Optimization of a Four-bar Spherical Homokinetic Linkage with Minimum Design Error

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Abstract—*Motion transmission between two shafts with intersecting axes at right angles is a recurrent problem in machine design. The mechanism should be able to accommodate the given layout of the driver and the driven shafts. Further, a constant velocity ratio, in our case 1:1, between the input and the output velocities is usually desired to ease the control algorithm. In this paper, a four-bar spherical linkage is optimally designed to transmit motion between two orthogonal intersecting shafts with an approximately constant 1:1 velocity ratio through a* $120°$ *rotation of its input link. This is done via minimizing the root-mean-square value of the design error at a sample of input-output values; the error is defined as the residue of synthesis equation at the prescribed set of input values. Optimization is reported here by means of a shifting of the zeros of the input and the output dials. To obtain the global minimum of this optimization problem, its first-order normality conditions are formulated. Eliminating all unknowns except for the two shift angles yields a set of two nonlinear equations in two unknowns, whose real solutions are found by a semigraphical approach.*

Keywords: homokinetic, spherical linkage, design error, optimization

I. Introduction

Transmission of motion between two intersecting shafts is a recurrent problem in mechanical design. So far, different types of coupling mechanisms have been designed in order to transmit motion between shafts in various layouts. Other design requirements might apply as well; for instance, in some applications, the coupling mechanism should be capable of handling small misalignments or limiting the transmitting torque below an upper bound. With this regard, couplings can be categorized into three groups, namely, rigid, flexible and torque limiting. While rigid couplings require a perfect geometric layout, flexible couplings can cope with misalignments. A widely used family of coupling mechanisms, being referred to as *constant velocity* or *homokinetic joints*, can transmit motion between two shafts with a constant velocity ratio of 1:1 [1]; this characteristic is highly desirable from the control viewpoint. A class of homokinetic joints allowing relative displacements of the input and output shafts are wellknown in the automotive industry. Particularly, in front-wheel drive cars, the motion of the suspension system as well as the steering mechanism continuously change the angle between the gearbox and the wheels; it is important that these variations be tolerated by the coupling mechanism in operation [2].

Universal joints are simple examples of couplings that transmit motion between intersecting shafts. One single universal joint can neither maintain a constant velocity ratio nor connect two orthogonal shafts. However, these shortcomings can be eliminated by means of pairs of universal joints to compensate for velocity ratio variations [3]. In fact, to connect two orthogonal axes, two universal joints and an intermediate shaft should be mounted such that the intermediate shaft makes an angle of $45°$ with the input and the output axes, as shown in Fig. 1. It was first realized by Robert Hooke that the coupling mechanism thus resulting is of the homokinetic type [2], [3], [4].

Rzeppa joints, whose components are displayed in Fig. 2, are another type of homokinetic joints, which can transmit motion between two intersecting shafts [5]. As shown in Fig. 2, the Rzeppa joint consists of an inner race, an outer race, a cage and six balls. The balls move inside six grooves located on the external and the internal peripheries of the inner and the outer races, respectively. The cage is mounted between the two races so as to keep the balls inside the grooves. The rotation of the input shaft, attached to the inner race, is transmitted to the output shaft, which is rigidly attached to the outer race, through the motion of the balls. The maximum angle allowed between the two shafts is about $50°$; hence, two Rzeppa joints should be implemented in series so as to uniformly transmit motion be-

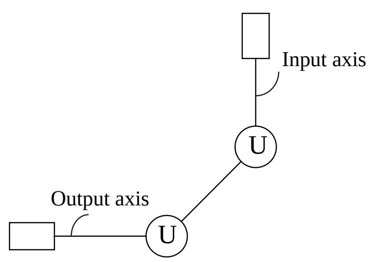

Fig. 1. Double universal joint mechanism

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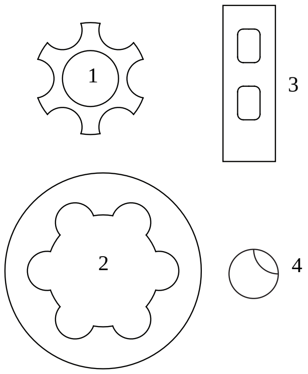

Fig. 2. Rzeppa joint: 1) inner race; 2) outer race; 3) cage; 4) ball

tween two orthogonal shafts. There are also other types of homokinetic joints, namely, the tripod and the Thompson mechanisms [6], [7].

Proposed in this paper is a spherical mechanism, which couples two orthogonal shafts with an approximately constant velocity ratio. Considering the kinematic relations governing the motion of spherical linkages [8], achieving a constant velocity ratio during the whole motion of their input link is not feasible. Nevertheless, we aim to design a spherical mechanism so as to exhibit, approximately, a constant velocity ratio of 1:1 in, at least, 120° rotation of its input link, which is large enough for many robotic applications. The design problem at hand can be formulated as an *unconstrained optimization* task [9], for which many solution algorithms are available in the literature. However, finding the global optimum of any optimization problem requires identification of all stationary points, defined as the points at which the first-order normality conditions (FONCs) are satisfied. In this paper, an elimination procedure is utilized to reduce the FONCs to a set of two nonlinear equations in terms of two unknowns, which is, then, solved via a graphical approach.

The paper is organized as follows: in Section II, the synthesis equations of the mechanism are derived. Next, the design problem is formulated as an optimization task, whose global minima are found thereafter. The paper closes with some concluding remarks.

II. Synthesis Equations of the Spherical Linkage

A four-bar spherical linkage, shown in Fig. 3, is a mechanism in which all points of the links move on the surfaces of concentric spheres, whose centre is located at the intersection of its joints axes. From geometry, the governing synthesis equation of the four-bar spherical linkage, is found as [8]:

$$F(\psi,\phi) \equiv k_1 + k_2\cos\psi + k_3\cos\psi\cos\phi \\ -k_4\cos\phi + \sin\psi\sin\phi = 0 \quad (1)$$

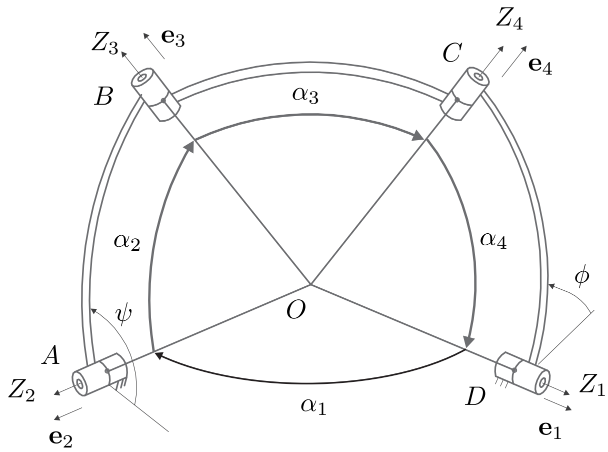

Fig. 3. The kinematic chain of a four-bar spherical linkage

where ψ and ϕ are the input and the output angles, shown in Fig. 3; k_i for $i = 1, ..., 4$, being the linkage Freudenstein parameters. The relations between the Freudenstein parameters and the link dimensions are given below:

$$\cos\alpha_1 - k_3 = 0 \quad (2a)$$

$$\cos\alpha_4\sin\alpha_1 - k_2\sin\alpha_4 = 0 \quad (2b)$$

$$\cos\alpha_2\sin\alpha_1 - k_4\sin\alpha_2 = 0 \quad (2c)$$

$$\cos\alpha_1\cos\alpha_2\cos\alpha_4 - \cos\alpha_3 \\ -k_1\sin\alpha_2\sin\alpha_4 = 0 \quad (2d)$$

where α_i for $i = 1, ..., 4$ are the link arcs.

Since we aim to synthesize a spherical linkage for transmitting the motion between two orthogonal axes, $\alpha_1 = \pi/2$ is substituted into eq. (2a), which makes $k_3 = 0$. Moreover, our engineering insight into the 1:1 velocity ratio requirement reveals that the mechanism should have a symmetric architecture, with $\alpha_2 = \alpha_4$, and hence, $k_2 = k_4$. Therefore, the synthesis equations of the spherical coupling mechanism can be cast in vector form as:

$$\mathbf{Sk} = \mathbf{b} \quad (3)$$

where,

$$\mathbf{S} = \begin{bmatrix} 1 & \cos\psi_1^* - \cos\phi_1^* \\ 1 & \cos\psi_2^* - \cos\phi_2^* \\ \vdots & \vdots \\ 1 & \cos\psi_m^* - \cos\phi_m^* \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \\ \mathbf{b} = \begin{bmatrix} -\sin\psi_1^*\sin\phi_1^* \\ -\sin\psi_2^*\sin\phi_2^* \\ \vdots \\ -\sin\psi_m^*\sin\phi_m^* \end{bmatrix} \quad (4)$$

with

$$\psi_i^* = \psi_i + \zeta, \quad \phi_i^* = \phi_i + \eta, \quad \text{for } i = 1, ..., m \quad (5)$$

in which $\{\psi_i, \phi_i\}_1^m$ is the set of m prescribed input-output (IO) pairs lying on a line with unit slope in the ψ-ϕ plane, where ζ and η are the shift angles to be determined optimally, and $\{\psi_i^*, \phi_i^*\}_1^m$ are the shifted values of the prescribed IO pairs.

III. Optimization of the Spherical Linkage

The objective is thus to design a four-bar spherical linkage to transmit motion between two orthogonal shafts with a velocity ratio between the input and the output rates as close as possible to 1:1. Moreover, the mechanism should be capable of maintaining this velocity ratio in, at least, 120° of rotation of its input link. Hence, the optimization problem can be stated as: *minimize the rms value of the components of the design error vector, defined as* $\mathbf{e} = \mathbf{b} - \mathbf{SK}$*, or its square for that matter, over the design variables* $\mathbf{x} = \begin{bmatrix}\mathbf{k}^T & \zeta & \eta\end{bmatrix}^T$*, subject to no constraints. This can be algebraically written as:*

$$\min f, \quad f \equiv \frac{1}{m}\|\mathbf{e}\|^2$$

To solve this unconstrained optimization problem, first, a uniformly distributed set of m angle values in the interval $[-60^\circ, 60^\circ]$ is chosen as the prescribed set of input and output angles. Upon substituting these prescribed values into eq. (4), the FONCs of the optimization problem are formulated by zeroing the partial derivatives of the objective function f with respect to $\mathbf{k}$, ζ and η, respectively. The FONCs obtained are, thus,

$$\frac{\partial f}{\partial \mathbf{k}} = \mathbf{0} \Rightarrow \mathbf{S}^T\mathbf{S}\mathbf{k} - \mathbf{S}^T\mathbf{b} = \mathbf{0} \tag{6a}$$

$$\frac{\partial f}{\partial \zeta} = 0 \Rightarrow \left(\frac{\partial \mathbf{b}^T}{\partial \zeta} - \mathbf{k}^T\frac{\partial \mathbf{S}^T}{\partial \zeta}\right)(\mathbf{b} - \mathbf{SK}) = 0 \tag{6b}$$

$$\frac{\partial f}{\partial \eta} = 0 \Rightarrow \left(\frac{\partial \mathbf{b}^T}{\partial \eta} - \mathbf{k}^T\frac{\partial \mathbf{S}^T}{\partial \eta}\right)(\mathbf{b} - \mathbf{SK}) = 0 \tag{6c}$$

where $\mathbf{0}$ is the two-dimensional zero vector.

The stationary points of the foregoing problem can be found by solving the system of equations (6), which consists of four nonlinear equations in terms of four unknowns, and hence, is solvable by a host of numerical methods, e.g. the Newton-Raphson's. However, such methods are dependent on the choice of an initial guess. Therefore, upon convergence, only one local stationary point is found, which can be a minimum, a maximum or a saddle point. A global minimum can be obtained provided that all stationary points are found; this calls for identification of all solutions of the FONCs at hand, which is not possible using the Newton-Raphson method or its numerical counterparts.

Not to be confined to the local minima of the problem, an elimination procedure is utilized so as to eliminate all the unknowns except two in eqs. (6). The intersections of the contour plots of the equations thus resulting, in the plane of these two unknowns are the set of real stationary points sought. Thus, we, first, find $\mathbf{k}$ from eq. (6a), using computer algebra, as:

$$\mathbf{k} = \left(\mathbf{S}^T\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{b} \tag{7}$$

Substituting $\mathbf{k}$ back into eqs. (6b & c) yields a system of two equations in two unknowns, ζ and η.

Choosing $m = 11$ input and output values, uniformly distributed in the interval $[-60^\circ, 60^\circ]$, the Freudentein parameters are eliminated between the FONCs (6) using eq. (7). The two equations thus resulting are only functions of ζ and η; henceforth, their solutions can be obtained from the intersections of their contours, which are illustrated in Fig. 4.

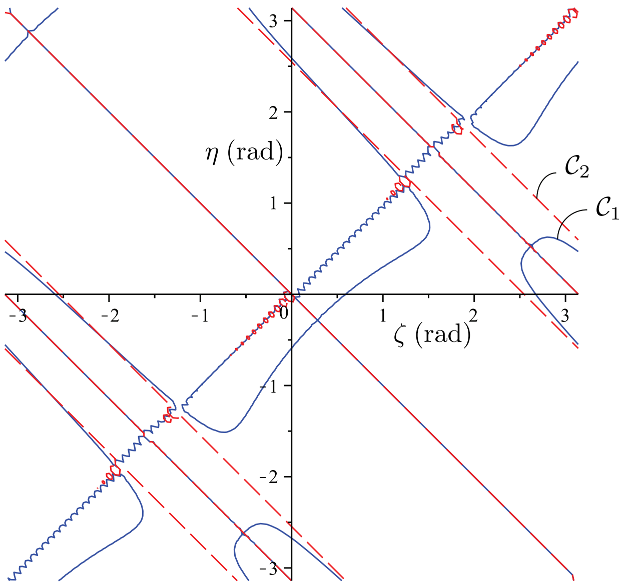

Fig. 4. The contour plots of the FONCs, $\mathcal{C}_1$: the second FONC, $\mathcal{C}_2$: the third FONC

The coordinates of the intersections, being simply extracted by looking at the plots shown in Fig. 4, and their corresponding values of the objective function are tabulated in Table I. If higher precision is required, the coordinates of each intersection in Table I should be submitted to the Newton-Raphson algorithm as an initial guess. It is noteworthy that, in addition to four isolated solutions, three

TABLE I. Stationary points

k	ζ_k (rad)	η_k (rad)	f
1	0.5	2.1	0.0026
2	-0.5	-2.1	0.0026
3	1.15	2.6	0.0026
4	-1.15	-2.6	0.0026
5	$\zeta_k = -\eta_k$	η_k	0.0704
6	$\zeta_k = \pi - \eta_k$	η_k	0.0005
7	$\zeta_k = -\pi - \eta_k$	η_k	0.0005

families of solutions exist, as recorded in the last three rows of Table I.

The type of each stationary point can be determined by means of the second-order normality conditions. However, upon comparing the values of the objective function at the stationary points, it is readily concluded that the global minimum occurs at every pair $[\zeta, \eta]^T$ verifying either $\zeta = \pi - \eta$ or $\zeta = -\pi - \eta$, with the global minimum of the objective function being 0.0005.

Since a family of global minima is obtained, let us arbitrarily choose a pair of $[\zeta, \eta]^T$, which satisfies one of the aforementioned conditions, as $[140°, 40°]^T$. Substituting back these values into eq. (7), the Freudenstein parameters of the linkage are found as $\mathbf{k} = [1.1161, 0.9790]^T$. Here, a word of caution is in order: equation (7), in general, is a formula, but not an algorithm, to calculate the least-square approximation of an overdetermined system of linear equations. In fact, the verbatim application of this formula is prone to round-off error amplification, and hence, should be avoided [10]. Several stable numerical algorithms are available to safely handle linear least square problems. However, in our specific case, we have computed $\mathbf{k}$ in closed-form by using Maple. Therefore, the numerical calculations are limited to substituting back ζ and η into the formula and does not involve any matrix inversions.

The link arcs of the optimum spherical coupling mechanism are obtained as:

$$\alpha_1 = 90°, \quad \alpha_2 = 45.6078°, \quad \alpha_3 = 124.7414°,$$
$$\alpha_4 = 45.6078°$$

The CAD model of this mechanism is depicted in Fig. 5.

In Fig. 6, the generated output angles are plotted versus a given set of input angles, uniformly distributed in the interval $[-60°, 60°]$. As expected, the plot is very close to a unit slope line showing consistency with our requirement of having constant 1:1 velocity ratio.

IV. Conclusions

A spherical four-bar linkage, with its input and output axes intersecting at $90°$, was proposed to transmit homokinetic motion between two intersecting shafts. A design requirement was set such that the velocity ratio between the input and the output rates be as close as possible to 1:1 in, at least, $120°$ rotation of the input link. To this end, an unconstrained optimization problem was formulated to minimize the design error by varying the zeros of the input and output dials. Formulating the FONCs, a set of four nonlinear equations in four unknowns, namely, the Freudenstein parameters involved, and the values of the shift angles sought, was obtained. Since we aimed to find the global minimum of the problem, the Freudenstein parameters were eliminated to derive a set of two nonlinear equations in terms of two unknowns. This set of equations was, then, solved by a graphical approach. The CAD model and the plot of the output versus the input angles were illustrated as well. The results showed consistency with the constant velocity ratio prescribed at the outset.

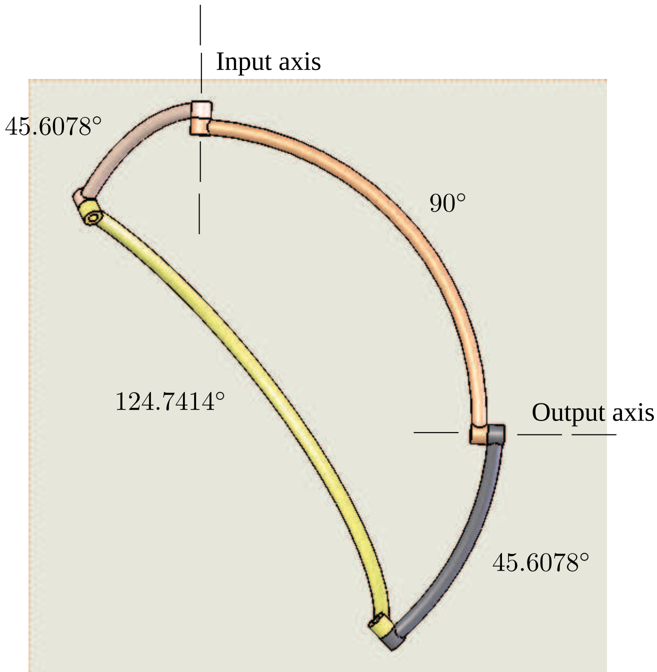

Fig. 5. The CAD model of an optimum linkage

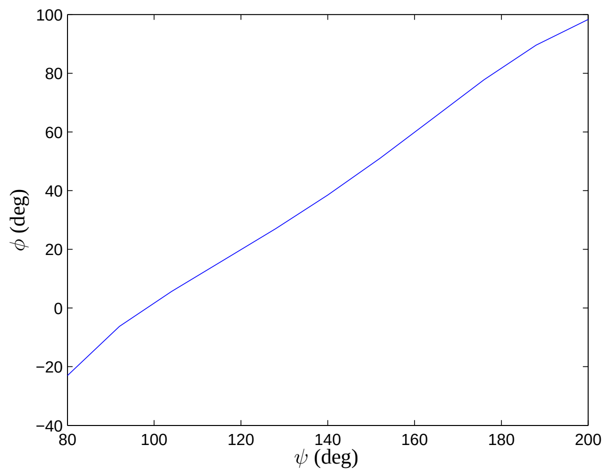

Fig. 6. The generated output angles of the mechanism versus its input angles

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References

[1] Bellomo, N. and Montanari, P. The general theory of the mechanics of a large class of multi-bodied systems for constant velocity transmission between intersecting axes. *Mechanism and Machine Theory*, 13:361–368, 1978.

[2] Kocabas, H. Design and analysis of a spherical constant velocity coupling mechanism. *ASME Journal of Mechanical Design*, 129:991–998, 2007.

[3] Johnson, D.A. and Willems, P.Y. On the necessary and sufficient conditions for homokinetic transmission in chains of cardan joints. *ASME Journal of Mechanical Design*, 115:255–261, 1993.

[4] Brutti, C. and Pennestri, E. and Biancolini, M.E. On the dynamics of transmission with a double Cardan joint. In *Tenth World Congress on the Theory of Machines and Mechanisms*. Oula, Finland, 1999.

[5] Rzeppa, A. Universal joint. *United States Patent*, 1934.

[6] Watanabe, K., Kawakatsu, T., and Nakao, S. Kinematic and static analyses of tripod constant velocity joints of the spherical end spider type. *ASME Journal of Mechanical Design*, 127:1137–1144, 2005.

[7] Thompson, G. Constant velocity coupling and Control System Therefor. *United States Patent*, 2006.
Constant velocity coupling and Control System Therefor

[8] Chiang, C.H. *Kinematics of Spherical Mechanisms*. Cambridge University Press, 1988.

[9] Rao, S.S. *Engineering Optimization*. John Wiley and Sons, 3rd ed., 1996.

[10] Golub, G.H. and Van Loan, C.F. *Matrix Computations*. The Johns Hopkins University Press, Baltimore.