = 0 \tag{10}$$

Hence, one of the two vectors $\mathbf{a}_0$ and $\mathbf{b}$ can be prescribed arbitrarily. If, for example, the former is prescribed, then $\mathbf{g}_1$ and $\mathbf{g}_2$ are known. The conditions of eq.(10) can thus be verified if $\mathbf{b}$ is computed as the cross product of the two other vectors in those equations, i.e.,

$$\mathbf{b} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\|\mathbf{g}_1 \times \mathbf{g}_2\|} \tag{11}$$

where $\mathbf{b}$ has been normalized to render it of unit magnitude.

A similar reasoning follows if $\mathbf{b}$ is prescribed, if with obvious modifications.

3 Four Poses

Now we have $m = 3$, the constraints being

$$\mathbf{g}_1^T\mathbf{b} = 0 \quad \mathbf{g}_2^T\mathbf{b} = 0 \quad \text{and} \quad \mathbf{g}_3^T\mathbf{b} = 0 \tag{12}$$

In order to be able to find a vector $\mathbf{b}$ simultaneously perpendicular to all three vectors $\mathbf{g}_j$ in the above equation, these three vectors must be coplanar, and hence,

$$F(\mathbf{a}_0) \equiv \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = 0 \tag{13}$$

which is a product of three factors that are linearly homogeneous in $\mathbf{a}_0$, as per eq.(8), and hence, the *synthesis equation* (13) is cubic and homogeneous in $\mathbf{a}_0$. Moreover, the synthesis equation represents a *conical surface* $\mathcal{K}$ with apex at the origin, of third degree. This surface can be termed, in analogy with the planar case, the *circlepoint conical surface*. The surface intersects the unit sphere centred at the origin along a spherical curve of third degree. The curve can be regarded as the *generatrix* of the conical surface, each of whose *elements* defines the axis of a revolute that verifies the synthesis equations, the foregoing joint thus being the moving revolute of the spherical dyad of the linkage sought.

By a similar reasoning, the *centrepoint conical surface* $\mathcal{M}$ is obtained likewise. Any element of this surface can play the role of the axis of the fixed revolute of the same dyad.

4 Five Poses

For $m = 4$, the synthesis equations lead to a homogeneous system of four homogeneous bilinear equations in the unknown vectors $\mathbf{a}_0$ and $\mathbf{b}$. As these are three-dimensional vectors, the total number of unknowns at hand is six, but then again, two additional equations are available, namely, eqs.(1), and the problem is fully determined. The four homogeneous equations can then be cast in the form

$$\underbrace{\begin{bmatrix} \mathbf{a}_0^T\mathbf{E}_1[s_1\mathbf{1} - (1-c_1)\mathbf{E}_1] \\ \mathbf{a}_0^T\mathbf{E}_2[s_2\mathbf{1} - (1-c_2)\mathbf{E}_2] \\ \mathbf{a}_0^T\mathbf{E}_3[s_3\mathbf{1} - (1-c_3)\mathbf{E}_3] \\ \mathbf{a}_0^T\mathbf{E}_4[s_4\mathbf{1} - (1-c_4)\mathbf{E}_4] \end{bmatrix}}_{\equiv \mathbf{G}} \mathbf{b} = \mathbf{0}_4 \tag{14}$$

in which $\mathbf{G}$ is a 4×3 matrix whose jth row is apaprently $\mathbf{g}_j^T$. In light of the second of equations (1), moreover, the trivial solution of eqs.(14) is not acceptable, and hence, $\mathbf{G}$ must be *rank-deficient*, i.e., its three columns must be linearly dependent. This happens if and only if the four independent 3×3 determinants obtained by taking three rows of $\mathbf{G}$ at a time vanish. Let

$$\Delta_j(\mathbf{a}_0) \equiv \det(\mathbf{G}_j), \quad j = 1, \ldots, 4 \tag{15}$$

with $\mathbf{G}_j$ denoting the 3×3 matrix obtained from $\mathbf{G}$ upon deleting its jth row. From Section 3 it is known that each of the four determinants defines a conical cubic surface whose apex is the origin. The common intersections of all four surfaces are common elements of these surfaces; they are the multiple moving-revolute axes that are capable of guiding the coupler link through the five prescribed poses.

The Bezout number of the four cubic equations (15) is $3^4 = 81$, which is an overestimate of the actual number of possible solutions; this is known to be six (Chiang, 1988; McCarthy, 2000). This statement is proven below, in following McCarthy. To this end, a result from algebra is first recalled:

Lemma 1 *Let* $\mathbf{M}$ *be a* $n \times n$ *matrix function of the real scalar* x*, its* i*th column containing entries which are all polynomials in* x *of degree* d_i*. Then,* $\det(\mathbf{M})$ *is a polynomial in* x *of degree* $d = d_1 + d_2 + \ldots + d_n$*.*

An informal proof is given below: Consider $\mathbf{M}$ of 3×3, its (i, j) entry being labelled m_{ij}. If $\det(\mathbf{M})$ is expanded by cofactors of its first column, then

$$\det(\mathbf{M}) = m_{11}(m_{22}m_{33} - m_{23}m_{32}) - m_{21}(m_{12}m_{33} - m_{13}m_{32}) + m_{31}(m_{12}m_{23} - m_{13}m_{22}) \tag{16}$$

Each term of the above summation is the product of a term m_{j1} of degree d_1 by a difference of products of degree $d_2 + d_3$, the summation then being of degree $d_1 + d_2 + d_3$. The reader should be able to apply this analysis to any $n \times n$ matrix with the structure of $\mathbf{M}$.

What McCarthy does is, first, let $\mathbf{a}_0 = [x,\, y,\, z]^T$, then relax condition (1)[1], and let $z = 1$, which yields

$$D_j(\mathbf{a}_0) \equiv A_{j3}y^3 + A_{j2}y^2 + A_{j1}y + A_{j0} = 0, \quad j = 1, \ldots, 4 \tag{17}$$

where A_{jk} is a polynomial in x of degree $3 - k$. In the final step, the four equations above are cast in linear homogeneous form in the first four powers of y, including $y^0 = 1$:

$$\mathbf{A}\mathbf{y} = \mathbf{0}_4 \tag{18}$$

with

$$\mathbf{A} \equiv \begin{bmatrix} A_{13} & A_{12} & A_{11} & A_{10} \\ A_{23} & A_{22} & A_{21} & A_{20} \\ A_{33} & A_{32} & A_{31} & A_{30} \\ A_{43} & A_{42} & A_{41} & A_{40} \end{bmatrix}, \quad \mathbf{y} \equiv \begin{bmatrix} y^3 \\ y^2 \\ y \\ 1 \end{bmatrix} \tag{19}$$

In light of the shape of $\mathbf{y}$—its fourth entry is $1 \neq 0$—the system (18) does not admit the trivial solution, and hence, $\mathbf{A}$ must be singular, i.e.,

$$\det(\mathbf{A}) = 0 \tag{20}$$

[1]This is possible because one is interested in finding a line, the axis of the moving-revolute joint; any point on the axis suffices to find the axis, not only the one lying a unit distance from the origin.

Apparently, the first column of $\mathbf{A}$ is cubic, the second quadratic, the third linear and the fourth of degree 0 in x, whence $\det(\mathbf{A})$ is a polynomial in x of degree $3+2+1+0=6$, q.e.d.

As a consequence, the problem admits six, four, two or zero circlepoint solutions. The same reasoning leads to the conclusion that the problem also admits six, four, two or zero centrepoint solutions. Therefore, the number of possible dyads that solve the problem is six, four, two or zero. Correspondingly, the number of spherical four-bar linkages that can guide their coupler link through the five given attitudes is the number of combinations of six, four or two objects taking two at a time, i.e., 15, 6 or 1.

Remark: If $\mathbf{a}_0$ happens to lie in the X-Y plane, then $z=0$, and the substitution $z=1$ does not work. In this case, simply choose an alternative coordinate, e.g., $y=1$, and the proof should work.

5 Computational Algorithm

The foregoing discussion is intended to lay down the principles underlying the Spherical Burmester Problem. It is by no means intended to be an algorithm to compute the coordinates of points B and A_0 on the unit sphere. This is most simply done using *spherical coordinates* on the unit sphere, namely, *longitude* and *latitude*. Let, then, ϑ_A and φ_A[2] be the longitude and the latitude of A_0, ϑ_B and φ_B those of B. Hence,

$$\mathbf{a}_0 = \begin{bmatrix} \cos\varphi_A\cos\vartheta_A \\ \cos\varphi_A\sin\vartheta_A \\ \sin\varphi_A \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \cos\varphi_B\cos\vartheta_B \\ \cos\varphi_B\sin\vartheta_B \\ \sin\varphi_B \end{bmatrix} \tag{21}$$

Now the four determinant equations (15) in $\mathbf{a}_0$ become equations in the harmonic functions of φ_A and ϑ_A. Hence, any two of the four equations thus resulting can be used to find the spherical coordinates of $\mathbf{a}_0$. Again, as in the planar case, it is convenient to use all four equations to add robustness to the solution. The method recommended here to compute all real solutions is based on a semigraphical approach: the jth determinant equation (15) defines a contour $\mathcal{C}_j$ in the φ_A-ϑ_A plane. If the four contours are plotted in the rectangle $-\pi \le \varphi_A \le \pi$, $-\pi \le \vartheta_A \le \pi$, then the intersections of the four contours yield *all the solutions* sought. These intersections can be estimated by inspection with two digits of precision, which can be good enough for most engineering problems. If higher precision is required, these estimates can be fine-tuned by means of the Newton-Gauss method for nonlinear least-square problems, as only two unknowns are to be found from four nonlinear equations. Moreover, if the estimates are given as initial guesses to the Newton-Gauss method, then the code implementing the method should converge within a couple of iterations.

6 Example

An example of spherical rigid-body guidance is given by Chiang (1988). The four cubic conical surfaces resulting from these data are produced and displayed in `Chiang5Poses.mw`

The above Maple worksheet is available on the course website:

`http://www.cim.mcgill.ca/~rmsl/Index/index.htm`

When visiting that site, look four **Courses** and then **MECH 541**.

[2]Literals ϑ and φ are read *varphi* and *vatheta*, respectively.