A Robust Solution of the Spherical Burmester Problem

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The Spherical Burmester Problem

What is the Burmester problem?

- The classical Burmester problem
  Are there any points in a rigid body whose corresponding position lies on a circle of the fixed plane for the four arbitrarily prescribed positions?–Burmester, 1888
- The spherical Burmester problem
  An extension of the classical Burmester problem, in which we are interested in synthesizing a spherical four-bar linkage to visit a discrete set of orientations of the coupler link

Revelent Works

The nature of the problem is to solve a system of trigonometric equations. The challenges can be seen in

- The numerics
- Branch-defect detection
- Consideration of joint type, R- or P- joint

The problem has been studied by different approaches:

- Geometric: using the Burmester-Roth Theorem to construct center axes—Roth, 1967
- Polynomial approach: sixth degree polynomial can be found through variable elimination—Ruth and McCarthy, 1996

The proposed approach in this work

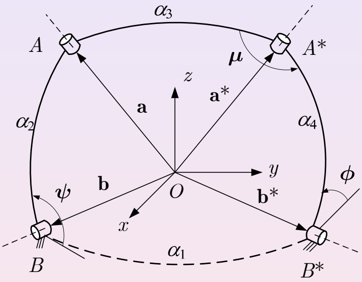
Proposed here is a semigraphical approach, consisting of three steps:

1. Algebraic formulation—problem definition
2. Graphic display+inspection—raw solution estimates
3. Numerical solution—accurate result

It eliminates spurious solutions and filters complex solutions

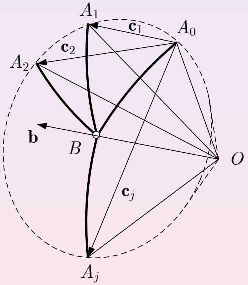
An extension of a previous work on the classical Burmester problem—Chen, Bai and Angeles 2008

Problem Formulation



Find a spherical four-bar linkage that will conduct its coupler link through a set $\mathcal{S}$ of m attitudes given by the orthogonal matrices $\left\{\mathbf{Q}_j\right\}_1^m$, defined with respect to a reference attitude given by $\mathbf{Q}_0 = \mathbf{1}$, the 3×3 identity matrix.

Synthesis Equations



Assuming all links are rigid, all angles α_i remain constant

$$\mathbf{a}_j^T\mathbf{b} = \mathbf{a}_0^T\mathbf{b} \quad \text{or} \quad (\mathbf{a}_j - \mathbf{a}_0)^T\mathbf{b} = 0; j = 1, \ldots, m \tag{1}$$

where

$$\mathbf{a}_j = \mathbf{Q}_j\mathbf{a}_0 \tag{2}$$

In a compact form, (1) becomes

$$\mathbf{c}_j^T\mathbf{b} = 0, \quad j = 1, \ldots, m \tag{3}$$

with

$$\mathbf{c}_j \equiv (\mathbf{Q}_j - \mathbf{1})\mathbf{a}_0 \tag{4}$$

Synthesis Equation (cont'd)

In *invariant form*,

$$\mathbf{Q}_j = \mathbf{1} + s_j \mathbf{E}_j + (1 - c_j)\mathbf{E}_j^2, \quad c_j \equiv \cos\phi_j, \quad s_j \equiv \sin\phi_j \qquad (5)$$

where $\mathbf{E}_j$ denotes the *cross-product matrix* (CPM) of $\mathbf{e}_j$, the unit vector that defines the direction of the axis of rotation of $\mathbf{Q}_j$, and ϕ_j the angle of rotation. The synthesis equation becomes

$$\mathbf{a}_0^T \mathbf{E}_j [s_j \mathbf{1} - (1 - c_j)\mathbf{E}_j]\mathbf{b} = 0, \quad j = 1, \ldots, m \qquad (6)$$

whose solution depends on the number m of prescribed poses.

Synthesis with three poses

In this case, $m = 2$, i.e., two constraint equations occur:

$$\mathbf{c}_1^T \mathbf{b} = 0, \quad \text{and} \quad \mathbf{c}_2^T \mathbf{b} = 0 \tag{7}$$

One of the two vectors $\mathbf{a}_0$ and $\mathbf{b}$ can be prescribed arbitrarily. In the case of $\mathbf{a}_0$ known, then

$$\mathbf{b} = \frac{\mathbf{c}_1 \times \mathbf{c}_2}{\|\mathbf{c}_1 \times \mathbf{c}_2\|} \tag{8}$$

A similar reasoning, with obvious modifications, follows if $\mathbf{b}$ is prescribed.

Synthesis with four poses

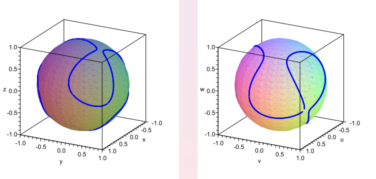
Now we have $m = 3$, the constraints being

$$\mathbf{c}_1^T \mathbf{b} = 0, \quad \mathbf{c}_2^T \mathbf{b} = 0 \quad \text{and} \quad \mathbf{c}_3^T \mathbf{b} = 0 \tag{9}$$

All three vectors $\mathbf{c}_j$ must be coplanar, and hence

$$F(\mathbf{a}_0) \equiv \mathbf{c}_1 \times \mathbf{c}_2 \cdot \mathbf{c}_3 = 0 \tag{10}$$

This is a cubic, homogeneous equation in $\mathbf{a}_0$, which defines a conical cubic surface whose apex is the origin.



(a) (b)

Figure: (a) the circlepoint curve and (b) the centerpoint curve

Synthesis with five poses

Now $m = 4$, the synthesis equations can be cast into the form

$$\underbrace{\begin{bmatrix} \mathbf{a}_0^T\mathbf{E}_1[s_1\mathbf{1} - (1-c_1)\mathbf{E}_1] \\ \mathbf{a}_0^T\mathbf{E}_2[s_2\mathbf{1} - (1-c_2)\mathbf{E}_2] \\ \mathbf{a}_0^T\mathbf{E}_3[s_3\mathbf{1} - (1-c_3)\mathbf{E}_3] \\ \mathbf{a}_0^T\mathbf{E}_4[s_4\mathbf{1} - (1-c_4)\mathbf{E}_4] \end{bmatrix}}_{\equiv \mathbf{C}} \mathbf{b} = \mathbf{0}_4 \tag{11}$$

Non-trivial solutions require a rank-deficient $\mathbf{C}$:

$$\Delta_j(\mathbf{a}_0) \equiv \det(\mathbf{C}_j) = 0, \quad j = 1, \ldots, 4 \tag{12}$$

$\mathbf{C}_j$ being the 3×3 matrix obtained from $\mathbf{C}$ upon deleting its jth row. Equations (12) define four conical cubic surfaces with one common apex.

Synthesis with one P-joint

A spherical linkage may end up with a P-joint, i.e., a slider moving on a circular guide, similar to the planar crank-slider mechanism.

We consider this a special case, with $\alpha_4 = 90\,\text{deg}$. Thus

$$\mathbf{b}^T\mathbf{Q}_j\mathbf{a}_0 = 0, \quad j = 0, \ldots, m \tag{13}$$

The above m equations can be written as

$$\mathbf{H}\mathbf{a}_0 = \mathbf{0}_n \tag{14}$$

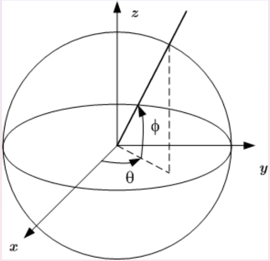
where $\mathbf{H}$ is a $n \times 3$ matrix with $n = m + 1$. Non-trivial solutions for $\mathbf{b}$ require a rank-deficient $\mathbf{H}$, i.e., for $m = 3$,

$$\det(\mathbf{H}_i) = 0, \quad i = 1, \ldots, 4 \tag{15}$$

where the 3×3 matrix $\mathbf{H}_i$ is obtained by removing the ith row from matrix $\mathbf{H}$.

Representation of the axes of rotation

We shall use *spherical coordinates* on the unit sphere, namely, *longitude* and *latitude*.



Let ϑ_A and φ_A be the longitude and the latitude of A_0, ϑ_B and φ_B those of B. Hence,

$$\mathbf{a}_0 = \begin{bmatrix} \cos\varphi_A \cos\vartheta_A \\ \cos\varphi_A \sin\vartheta_A \\ \sin\varphi_A \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \cos\varphi_B \cos\vartheta_B \\ \cos\varphi_B \sin\vartheta_B \\ \sin\varphi_B \end{bmatrix} \tag{16}$$

The ranges of all spherical coordinates: $\{\varphi_A, \vartheta_A, \varphi_B, \vartheta_B\} \in [-\pi/2, \pi/2]$.

Branching detection

Branching-defect can occur in spherical linkages.

The branching defect is detected here by means of the sign of the sine of the transmission angle, in analogy with the planar case:

$$\sin\mu = \| \overline{\mathbf{a}} \times \overline{\mathbf{b}} \| / (\| \overline{\mathbf{a}} \| \| \overline{\mathbf{b}} \|) \tag{17}$$

where $\overline{\mathbf{a}} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{a}^\star)\mathbf{a}^\star$ and $\overline{\mathbf{b}} = \mathbf{b} - (\mathbf{b} \cdot \mathbf{a}^\star)\mathbf{a}^\star$, with
$\mathbf{a} = \mathbf{Q}\mathbf{a}_0, \quad \mathbf{a}^\star = \mathbf{Q}\mathbf{a}_0^\star$.

The sign of the sine of the transmission angle is given by

$$\mathrm{sgn}(\mu) = \mathrm{sgn}\left[(\overline{\mathbf{a}} \times \overline{\mathbf{b}}) \cdot \mathbf{a}^\star\right] \tag{18}$$

Example 1

Prescribed poses described by axes and angles of rotations with reference to the initial pose.

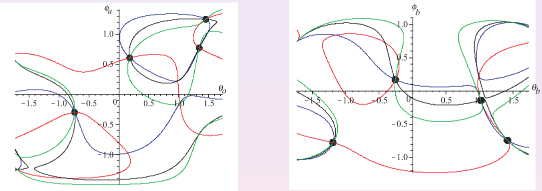
Table: Five poses for Example 1

ϕ_j[rad]	$\mathbf{e}^T$	$[\beta_1, \beta_2, \beta_3]$[deg]
0	[0,0,1]	[0,0,0]
0.2034	$[-0.0449, -0.5133, -0.8569]$	$[-10.0, -6.0, 0]$
1.1957	$[0.1827, 0.7709, -0.6101]$	$[-51.0, -52.0, -12.0]$
1.1932	$[0.5212, 0.8414, -0.1422]$	$[15.0, -56.0, 47.0]$
1.0512	$[0.5384, 0.8114, 0.2271]$	$[33.0, -40.0, 47.0]$

Note: β_1, β_2 and β_3 are the corresponding angles of longitude and latitude, respectively, which are also given with the purpose of helping the visualization of the prescribed attitudes.

Example 1 (cont'd)

Four sets of real solutions are found from the contour plots of angles of longitude and latitude

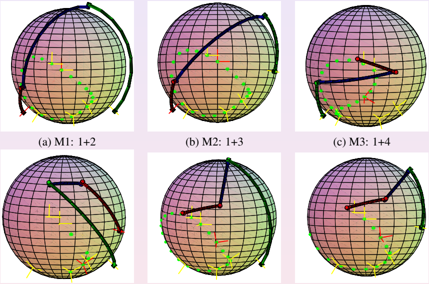


(a) (b)

	$\mathbf{a}_0$(or $\mathbf{a}_0^\star$)	$\mathbf{b}$(or $\mathbf{b}^\star$)
# 1	[0.7085, −.6418, −.2932]	[0.2640, −.6636, −.6998]
# 2	[0.0385, 0.3163, 0.9478]	[0.1143, 0.7263, −.6777]
# 3	[0.1642, 0.6977, 0.6972]	[0.5218, 0.8413, −.1403]
# 4	[0.8077, 0.1493, 0.5702]	[0.9524, −.2535, 0.1686]

Six spherical linkages for Example 1

Altogether, six linkages can be synthesized for this example.



(a) M1: 1+2 (b) M2: 1+3 (c) M3: 1+4

(d) M4: 2+3 (e) M5: 2+4 (f) M6: 3+4

Note:

- All task orientations shown in yellow
- The frames in red show the orientations of the coupler
- Branch defects were detected for M3 and M4.

Example 2: Synthesis with one P-joint

Five poses and solutions to the linkage with one P-joint

ϕ_j [rad]	$\mathbf{e}^T$	$[\beta_1, \beta_2, \beta_3]$[deg]
0	[0,0,1]	[0,0,0]
0.2563	$[-0.2280, -0.4553, -0.8606]$	$[-12.5, 7.0, -2.6]$
1.1307	$[-0.0578, 0.2370, -0.9697]$	$[-64.2, -10.5, -10.8]$
1.1938	$[0.5049, 0.8505, -0.1468]$	$[14.3, -56.9, 45.6]$
1.3665	$[0.7119, 0.6601, 0.2393]$	$[45.1, -30.7, 73.2]$

	$\mathbf{a}_0$	$\mathbf{b}$
# 1	$[0.2845, 0.3863, 0.8773]$	$[0.1219, -.7089, -.6946]$
# 2	$[0.7226, 0.5295, 0.4442]$	$[0.2309, 0.4566, 0.8591]$
# 3	$[0.9573, -.2433, 0.1555]$	$[0.8134, 0.1643, 0.5579]$
	$\mathbf{a}_0^\star$	$\mathbf{b}^\star$
# 4	$[0.5221, 0.8442, -.1208]$	$[0.0655, 0.1015, 0.9926]$

Mechanism synthesized for Example 2

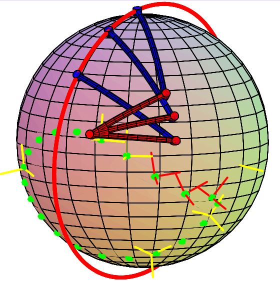


Figure: One synthesized mechanism with a P-joint, showing three link attitudes

Conclusions

- A robust solution of synthesis equation for the spherical Burmester problem was formulated
- A semigraphical solution method was developed
- Branch defect detection and synthesis with one P-joint were considered
- The method was demonstrated with two examples
- Future work? spatial Burmester problem

References

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