On the Home Posture of the McGill Schönflies Motion Generator

TR-CIM-09-01 January 2009

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Abstract

In the determination of the *home posture* of a two-limb Schönflies Motion Generator (SMG), which is defined here as that at which the forward-kinematics Jacobian attains its minimum condition number, the geometry and velocity analysis of the robot is recalled from a previous publication. Given the simplicity of this parallel-robot architecture, it is possible to obtain a closed-form expression of its condition number based on the matrix Frobenius norm. By making intensive use of both linear-algebra identities and results specific to the kinematics of the Schönflies subgroup, the normality conditions associated with the minimization of the condition number of interest are derived in frame-invariant form. The frame-invariant form lends itself to geometric interpretations that would not be possible with lengthy componentwise expressions.

Contents

1 Introduction

The *home posture*, sometimes referred to as the *home configuration* or, even erroneously as the *home pose*$^{\#}$, is that posture at which the robot is set when not in operation. There are no rules to define this posture, but this is usually chosen as a retracted one, occupying a minimum volume. In this report we propose to define the home posture as that at which the forward-kinematics Jacobian attains a minimum condition number. As condition number minimization is a rather complex problem, in that gradient evaluations, needed to geometrically characterize the posture of interest, are particularly challenging, we start by discussing the computational issues surrounding the derivation of the first-order normality conditions of the optimization problem at hand. The discussion takes place within the framework of computational kinematics.

The use of computer algebra in robot kinematics can be traced back to the late seventies, during the intensive quest for finding the *characteristic polynomial* of the general six-revolute robotic manipulator with serial architecture. Early attempts to derive this polynomial were reported by Duffy and Derby (1979), Duffy and Crane (1980), Albala (1982), and Alizade et al. (1983), who derived a 32nd-degree polynomial, but suspected that this polynomial was not minimal in the sense that the manipulator at hand might not be able to admit up to 32 postures for a given end-effector (EE) pose. Tsai and Morgan (1985) used a technique known as *polynomial continuation* (Morgan, 1987) to solve *numerically* the nonlinear displacement equations, cast in the form of a system of quadratic equations. These researchers found that no more than 16 solutions were to be expected. Primrose (1986) proved conclusively that the problem under discussion admits at most 16 solutions, while Lee and Liang (1986) showed that the same problem leads to a 16th-degree univariate polynomial. Using different elimination procedures, Li (1990) and Raghavan and Roth (1990, 1993) devised different procedures for the computation of the coefficients of the univariate polynomial. More recently, Husty et al. (2007) reported a geometric, streamlined approach to the derivation of the minimal characteristic polynomial.

While the derivation of the 16th-degree polynomial associated with serial robots can be considered essentially accomplished since the early nineties, kinematicians and geometers embarked in the eighties on a more challenging problem, the derivation of the *minimal characteristic polynomial* of the general Stewart-Gough (SG) manipulator. A breakthrough in the solution of the direct kinematics of parallel manipulators of the general type was reported by Raghavan (1993), who resorted to polynomial continuation for computing up to 40 poses of the moving platform for given leg-lengths of the SG manipulator with attachment points at both the moving and the base platform with an arbitrary layout. What Raghavan did not derive is the characteristic 40th-degree polynomial of the general SG manipulator. Independently, Wampler (1996) and Husty (1996) devised procedures to derive this polynomial, although Wampler did not pursue the univariate polynomial approach and preferred to cast the problem in a form suitable for a solution by means of polynomial continuation. Husty did derive the 40th-degree polynomial for several examples, but stayed short of showing that his polynomial was minimal in that manipulator architectures are possible that exhibit up to 40 actual solutions. Dietmaier (1998) did this by devising an algorithm that would iteratively increase the number of real solutions of a given architecture. With this paper, Dietmaier proved conclusively that Husty's 40th-degree polynomial is, in fact, minimal.

Methods of polynomial continuation (Sommese and Wampler, 2005) aside, the above works rely on the *elimination procedures* used by computer-algebra software that are capable of eliminating symbolically all unknowns but one, thereby deriving a univariate polynomial that is the characteristic polynomial sought. These methods are applicable because, by virtue of what is known as the tan-half trigonometric identities[1], the trigonometric equations of the kinematic model involved are transformed into polynomial equations.

In fact, commercial software is capable of handling cumbersome expressions of scalar quantities in symbolic form. However, this kind of software is incapable of handling vectors and tensors in invariant, coordinate-free form. This shortcoming can be overcome if fundamental linear-algebra relations, well known in the realm of system theory, but rather unknown in computational kinematics, are recalled. In this report some of these relations are extensively used in determination of the *home posture* of a class of two-limb Schönflies-motion generators (SMG). In Sections 2 and 3, the geometry and the velocity analysis of the SMG are recalled from a previous publication (Gauthier et al., 2009). Then, the constrained kinematics Jacobian for the SMG is formulated. The frame-invariant expression for the condition number is derived and minimized in Section 4, thereby the home posture of the robot of interest is found in Section 5. The

[1] $\cos x = (1 - T^2)/(1 + T^2)$, $\sin x = 2T/(1 + T^2)$, with $T \equiv \tan(x/2)$

report ends with some concluding remarks.

2 Geometry of the SMG

The SMG is a two-limb parallel robot with four degrees of freedom. Its *moving platform* (MP) motion belongs to the *Schönflies displacement subgroup* (Angeles, 2004), which consists of three translations in the Cartesian space and one rotation about a fixed-orientation axis. Each limb comprises four joints forming a $R\Pi\Pi R$ kinematc chain, where R and Π denote revolute and parallogram joints, respectively. Since the planar parallelogram linkage produces a pure translation of its coupler link with respect to its fixed link, the linkage can be regarded as a kinematic pair, which is sometimes referred to as the Π pair. It has been studied as such in the literature (Hervé and Sparacino, 1992; Wohlhart, 1992; Dietmaier, 1998). The axes of two revolutes in both limbs are parallel to the vertical axis. Hence, each limb can produce translation in the three-dimensional Cartesian space and one rotation about the vertical axis, as stated in the definition of the Schönflies displacement subgroup. Two limbs are connected to the MP by means of the end revolutes via an unlimited-rotaion mechanism. The side and top views of the SMG are illustrated in Figs. 1 and 2. Before proceeding to derive the geometric relations of the SMG, the notation used in the balance of the report is listed in Table 1. Vectors and matrices are denoted in the report by bold lower-cases and bold upper-cases, respectively.

Table 1: Notation

Notation	Description
θ_{Ji}	The ith angle in the Jth limb for $J = I, II$ and $i = 1 \dots 5$
ϕ	The angle of rotation of the moving platform
$\begin{bmatrix} x & y & z \end{bmatrix}^T$	Cartesian coordinates of point P on the moving platform
c_{Ji}	$\cos\theta_{Ji}$
s_{Ji}	$\sin\theta_{Ji}$
$\mathbf{a}_{Ji}$	Vector $\overrightarrow{O_{J(i-1)}O_{Ji}}$ for $J = I, II$ and $i = 1, 2, 3$
$\mathbf{a}_{I4}$	Vector $\overrightarrow{O_{I3}P'}$
$\mathbf{a}_{II4}$	Vector $\overrightarrow{O_{II3}O_{II4}}$
$\mathbf{a}_{II5}$	Vector $\overrightarrow{O_{II4}P'}$

Considering the side view of the SMG shown in Fig. 1, the position vectors $\mathbf{p}_I$ and $\mathbf{p}_{II}$ of two points, P' and O_{II4}, respectively, are found as

$$\mathbf{p}_I = \mathbf{a}_{I1} + \mathbf{a}_{I2} + \mathbf{a}_{I3} + \mathbf{a}_{I4}, \quad \mathbf{p}_{II} = \mathbf{a}_{II1} + \mathbf{a}_{II2} + \mathbf{a}_{II3} + \mathbf{a}_{II4} + l_0\mathbf{i} \tag{1}$$

where

$$\mathbf{a}_{J1} = l_1 \begin{bmatrix} c_{J1} \\ s_{J1} \\ 0 \end{bmatrix}, \quad \mathbf{a}_{J2} = l_2 \begin{bmatrix} c_{J2}c_{J1} \\ c_{J2}s_{J1} \\ s_{J2} \end{bmatrix}, \quad \mathbf{a}_{J3} = l_3 \begin{bmatrix} c_{J3}c_{J1} \\ c_{J3}s_{J1} \\ s_{J3} \end{bmatrix}, \quad \mathbf{a}_{J4} = a_{J4} \begin{bmatrix} c_{J1} \\ s_{J1} \\ 0 \end{bmatrix}$$

Moreover, the relation between $\mathbf{p}_I$ and $\mathbf{p}_{II}$ is obtained from the top view as

$$\mathbf{p}_I = \mathbf{p}_{II} + \mathbf{a}_{II5}, \quad \mathbf{a}_{II5} = a_{II5} \begin{bmatrix} -c_\phi & s_\phi & 0 \end{bmatrix}^T \tag{2}$$

Apparently, the x and y Cartesian coordinates of the MP at operation point P are the same as those of point P', while the third coordinates obey the relation $z = z' + h$.

The coordinates of point P' in Cartesian space can be simply obtained from the first relation of eq. (1):

$$x = (l_1 + l_2c_{I2} + l_3c_{I3} + a_{I4})c_{I1}, \quad y = (l_1 + l_2c_{I2} + l_3c_{I3} + a_{I4})s_{I1}, \quad z' = l_2s_{I2} + l_3s_{I3} \tag{3}$$

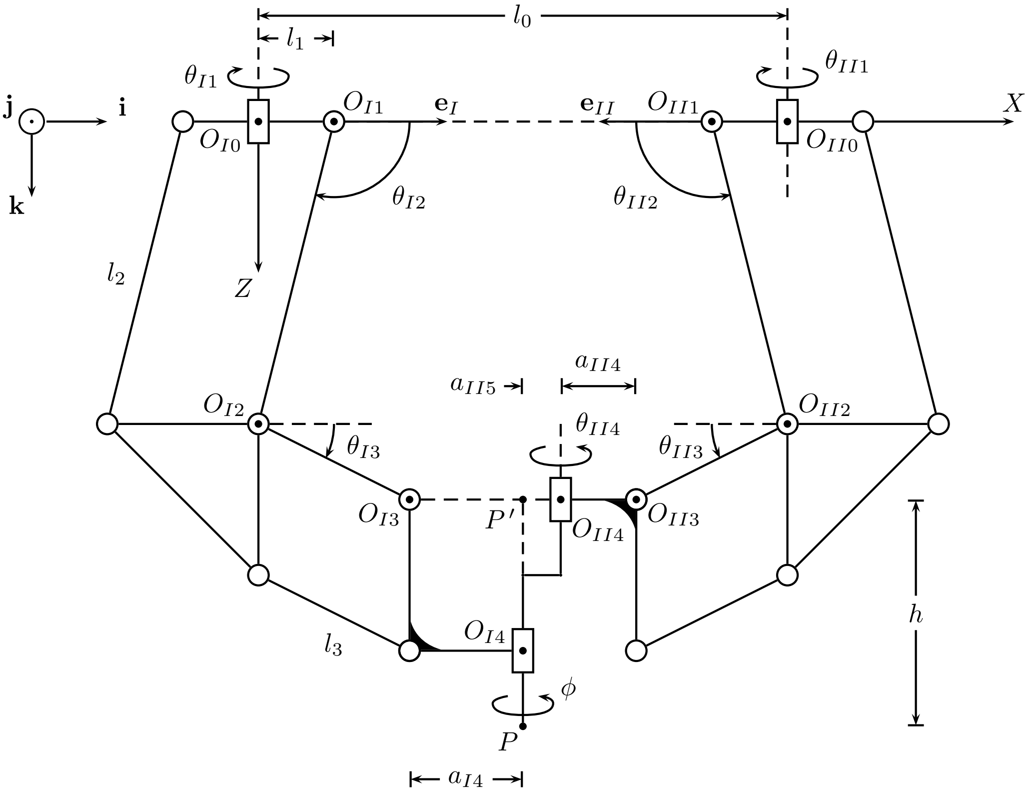

Figure 1: Front view of the kinematic chain of the SMG with the two leg-planes coincident with the X-Z plane (Gauthier et al., 2009)

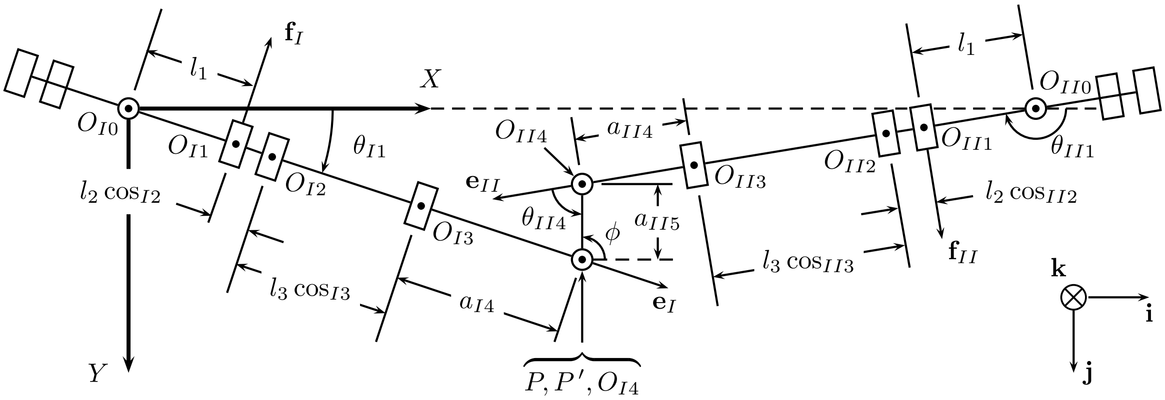

Figure 2: Top view of the kinematic chain of the SMG at an arbitrary posture (Gauthier et al., 2009)

Substituting $\mathbf{p}_{II}$ from eq. (2) into the second relation of eq. (1) yields

$$x = (l_1 + l_2 c_{II2} + l_3 c_{II3} + a_{II4}) c_{II1} - a_{II5} c_\phi + l_0 \tag{4a}$$

$$y = (l_1 + l_2 c_{II2} + l_3 c_{II3} + a_{II4}) s_{II1} + a_{II5} s_\phi \tag{4b}$$

$$z' = l_2 s_{II2} + l_3 s_{II3} \tag{4c}$$

The angle of rotation ϕ of the MP is determined from the top view in Fig. 2, namely,

$$\phi = \theta_{II4} - \theta_{II1} + \pi, \quad \phi = \theta_{I4} - \theta_{I1} \tag{5}$$

3 Velocity Analysis of the SMG

In order to find the relation between the MP twist and the joint rates, both sides of eqs. (1) are differentiated with respect to time, whence,

$$\dot{\mathbf{p}}_J = \dot{\mathbf{a}}_{J1} + \dot{\mathbf{a}}_{J2} + \dot{\mathbf{a}}_{J3} + \dot{\mathbf{a}}_{J4} \quad \text{for } J = I, II \tag{6}$$

The time-rates of change of vectors $\mathbf{a}_{Ji}$ are found as

$$\dot{\mathbf{a}}_{Ji} = \begin{cases} \dot{\theta}_{Ji}\mathbf{k} \times \mathbf{a}_{Ji} & \text{for } J = I, II \text{ and } i = 1 \\ (\dot{\theta}_{J1}\mathbf{k} + \dot{\theta}_{Ji}\mathbf{f}_J) \times \mathbf{a}_{Ji} & \text{for } J = I, II \text{ and } i = 2, 3 \\ \dot{\theta}_{J1}\mathbf{k} \times \mathbf{a}_{Ji} & \text{for } J = I, II \text{ and } i = 4 \\ -\dot{\phi}\mathbf{k} \times \mathbf{a}_{Ji} & \text{for } J = II \text{ and } i = 5 \end{cases} \tag{7}$$

Substituting $\dot{\mathbf{a}}_{Ji}$ from the equation above into eq. (6) yields

$$\dot{\mathbf{p}}_J = \dot{\theta}_{I1}\mathbf{k} \times (\mathbf{a}_{J1} + \mathbf{a}_{J2} + \mathbf{a}_{J3} + \mathbf{a}_{J4}) + \dot{\theta}_{J2}\mathbf{f}_J \times \mathbf{a}_{J2} + \dot{\theta}_{J3}\mathbf{f}_J \times \mathbf{a}_{J3} \quad \text{for } J = I, II \tag{8}$$

Moreover, the relation between $\dot{\mathbf{p}}_I$ and $\dot{\mathbf{p}}_{II}$ follows from differentiating eq. (2) with respect to time, i.e.,

$$\dot{\mathbf{p}}_{II} = \dot{\mathbf{p}}_I + \dot{\phi}\mathbf{k} \times \mathbf{a}_{II5} \tag{9}$$

Before stating the relation between the MP twist and the joint rates in invariant form, a useful result, already included in (Gauthier et al., 2009) is recalled below in theorem form:

Theorem 1 *If a rigid body moves within the Schönflies displacement subgroup, the velocities of all points lying on lines parallel to the axis of rotation are identical.*

The twist $\mathbf{t}$ of the MP is defined as an array of four scalars consisting of three Cartesian components of the velocity of point P and the angular velocity $\dot{\phi}$, i.e., $\mathbf{t} = \begin{bmatrix}\dot{\mathbf{p}}_I^T & \dot{\phi}\end{bmatrix}^T$. Theorem 1 implies that the velocities of P and P' are identical. However, the velocity of point P', denoted by $\dot{\mathbf{p}}_I$, has been obtained in eq. (8) for $J = I$. The relation for $\dot{\phi}$ in terms of the other joint rates can be derived by substituting $\dot{\mathbf{p}}_{II}$ from eq. (9) into eq. (8). The passive joint rates in both equations are eliminated from the cross-product of each equation with the vector coefficient of the passive-joint-rate term in that equation, namely,

$$(\mathbf{f}_I \times \mathbf{a}_{I3}) \times \dot{\mathbf{p}}_I = \mathbf{v}_I\dot{\theta}_{I1} + \Delta_I\mathbf{f}_I\dot{\theta}_{I2} \tag{10a}$$

$$(\mathbf{f}_{II} \times \mathbf{a}_{II3}) \times \left[\dot{\mathbf{p}}_I + \dot{\phi}\mathbf{k} \times \mathbf{a}_{II5}\right] = \mathbf{v}_{II}\dot{\theta}_{II1} + \Delta_{II}\mathbf{f}_{II}\dot{\theta}_{II2} \tag{10b}$$

where

$$\mathbf{v}_J = (\mathbf{f}_J \times \mathbf{a}_{J3}) \times (\mathbf{k} \times \mathbf{r}_{J14}), \quad \Delta_J = (\mathbf{f}_J \times \mathbf{a}_{J3}) \cdot \mathbf{a}_{J2}, \quad \mathbf{r}_{J14} = \mathbf{a}_{J1} + \mathbf{a}_{J2} + \mathbf{a}_{J3} + \mathbf{a}_{J4} \tag{11}$$

The relation between the MP twist $\mathbf{t}$ and the actuated-joint-rate vector $\dot{\boldsymbol{\theta}} = \begin{bmatrix}\dot{\theta}_{I1} & \dot{\theta}_{I2} & \dot{\theta}_{II1} & \dot{\theta}_{II2}\end{bmatrix}^T$ can be expressed as

$$\mathbf{A}\mathbf{t} = \mathbf{B}\dot{\boldsymbol{\theta}} \tag{12}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the *forward-* and the *inverse-kinematics* Jacobian matrices, namely,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\Psi}_I \\ \boldsymbol{\Psi}_{II}(\mathbf{k} \times \mathbf{a}_{II5}) & \boldsymbol{\Psi}_{II} \end{bmatrix} \in \mathbb{R}^{6\times4}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{v}_I & \Delta_I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_{II} & \Delta_{II} \end{bmatrix} \in \mathbb{R}^{6\times4} \tag{13}$$

with $\mathbf{0}$ denoting the three-dimensional zero vector and $\boldsymbol{\Psi}_J$ the *cross-product-matrix*[2] (Angeles, 2007) of vector $\boldsymbol{\psi}_J = \mathbf{f}_J \times \mathbf{a}_{J3}$.

[2]The cross product matrix $\mathbf{M}$ of a vector $\mathbf{m} \in \mathbb{R}^3$ is defined as $\mathbf{M} \equiv \mathrm{CPM}(\mathbf{m}) = \partial(\mathbf{m} \times \mathbf{v})/\partial\mathbf{v},\ \forall\mathbf{v} \in \mathbb{R}^3$

3.1 Constrained Kinematics of the SMG

In this subsection we derive the Jacobians relating the passive and the active joint rates. These Jacobians will be used in the derivation of the *first-order normality conditions* (FONC), pertaining to the minimization of the condition number of the Jacobians of interest. To this end, vectors $\dot{\mathbf{p}}_I$ and $\dot{\mathbf{p}}_{II}$ are substituted from eq. (8) into eq. (9), which yields

$$\begin{aligned}\dot{\phi}\mathbf{k} \times \mathbf{a}_{II5} = \dot{\theta}_{II1}\mathbf{k} \times \mathbf{r}_{II14} + \dot{\theta}_{II2}\mathbf{f}_{II} \times \mathbf{a}_{II2} + \dot{\theta}_{II3}\mathbf{f}_{II} \times \mathbf{a}_{II3} \\ - \dot{\theta}_{I1}\mathbf{k} \times \mathbf{r}_{I14} - \dot{\theta}_{I2}\mathbf{f}_I \times \mathbf{a}_{I2} - \dot{\theta}_{I3}\mathbf{f}_I \times \mathbf{a}_{I3}\end{aligned} \tag{14}$$

To eliminate $\dot{\phi}$, eqs. (5) are differentiated with respect to time:

$$\dot{\phi} = \dot{\theta}_{II4} - \dot{\theta}_{II1}, \quad \dot{\phi} = \dot{\theta}_{I4} - \dot{\theta}_{I1} \tag{15}$$

Substituting the first of eqs. (15) into eq. (14) and then rearranging the result yields

$$\dot{\theta}_{II3}\boldsymbol{\psi}_{II} - \dot{\theta}_{II4}\boldsymbol{\sigma} - \dot{\theta}_{I3}\boldsymbol{\psi}_I = \dot{\theta}_{I1}\mathbf{k} \times \mathbf{r}_{I14} + \dot{\theta}_{I2}\mathbf{f}_I \times \mathbf{a}_{I2} - \dot{\theta}_{II1}\mathbf{k} \times (\mathbf{r}_{II14} + \mathbf{a}_{II5}) - \dot{\theta}_{II2}\mathbf{f}_{II} \times \mathbf{a}_{II} \tag{16}$$

Moreover, equating the right hand sides of the two eqs. (15) gives

$$\dot{\theta}_{I4} - \dot{\theta}_{II4} = \dot{\theta}_{I1} - \dot{\theta}_{II1} \tag{17}$$

Equations (16) and (17) can be cast in compact form as

$$\mathbf{C}\dot{\boldsymbol{\theta}}_p = \mathbf{D}\dot{\boldsymbol{\theta}}_a \tag{18}$$

where $\dot{\boldsymbol{\theta}}_p = \begin{bmatrix}\dot{\theta}_{I4} & \dot{\theta}_{I3} & \dot{\theta}_{II4} & \dot{\theta}_{II3}\end{bmatrix}^T$ and $\dot{\boldsymbol{\theta}}_a = \begin{bmatrix}\dot{\theta}_{I1} & \dot{\theta}_{I2} & \dot{\theta}_{II1} & \dot{\theta}_{II2}\end{bmatrix}^T$ are the vectors of the passive and the active joint rates, respectively[3]. The 4×4 matrices $\mathbf{C}$ and $\mathbf{D}$ are

$$\begin{aligned}\mathbf{C} &= \begin{bmatrix}\mathbf{0} & -\boldsymbol{\psi}_I & -\boldsymbol{\sigma} & \boldsymbol{\psi}_{II} \\ -1 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{G} \\ -1 & \mathbf{j}^T\end{bmatrix}, \quad \boldsymbol{\sigma} \equiv \mathbf{k} \times \mathbf{a}_{II5} \\ \mathbf{D} &= \begin{bmatrix}\mathbf{k} \times \mathbf{r}_{I14} & \mathbf{f}_I \times \mathbf{a}_{I2} & -\mathbf{k} \times (\mathbf{r}_{II14} + \mathbf{a}_{II5}) & -\mathbf{f}_{II} \times \mathbf{a}_{II2} \\ -1 & 0 & 1 & 0\end{bmatrix}\end{aligned} \tag{19}$$

where $\mathbf{j}$ was introduced in Fig. 1 and $\mathbf{G}$ is the 3×3 matrix given by

$$\mathbf{G} = \begin{bmatrix}-\boldsymbol{\psi}_I & -\boldsymbol{\sigma} & \boldsymbol{\psi}_{II}\end{bmatrix} \tag{20}$$

Explicit expressions for the passive joint rates in terms of the active ones are obtained from eq. (18):

$$\boldsymbol{\theta}_p = \mathbf{J}_C\theta_a, \quad \mathbf{J}_C = \mathbf{C}^{-1}\mathbf{D}, \quad \mathbf{C}^{-1} = \begin{bmatrix}\mathbf{j}^T\mathbf{G}^{-1} & -1 \\ \mathbf{G}^{-1} & \mathbf{0}\end{bmatrix} \tag{21}$$

where $\mathbf{J}_C$ is henceforth referred to as the *constrained-kinematics Jacobian*. Moreover, $\mathbf{G}^{-1}$ is obtained in invariant form by resorting to the concept of *reciprocal bases* (Brand, 1965) as

$$\mathbf{G}^{-1} = \frac{1}{\delta}\begin{bmatrix}-(\boldsymbol{\sigma} \times \boldsymbol{\psi}_{II})^T \\ -(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T \\ (\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T\end{bmatrix}, \quad \delta = \boldsymbol{\psi}_I \times \boldsymbol{\sigma} \cdot \boldsymbol{\psi}_{II} \tag{22}$$

Subtituting eq. (22) into eq. (21) yields

$$\begin{aligned}&\mathbf{J}_C = \begin{bmatrix}j_{11} & j_{12} & j_{13} & j_{14} \\ \mathbf{j}_{21} & \mathbf{j}_{22} & \mathbf{j}_{23} & \mathbf{j}_{24}\end{bmatrix}, \quad j_{11} = \mathbf{j}^T\mathbf{G}^{-1}(\mathbf{k} \times \mathbf{r}_{I4}) + 1, \quad j_{12} = \mathbf{j}^T\mathbf{G}^{-1}(\mathbf{f}_I \times \mathbf{a}_{I2}) \\ &j_{13} = -\mathbf{j}^T\mathbf{G}^{-1}(\mathbf{k} \times \mathbf{r}_{II14} + \boldsymbol{\sigma}) - 1, \quad j_{14} = -\mathbf{j}^T\mathbf{G}^{-1}(\mathbf{f}_{II} \times \mathbf{a}_{II2}), \quad \mathbf{j}_{21} = \mathbf{G}^{-1}(\mathbf{k} \times \mathbf{r}_{I14}) \\ &\mathbf{j}_{22} = \mathbf{G}^{-1}(\mathbf{f}_I \times \mathbf{a}_{I2}), \quad \mathbf{j}_{23} = -\mathbf{G}^{-1}(\mathbf{k} \times \mathbf{r}_{I14} + \boldsymbol{\sigma}), \quad \mathbf{j}_{24} = -\mathbf{G}^{-1}(\mathbf{f}_{II} \times \mathbf{a}_{II2})\end{aligned} \tag{23}$$

[3]The definition of $\dot{\boldsymbol{\theta}}_p$ may look awkward in light of that of $\dot{\boldsymbol{\theta}}_a$, the reason behind this unusual definition being the ease with which the matrix $\mathbf{C}$ thus resulting can be inverted.

4 Kinetostatic Conditioning of the SMG

The kinetostatics of parallel robots in general is briefly recalled in (Gauthier et al., 2009), with special attention to the McGill Schönflies Motion Generator motivating this paper. We will not dwell on this issue further here. Rather, the effective computational issues related to kinetostatic conditioning are stressed below.

4.1 Condition Number of the Forward Kinematics Jacobian

Before deriving the condition number of Jacobian $\mathbf{A}$, it is necessary to non-dimensionalize all its entries in order to cope with the dimensional inhomogeneity of the matrix. A dimensionless Jacobian can be achieved by introducing the *characteristic length* L, as yet to be determined, so that the dimensionless vectors below are introduced:

$$\boldsymbol{\xi}_J = \frac{\mathbf{p}_J}{L}, \quad \boldsymbol{\rho}_{J1} = \frac{\mathbf{r}_{J14}}{L}, \quad \boldsymbol{\rho}_{J2} = \frac{\mathbf{a}_{J2}}{L}, \quad \boldsymbol{\rho}_{J3} = \frac{\mathbf{a}_{J3}}{L}, \quad \boldsymbol{\rho}_{II5} = \frac{\mathbf{a}_{II5}}{L}, \quad \overline{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{L} \tag{24}$$

Furthermore, to simplify the manipulation of the equations, $\boldsymbol{\rho}_{J3}$ is also normalized as

$$\tilde{\boldsymbol{\rho}}_{J3} = \frac{L}{l_3}\boldsymbol{\rho}_{J3}, \quad \|\tilde{\boldsymbol{\rho}}_{J3}\| = 1 \tag{25}$$

Upon replacing in the original matrix $\mathbf{A}$ the entries bearing units of length with the above vectors, the dimensionless forward-kinematics Jacobian is obtained as

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \overline{\boldsymbol{\Psi}}_I \\ \overline{\boldsymbol{\Psi}}_{II}\left(\mathbf{k} \times \boldsymbol{\rho}_{II5}\right) & \overline{\boldsymbol{\Psi}}_{II} \end{bmatrix} \in \mathbb{R}^{6\times4} \tag{26}$$

where $\overline{\boldsymbol{\Psi}}_J$ is the cross-product matrix of the unit vector $\overline{\boldsymbol{\psi}}_J = \mathbf{f}_J \times \tilde{\boldsymbol{\rho}}_{J3}$.

Different definitions of the norm can be used to evaluate the condition number of any matrix. By adopting the *weighted Frobenius norm*, the condition number of a $n \times n$ matrix $\mathbf{M}$ is obtained as

$$\kappa^2\left(\mathbf{M}\mathbf{M}^T\right) = \frac{1}{n^2}\mathrm{tr}(\mathbf{M}\mathbf{M}^T)\mathrm{tr}\left(\mathbf{M}\mathbf{M}^T\right)^{-1}, \quad \mathrm{tr}(\cdot) = \text{trace of}\,(\cdot) \tag{27}$$

Thus, all we need to implement eq. (27) is the trace of $\overline{\mathbf{A}}^T\overline{\mathbf{A}}$ and of its inverse. The product $\overline{\mathbf{A}}^T\overline{\mathbf{A}}$ is readily calculated as

$$\overline{\mathbf{A}}^T\overline{\mathbf{A}} = \begin{bmatrix} a_{11} & \mathbf{a}_{21}^T \\ \mathbf{a}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad a_{11} = \|\mathbf{m}_{21}\|^2, \quad \mathbf{m}_{21} = \overline{\boldsymbol{\Psi}}_{II}\overline{\boldsymbol{\sigma}}, \quad \mathbf{a}_{21} = \overline{\boldsymbol{\Psi}}_{II}^T\mathbf{m}_{21}, \quad \mathbf{A}_{22} = \overline{\boldsymbol{\Psi}}_I^T\overline{\boldsymbol{\Psi}}_I + \overline{\boldsymbol{\Psi}}_{II}^T\overline{\boldsymbol{\Psi}}_{II} \tag{28}$$

Apparently,

$$\mathrm{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right) = a_{11} + \mathrm{tr}\left(\mathbf{A}_{22}\right) \tag{29}$$

while $\mathbf{A}_{22}$ can be expanded as

$$\mathbf{A}_{22} = \mathbf{f}_I\mathbf{f}_I^T + \tilde{\boldsymbol{\rho}}_{I3}\tilde{\boldsymbol{\rho}}_{I3}^T + \mathbf{f}_{II}\mathbf{f}_{II}^T + \tilde{\boldsymbol{\rho}}_{II3}\tilde{\boldsymbol{\rho}}_{II3}^T \tag{30}$$

Since $\mathbf{f}_J$ and $\tilde{\boldsymbol{\rho}}_{J3}$ for $J = I, II$ are all unit vectors,

$$\mathrm{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right) = a_{11} + 4 \tag{31}$$

Matrix $\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1}$ is found from the formula for the inversion of block matrices (Beyer, 1987):

$$\begin{aligned} \left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1} &= \begin{bmatrix} i_{11} & \mathbf{i}_{21}^T \\ \mathbf{i}_{21} & \mathbf{I}_{22} \end{bmatrix}, \quad i_{11} = \frac{1}{a_{11} - \mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\mathbf{a}_{21}}, \quad \mathbf{i}_{21} = -\mathbf{A}_{22}^{-1}\mathbf{a}_{21}i_{11} \\ \mathbf{I}_{22} &= \left(\mathbf{A}_{22} - \frac{1}{a_{11}}\mathbf{a}_{21}\mathbf{a}_{21}^T\right)^{-1} \end{aligned} \tag{32}$$

An expression for $\mathbf{I}_{22}$ is derived upon recalling the *Matrix Inversion Lemma* (MIL):

Lemma 1 (Bryson and Ho, 1975) *Let* $\mathbf{Q}$ *and* $\mathbf{R}$ *be* $m \times m$ *and* $n \times n$ *matrices, respectively. If* $\mathbf{G}$ *is an arbitrary* $n \times m$ *matrix, then,*

$$\mathbf{Q} - \mathbf{Q}\mathbf{G}\left(\mathbf{R} + \mathbf{G}^T\mathbf{Q}\mathbf{G}\right)^{-1}\mathbf{G}^T\mathbf{Q} \equiv \left[\mathbf{Q}^{-1} + \mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T\right]^{-1} \tag{33}$$

We exploit Lemma 1 to compute $\mathbf{I}_{22}$ by recalling the substitutions proposed by (Gauthier et al., 2009):

$$\mathbf{Q} = \mathbf{A}_{22}, \quad \mathbf{G} = \left(\mathbf{A}_{22}\right)^{-1}\mathbf{a}_{21}, \quad \mathbf{R} = a_{11} - \mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\mathbf{a}_{21} \tag{34}$$

As $\mathbf{R}$ is apparently a scalar, it is replaced henceforth by r, to be consistent with our notation. Prior to implementing the MIL, we have to verify the sign-definition of $\mathbf{Q}$ and r. From eq. (30), it is apparent that $\mathbf{A}_{22}$ is symmetric and positive-definite, and so is its inverse. Hence, $\mathbf{Q}$ is positive-definite. The scalar r is also positive because

$$\det\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right) = \det\left(\mathbf{A}_{22}\right)\left(a_{11} - \mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\mathbf{a}_{21}\right) = \Delta_{22}\, r > 0, \quad \Delta_{22} \equiv \det\left(\mathbf{A}_{22}\right) \tag{35}$$

whence,

$$\mathbf{Q} - \mathbf{Q}\mathbf{G}\left(\mathbf{R} + \mathbf{G}^T\mathbf{Q}\mathbf{G}\right)^{-1}\mathbf{G}^T\mathbf{Q} = \mathbf{A}_{22} - \frac{1}{a_{11}}\mathbf{a}_{21}\mathbf{a}_{21}^T \tag{36}$$

Considering eqs. (28), (36) and the MIL, $\mathbf{I}_{22}$ turns out to be

$$\mathbf{I}_{22} = \mathbf{A}_{22}^{-1} + \frac{1}{r}\left(\mathbf{A}_{22}^{-1}\mathbf{a}_{21}\mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\right) \tag{37}$$

The trace of $\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1}$ is obtained from eqs. (32) and (37) as

$$\mathrm{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1} = \frac{1}{r}\left(1 + \mathbf{a}_{21}^T\mathbf{A}_{22}^{-2}\mathbf{a}_{21}\right) + \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right) \tag{38}$$

To find $\mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)$, the eigenvalues of $\mathbf{A}_{22}$ are sought. Recalling the expression for $\mathbf{A}_{22}$ in eq. (28) and the skew-symmetry of cross-product matrices, we have

$$\mathbf{A}_{22} = -\left(\overline{\mathbf{\Psi}}_I^2 + \overline{\mathbf{\Psi}}_{II}^2\right) \tag{39}$$

Since $\overline{\boldsymbol{\psi}}_J$, for $J = I, II$, are unit vectors, the identity

$$\overline{\mathbf{\Psi}}_J^2 = -\mathbf{1} + \overline{\boldsymbol{\psi}}_J\overline{\boldsymbol{\psi}}_J^T \tag{40}$$

follows, with $\mathbf{1}$ denoting the 3×3 identity matrix. Substituting eq. (40) into eq. (39) yields

$$\mathbf{A}_{22} = (2)\mathbf{1} - \overline{\boldsymbol{\psi}}_I\overline{\boldsymbol{\psi}}_I^T - \overline{\boldsymbol{\psi}}_{II}\overline{\boldsymbol{\psi}}_{II}^T \tag{41}$$

One can verify that the three eigenvalues of matrix $\mathbf{A}_{22}$ and their corresponding eigenvectors are

$$\begin{aligned} &\lambda_1 = 2, && \lambda_2 = 1 - \overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}, && \lambda_3 = 1 + \overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II} \\ &\mathbf{v}_1 = \frac{\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}}{\|\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}\|}, && \mathbf{v}_2 = \frac{\overline{\boldsymbol{\psi}}_I + \overline{\boldsymbol{\psi}}_{II}}{\|\overline{\boldsymbol{\psi}}_I + \overline{\boldsymbol{\psi}}_{II}\|}, && \mathbf{v}_3 = \frac{\overline{\boldsymbol{\psi}}_I - \overline{\boldsymbol{\psi}}_{II}}{\|\overline{\boldsymbol{\psi}}_I - \overline{\boldsymbol{\psi}}_{II}\|} \end{aligned} \tag{42}$$

Moreover, $\det\left(\mathbf{A}_{22}\right)$, denoted by Δ_{22}, is readily derived as the product of of its three eigenvalues, i.e.,

$$\Delta_{22} = 2\left[1 - \left(\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}\right)^2\right] = 2\|\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}\|^2 \tag{43}$$

Considering the expression for $\mathbf{A}_{22}$ in eq. (41), its inverse is very likely to be of the form

$$\mathbf{A}_{22}^{-1} = k_1\mathbf{1} + k_2\mathbf{U} + k_3\mathbf{V}, \quad \mathbf{U} = \overline{\boldsymbol{\psi}}_I\overline{\boldsymbol{\psi}}_I^T + \overline{\boldsymbol{\psi}}_{II}\overline{\boldsymbol{\psi}}_{II}^T, \quad \mathbf{V} = \overline{\boldsymbol{\psi}}_I\overline{\boldsymbol{\psi}}_{II}^T + \overline{\boldsymbol{\psi}}_{II}\overline{\boldsymbol{\psi}}_I^T \tag{44}$$

The unknown coefficients k_1, k_2 and k_3 are readily found from $\mathbf{A}_{22}\mathbf{A}_{22}^{-1} = \mathbf{1}$, which yields

$$\mathbf{A}_{22}^{-1} = \frac{1}{2}\mathbf{1} + \frac{1}{2\|\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}\|^2}\left(\overline{\boldsymbol{\psi}}_I\overline{\boldsymbol{\psi}}_I^T + \overline{\boldsymbol{\psi}}_{II}\overline{\boldsymbol{\psi}}_{II}^T\right) + \frac{\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}}{2\|\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}\|^2}\left(\overline{\boldsymbol{\psi}}_I\overline{\boldsymbol{\psi}}_{II}^T + \overline{\boldsymbol{\psi}}_{II}\overline{\boldsymbol{\psi}}_I^T\right) \tag{45}$$

where $\mathbf{1}$ denotes the 3×3 identity matrix. The trace of $\mathbf{A}_{22}^{-1}$ is now obtained as

$$\mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right) = \frac{3}{2} + \frac{2}{\Delta_{22}}\left[1 + \left(\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}\right)^2\right] \tag{46}$$

Upon substituting eqs. (31), (38) and (46) into eq. (27), the final form of the squared condition number sought is

$$\kappa_{\overline{\mathbf{A}}}^2 = \frac{1}{16}\left(a_{11} + 4\right)\left\{\frac{1}{r}\left[1 + \mathbf{a}_{21}^T\left(\mathbf{A}_{22}^{-1}\right)^2\mathbf{a}_{21}\right] + \frac{3}{2} + \frac{2}{\Delta_{22}}\left[1 + \left(\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}\right)^2\right]\right\} \tag{47}$$

4.2 Minimization of the Condition Number of the Forward-Kinematics Jacobian

The home posture of the robot under study is defined as its *farthest* possible configuration from singularities. One can regard the condition number of the Jacobian of the robot at any specific posture as an index of the above distance. The condition number of $\overline{\mathbf{A}}$ is minimized below over its posture variables, in order to find the home posture of the robot motivating this study.

Here, we are interested in minimizing $\kappa_{\overline{\mathbf{A}}}^2$ for the existing prototype of the SMG, whose geometry is given. The expression for $\kappa_{\overline{\mathbf{A}}}^2$ is given in eq. (47) as an explicit function of six variables, namely, $\theta_{I1}, \theta_{I2}, \theta_{II1}, \theta_{II2}, \phi$ and L. Since the SMG has four degrees of freedom, only four joint variables are independent. Hence, the foregoing set of variables is not independent. To formulate the problem as an *unconstrained optimization* task (Luenberger, 2003), we introduce the *design-variable* vector $\mathbf{x} = [\theta_{I1}, \theta_{I2}, \theta_{II1}, \theta_{II2}, L]^T$, consisting of four actuated joint rates plus the characteristic length. The posture with the minimum condition number of the forward-kinematics Jacobian will be found upon solving a system of five nonlinear algebraic equations, namely, the FONC, in terms of the design variables. Thus, it is required to derive the gradient of $\kappa_{\overline{\mathbf{A}}}$, or of its square for that matter, with respect to the five design variables.

Differentiation of the expression in eq. (47) is a trivial task if that expression is expanded in terms of components. However, such an expression will a) be unmanageably long and b) not lend itself to a geometric interpretation. For this reason, we do not follow this track. Instead, we take a frame-invariant approach, and keep vectors and matrices as such, without expanding them into components.

The FONC are found from the vanishing of the derivatives of $\kappa_{\overline{\mathbf{A}}}^2$, given by eq. (27), with $\mathbf{M} = \overline{\mathbf{A}}$:

$$\frac{\partial \mathrm{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})}{\partial x_i}\mathrm{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})^{-1} + \mathrm{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})\frac{\partial \mathrm{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})^{-1}}{\partial x_i} = 0, \quad \text{for } i = 1, \ldots, 5 \tag{48}$$

where x_i, for $i = 1, \ldots, 5$, is the ith design variable. Considering eqs. (28) and (31), the first derivative term of eq. (48) is calculated as

$$\frac{\partial \mathrm{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})}{\partial x_i} = 2\mathbf{m}_{21}^T\frac{\partial \mathbf{m}_{21}}{\partial x_i} \tag{49}$$

Substituting $\mathbf{m}_{21}$ from eq. (28) yields

$$\frac{\partial \mathbf{m}_{21}}{\partial x_i} = \frac{\partial \overline{\mathbf{\Psi}}_{II}}{\partial x_i}(\mathbf{k} \times \boldsymbol{\rho}_{II5}) + \overline{\mathbf{\Psi}}_{II}\left(\frac{\partial \mathbf{k}}{\partial x_i} \times \boldsymbol{\rho}_{II5} + \mathbf{k} \times \frac{\partial \boldsymbol{\rho}_{II5}}{\partial x_i}\right) \tag{50}$$

We thus need to find the derivatives of $\overline{\mathbf{\Psi}}_{II}$, $\mathbf{k}$ and $\boldsymbol{\rho}_{II5}$ with respect to each design variable.

Vector $\mathbf{k}$ being constant, its derivatives with respect to all design variables vanish. The derivatives of $\overline{\boldsymbol{\psi}}_{II}$, the vector of the cross-product matrix $\overline{\mathbf{\Psi}}_{II}$, are calculated below:

$$\frac{\partial \overline{\boldsymbol{\psi}}_{II}}{\partial x_i} = \frac{\partial \mathbf{f}_{II}}{\partial x_i} \times \tilde{\boldsymbol{\rho}}_{II3} + \mathbf{f}_{II} \times \frac{\partial \tilde{\boldsymbol{\rho}}_{II3}}{\partial x_i} \tag{51}$$

Let us begin with θ_{I1}. Since the orientation of the plane of the second limb does not depend on θ_{I1}, $\partial \mathbf{f}_{II}/\partial\theta_{I1}$ vanishes. From the chain rule, the next derivative, $\partial\tilde{\boldsymbol{\rho}}_{II3}/\partial\theta_{I1}$, is equal to $\partial\dot{\tilde{\boldsymbol{\rho}}}_{II3}/\partial\dot{\theta}_{I1}$. From eq. (7), vector $\dot{\tilde{\boldsymbol{\rho}}}_{II3}$ is found as

$$\dot{\tilde{\boldsymbol{\rho}}}_{II3} = \dot{\theta}_{II1}\mathbf{k} \times \tilde{\boldsymbol{\rho}}_{II3} + \dot{\theta}_{II3}\mathbf{f}_{II} \times \tilde{\boldsymbol{\rho}}_{II3} \tag{52}$$

whence,

$$\frac{\partial\dot{\tilde{\boldsymbol{\rho}}}_{II3}}{\partial\dot{\theta}_{I1}} = \frac{\partial\dot{\theta}_{II3}}{\partial\dot{\theta}_{I1}}\mathbf{f}_{II} \times \tilde{\boldsymbol{\rho}}_{II3} \tag{53}$$

The coefficient $\partial\dot{\theta}_{II3}/\partial\dot{\theta}_{I1}$ being nothing but the (4,1) entry of $\mathbf{J}_C$, in light of the definition introduced in Subsection 3.1 and hence,

$$\frac{\partial\dot{\theta}_{II3}}{\partial\dot{\theta}_{I1}} = \frac{(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{k} \times \mathbf{r}_{I14})}{\delta} \tag{54}$$

From eqs. (51), (53) and (54),

$$\frac{\partial\overline{\boldsymbol{\psi}}_{II}}{\partial\theta_{I1}} = p_1\mathbf{f}_{II} \times \overline{\boldsymbol{\psi}}_{II}, \quad p_1 \equiv \frac{(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{k} \times \mathbf{r}_{I14})}{\delta} \tag{55}$$

Recalling that $\overline{\boldsymbol{\psi}}_{II} = \mathbf{f}_{II} \times \tilde{\boldsymbol{\rho}}_{II3}$, the above equation can be simplified as

$$\frac{\partial\overline{\boldsymbol{\psi}}_{II}}{\partial\theta_{I1}} = p_1\left[(\mathbf{f}_{II}.\tilde{\boldsymbol{\rho}}_{II3})\mathbf{f}_{II} - (\mathbf{f}_{II}.\mathbf{f}_{II})\tilde{\boldsymbol{\rho}}_{II3}\right] \tag{56}$$

Since $\mathbf{f}_{II}$ is the unit normal vector to the plane of the second limb, it is also orthogonal to vector $\tilde{\boldsymbol{\rho}}_{II3}$, and hence,

$$\frac{\partial\overline{\boldsymbol{\psi}}_{II}}{\partial\theta_{I1}} = -p_1\tilde{\boldsymbol{\rho}}_{II3} \tag{57}$$

The last remaining term in eq. (50), $\partial\boldsymbol{\rho}_{II5}/\partial\theta_{I1}$, is also equal to $\partial\dot{\boldsymbol{\rho}}_{II5}/\partial\dot{\theta}_{I1}$ by virtue of the chain rule. The equations below are reproduced from the results of the velocity analysis in eqs. (5) and (7):

$$\dot{\boldsymbol{\rho}}_{II5} = -\dot{\phi}\mathbf{k} \times \boldsymbol{\rho}_{II5}, \quad \dot{\phi} = \dot{\theta}_{II4} - \dot{\theta}_{II1} \tag{58}$$

From the (3,1) entry of the constrained kinematics Jacobian in eq. (23), we obtain

$$\frac{\partial\dot{\theta}_{II4}}{\partial\dot{\theta}_{I1}} = \frac{-(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T (\mathbf{k} \times \mathbf{r}_{I14})}{\delta} \tag{59}$$

whence,

$$\frac{\partial\boldsymbol{\rho}_{II5}}{\partial\theta_{I1}} = \frac{\partial\dot{\boldsymbol{\rho}}_{II5}}{\partial\dot{\theta}_{I1}} = p_2\overline{\boldsymbol{\sigma}}, \quad p_2 \equiv \frac{(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T (\mathbf{k} \times \mathbf{r}_{I14})}{\delta} \tag{60}$$

Substituting eqs. (57) and (60) into eq. (50) yields

$$\frac{\partial\mathbf{m}_{21}}{\partial\theta_{I1}} = -p_1\tilde{\boldsymbol{\rho}}_{II3} \times \overline{\boldsymbol{\sigma}} + p_2\overline{\boldsymbol{\psi}}_{II} \times (\mathbf{k} \times \overline{\boldsymbol{\sigma}}) \tag{61}$$

Recalling $\overline{\boldsymbol{\sigma}} = \mathbf{k} \times \boldsymbol{\rho}_{II5}$, eq. (61) can be simplified as

$$\frac{\partial\mathbf{m}_{21}}{\partial\theta_{I1}} = -p_1\tilde{\boldsymbol{\rho}}_{II3} \times \overline{\boldsymbol{\sigma}} - p_2\overline{\boldsymbol{\psi}}_{II} \times \boldsymbol{\rho}_{II5} \tag{62}$$

Substituting eq. (62) into eq. (49) yields

$$\frac{\partial \mathrm{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)}{\partial\theta_{I1}} = 2\mathbf{m}_{21}^T\left(-p_1\tilde{\boldsymbol{\rho}}_{II3} \times \overline{\boldsymbol{\sigma}} - p_2\overline{\boldsymbol{\psi}}_{II} \times \boldsymbol{\rho}_{II5}\right) \tag{63}$$

In order to formulate the first FONC from eq. (48), we need $\partial \text{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1}/\partial\theta_{I1}$. From the expression for the trace of $\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1}$, given in eq. (38), its derivative with respect to θ_{I1} follows:

$$\begin{aligned}\frac{\partial \text{tr}(\overline{\mathbf{A}}^T\overline{\mathbf{A}})^{-1}}{\partial\theta_{I1}} &= \frac{\partial\,(1/r)}{\partial\theta_{I1}}\left(1+\mathbf{a}_{21}^T\mathbf{A}_{22}^{-2}\mathbf{a}_{21}\right)+\frac{1}{r}\left(2\mathbf{a}_{21}^T\mathbf{A}_{22}^{-2}\frac{\partial\mathbf{a}_{21}}{\partial\theta_{I1}}+\mathbf{a}_{21}^T\frac{\partial\mathbf{A}_{22}^{-2}}{\partial\theta_{I1}}\mathbf{a}_{21}\right)\\ &\quad-\frac{2}{\Delta_{22}^2}\frac{\partial\Delta_{22}}{\partial\theta_{I1}}\left[1+\left(\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}\right)^2\right]+\frac{4\overline{\boldsymbol{\psi}}_I^T\overline{\boldsymbol{\psi}}_{II}}{\Delta_{22}}\end{aligned} \tag{64}$$

Let us calculate each derivative in the above equation, separately. Using eq. (34), $\partial\,(1/r)\,/\partial\theta_{I1}$ is expanded as

$$\frac{\partial\,(1/r)}{\partial\theta_{I1}}=\frac{-1}{r^2}\left(\frac{\partial a_{11}}{\partial\theta_{I1}}-2\mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\frac{\partial\mathbf{a}_{21}}{\partial\theta_{I1}}-\mathbf{a}_{21}^T\frac{\partial\mathbf{A}_{22}^{-1}}{\partial\theta_{I1}}\mathbf{a}_{21}\right) \tag{65}$$

The term $\partial a_{11}/\partial\theta_{I1}$ is already available in eq. (63). Moreover,

$$\frac{\partial\mathbf{a}_{21}}{\partial\theta_{I1}}=\frac{\partial\overline{\mathbf{\Psi}}_{II}^T}{\partial\theta_{I1}}\mathbf{m}_{21}+\overline{\mathbf{\Psi}}_{II}^T\frac{\partial\mathbf{m}_{21}}{\partial\theta_{I1}} \tag{66}$$

Substituting $\partial\overline{\mathbf{\Psi}}_{II}/\partial\theta_{I1}$ and $\partial\mathbf{m}_{21}/\partial\theta_{I1}$ from eqs. (57) and eq. (62) into eq. (66) leads to:

$$\frac{\partial\mathbf{a}_{21}}{\partial\theta_{I1}}=p_1\tilde{\boldsymbol{\rho}}_{II3}\times\mathbf{m}_{21}+\overline{\mathbf{\Psi}}_{II}\left(p_1\tilde{\boldsymbol{\rho}}_{II3}\times\overline{\boldsymbol{\sigma}}+p_2\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\rho}_{II5}\right) \tag{67}$$

The partial derivative of the inverse of a square matrix $\mathbf{M}$ with respect to a scalar argument x is recalled below for quick reference:

$$\frac{\partial\mathbf{M}^{-1}}{\partial x}=-\mathbf{M}^{-1}\frac{\partial\mathbf{M}^{-1}}{\partial x}\mathbf{M}^{-1} \tag{68}$$

Applying eq. (68) for calculating $\partial\mathbf{A}_{22}^{-1}/\partial\theta_{I1}$ requires $\partial\mathbf{A}_{22}/\partial\theta_{I1}$. Upon recalling the expansion of $\mathbf{A}_{22}$ from eq. (30), its derivative is obtained as

$$\begin{aligned}\frac{\partial\mathbf{A}_{22}}{\partial\theta_{I1}} &= \frac{\partial\mathbf{f}_I}{\partial\theta_{I1}}\mathbf{f}_I^T+\mathbf{f}_I\frac{\partial\mathbf{f}_I^T}{\partial\theta_{I1}}+\frac{\partial\tilde{\boldsymbol{\rho}}_{I3}}{\partial\theta_{I1}}\tilde{\boldsymbol{\rho}}_{I3}^T+\tilde{\boldsymbol{\rho}}_{I3}\frac{\partial\tilde{\boldsymbol{\rho}}_{I3}^T}{\partial\theta_{I1}}+\frac{\partial\mathbf{f}_{II}}{\partial\theta_{I1}}\mathbf{f}_{II}^T+\mathbf{f}_{II}\frac{\partial\mathbf{f}_{II}^T}{\partial\theta_{I1}}\\ &\quad+\frac{\partial\tilde{\boldsymbol{\rho}}_{II3}}{\partial\theta_{I1}}\tilde{\boldsymbol{\rho}}_{II3}^T+\tilde{\boldsymbol{\rho}}_{II3}\frac{\partial\tilde{\boldsymbol{\rho}}_{II3}^T}{\partial\theta_{I1}}\end{aligned} \tag{69}$$

Considering Fig. 2, the derivative of vector $\mathbf{f}_J$ normal to the plane of the Jth limb turns out to be

$$\frac{\partial\mathbf{f}_J}{\partial\theta_{J1}}=\mathbf{k}\times\mathbf{f}_J\equiv\mathbf{e}_J,\quad\text{for } J=I,II \tag{70}$$

The term $\partial\tilde{\boldsymbol{\rho}}_{II3}/\partial\theta_{I1}$ is already available in eq (53). Again, by application of the chain rule, $\partial\tilde{\boldsymbol{\rho}}_{I3}/\partial\theta_{I1}$ equals $\partial\dot{\tilde{\boldsymbol{\rho}}}_{I3}/\partial\dot\theta_{I1}$ which calls for

$$\dot{\tilde{\boldsymbol{\rho}}}_{I3}=\dot\theta_{I1}\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3}+\dot\theta_{I3}\mathbf{f}_I\times\tilde{\boldsymbol{\rho}}_{I3} \tag{71}$$

whence,

$$\frac{\partial\dot{\tilde{\boldsymbol{\rho}}}_{I3}}{\partial\dot\theta_{I1}}=\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3}+\frac{\partial\dot\theta_{I3}}{\partial\dot\theta_{I1}}\mathbf{f}_I\times\tilde{\boldsymbol{\rho}}_{I3} \tag{72}$$

Moreover, $\partial\dot\theta_{I3}/\partial\dot\theta_{I1}$ is found from the (2,1) entry of the constrained-kinematics Jacobian in eq. (23) as

$$\frac{\partial\dot\theta_{I3}}{\partial\dot\theta_{I1}}=\frac{-\left(\boldsymbol{\sigma}\times\boldsymbol{\psi}_{II}\right)^T\left(\mathbf{k}\times\mathbf{r}_{I14}\right)}{\delta} \tag{73}$$

Hence,

$$\frac{\partial\tilde{\boldsymbol{\rho}}_{I3}}{\partial\theta_{I1}}=\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3}+p_3\overline{\boldsymbol{\psi}}_I,\quad p_3\equiv\frac{-\left(\boldsymbol{\sigma}\times\boldsymbol{\psi}_{II}\right)^T\left(\mathbf{k}\times\mathbf{r}_{I14}\right)}{\delta} \tag{74}$$

For future reference, it is helpful to find $\partial\overline{\boldsymbol{\psi}}_I/\partial\theta_{I1}$, namely,

$$\frac{\partial\overline{\boldsymbol{\psi}}_I}{\partial\theta_{I1}} = (\mathbf{k} + p_3\mathbf{f}_I) \times \overline{\boldsymbol{\psi}}_I = \mathbf{k} \times \overline{\boldsymbol{\psi}}_I - p_3\tilde{\boldsymbol{\rho}}_{I3} \tag{75}$$

Substituting eqs. (53), (70) and (74) back into eq. (69) yields

$$\begin{aligned}\frac{\partial\mathbf{A}_{22}}{\partial\theta_{I1}} &= \mathbf{f}_I\,(\mathbf{k}\times\mathbf{f}_I)^T + (\mathbf{k}\times\mathbf{f}_I)\,\mathbf{f}_I^T + \left(\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3} + p_3\overline{\boldsymbol{\psi}}_I\right)\tilde{\boldsymbol{\rho}}_{I3}^T \\ &\quad + \tilde{\boldsymbol{\rho}}_{I3}\left(\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3} + p_3\overline{\boldsymbol{\psi}}_I\right)^T + p_1\overline{\boldsymbol{\psi}}_{II}\tilde{\boldsymbol{\rho}}_{II3}^T + p_1\tilde{\boldsymbol{\rho}}_{II3}\overline{\boldsymbol{\psi}}_{II}^T\end{aligned} \tag{76}$$

Further, substituting eqs. (66) and (76) into eq. (65) gives $\partial\,(1/r)\,/\partial\theta_{I1}$ as

$$\begin{aligned}\frac{\partial\,(1/r)}{\partial\theta_{I1}} = \frac{-1}{r^2}\Bigg\{ & -2p_1\mathbf{m}_{21}^T\,(\tilde{\boldsymbol{\rho}}_{II3}\times\overline{\boldsymbol{\sigma}}) - 2p_2\mathbf{m}_{21}^T\,(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\rho}_{II5}) \\ & - 2\mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\left[p_1\tilde{\boldsymbol{\rho}}_{II3}\times\mathbf{m}_{21} + \overline{\boldsymbol{\Psi}}_{II}\,(p_1\tilde{\boldsymbol{\rho}}_{II3}\times\overline{\boldsymbol{\sigma}} + p_2\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\rho}_{II5})\right] \\ & + \mathbf{a}_{21}^T\mathbf{A}_{22}^{-1}\Big[(\mathbf{k}\times\mathbf{f}_I)^T + \mathbf{f}_I\,(\mathbf{k}\times\mathbf{f}_I)^T + \tilde{\boldsymbol{\rho}}_{I3}\,[(\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3}) + p_3\overline{\boldsymbol{\psi}}_I]^T \\ & + [(\mathbf{k}\times\tilde{\boldsymbol{\rho}}_{I3}) + p_3\overline{\boldsymbol{\psi}}_I]\,\tilde{\boldsymbol{\rho}}_{I3}^T + p_1\tilde{\boldsymbol{\rho}}_{II3}\overline{\boldsymbol{\psi}}_{II}^T + p_1\overline{\boldsymbol{\psi}}_{II}\tilde{\boldsymbol{\rho}}_{II3}^T\Big]\mathbf{A}_{22}^{-1}\mathbf{a}_{21}^T\Bigg\}\end{aligned} \tag{77}$$

Going back to eq. (64), the derivative of the squared inverse matrix and its determinant are as yet to be determined. The former is found by a straightforward application of eq. (68):

$$\frac{\partial(\mathbf{M}^{-1})^2}{\partial x} = \frac{\partial\mathbf{M}^{-2}}{\partial x} = -\mathbf{M}^{-1}\frac{\partial\mathbf{M}}{\partial x}\mathbf{M}^{-2} - \mathbf{M}^{-2}\frac{\partial\mathbf{M}}{\partial x}\mathbf{M}^{-1} \tag{78}$$

The derivative of the determinant of matrix $\mathbf{A}_{22}$ is obtained by recalling the expression for $\det(\mathbf{A}_{22})$ in eq. (43) as

$$\frac{\partial}{\partial\theta_{I1}}\det(\mathbf{A}_{22}) = \frac{\partial\Delta_{22}}{\partial\theta_{I1}} = 4\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left(\frac{\partial\overline{\boldsymbol{\psi}}_I}{\partial\theta_{I1}}\times\overline{\boldsymbol{\psi}}_{II} + \overline{\boldsymbol{\psi}}_I\times\frac{\partial\overline{\boldsymbol{\psi}}_{II}}{\partial\theta_{I1}}\right) \tag{79a}$$

Substituting $\partial\overline{\boldsymbol{\psi}}_I/\partial\theta_{I1}$ and $\partial\overline{\boldsymbol{\psi}}_{II}/\partial\theta_{I1}$ from eqs. (75) and (57) into the above equation leads to

$$\frac{\partial\Delta_{22}}{\partial\theta_{I1}} = 4\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left[(\mathbf{k}\times\overline{\boldsymbol{\psi}}_I - p_3\tilde{\boldsymbol{\rho}}_{I3})\times\overline{\boldsymbol{\psi}}_{II} - p_1\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right] \tag{79b}$$

All required derivatives in the expression for $\partial\mathrm{tr}\left(\overline{\mathbf{A}}^T\overline{\mathbf{A}}\right)^{-1}/\partial\theta_{I1}$ are now available.

5 Determination of the Home Posture of the SMG

The home posture of the SMG is found here via the minimization of the condition number of Jacobian $\mathbf{A}$, as given by eq. (47). This is done upon solving the five FONC introduced in subsection 4.2.

The first FONC can be found by substituting the obtained derivatives into eq. (48). All other FONC are derived likewise. Then, each FONC is rearranged as a polynomial in terms of the charectristic length L. Thus, the ith FONC turns out to be of the form

$$\eta_{i2}L^4 + \eta_{i1}L^2 + \eta_{i0} = 0, \quad \text{for } i = 1, \ldots, 5 \tag{80}$$

its polynomial coefficients η_{ij} being displayed in Appendix A.

The expressions thus resulting, although streamlined when compared to their expansion componentwise, still take three pages, which makes a solution of eqs. (80) numerically challenging. In order to solve eqs. (80),

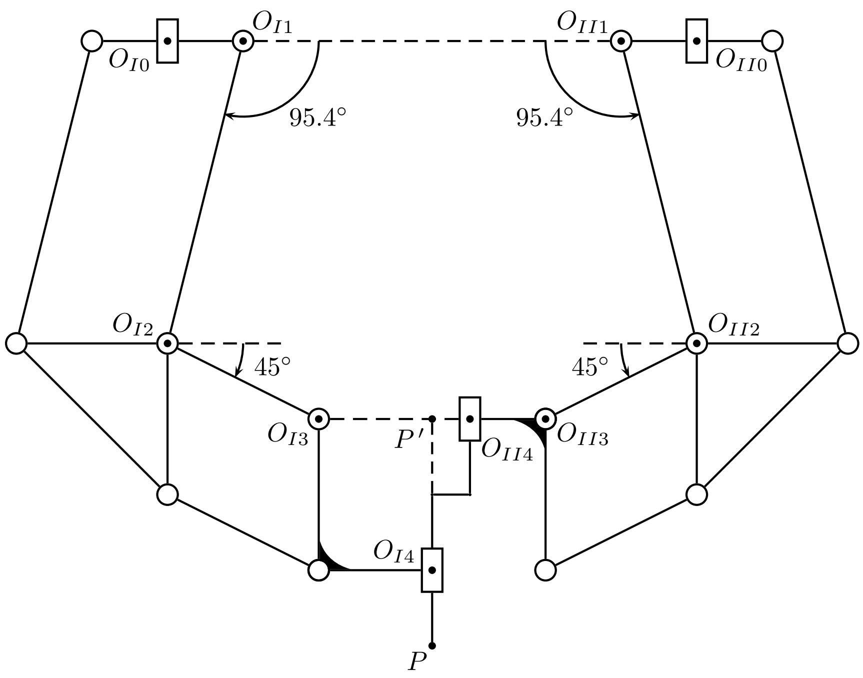

Figure 3: Front view of the kinematic chain of the SMG at the symmetric home posture

we resort to our engineering insight into the problem. Considering the symmetry of the architecture of the SMG, it is very likely that the home posture defined above is symmetric. Hence, the home posture is anticipated to occur at the configuration where the planes of the two limbs coincide. Therefore, θ_{I1} and θ_{II1} are substituted in the FONC with 0 and π, respectively. Moreover, the angle of rotation ϕ of the MP is assumed to vanish at the home posture. The system of five nonlinear algebraic equations of the FONC should be solved now for the three remaining unknowns, θ_{I3}, θ_{II3} and L. However, substituting the assumed values mentioned above for the three variables, θ_{I1}, θ_{II1} and ϕ, the first and the third FONC, $\partial\kappa^2_{\overline{\mathbf{A}}}/\partial\theta_{I1}$ and $\partial\kappa^2_{\overline{\mathbf{A}}}/\partial\theta_{II1}$, are satisfied identically. Thus, a reduced system of three equations in three unknowns is obtained:

$$4\frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{I2}}L^2 + \|\overline{\mathbf{\Psi}}_{II}\left(\mathbf{k}\times\mathbf{a}_{II5}\right)\|^2\frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{I2}} = 0 \tag{81a}$$

$$4\frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{II2}}L^2 + \|\overline{\mathbf{\Psi}}_{II}\left(\mathbf{k}\times\mathbf{a}_{II5}\right)\|^2\frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{II2}} = 0 \tag{81b}$$

$$\frac{\partial\kappa^2_{\overline{\mathbf{A}}}}{\partial L} = 0 \tag{81c}$$

Since $\mathbf{A}_{22}^{-1}$ is symmetric and positive-definite, its trace is also positive. Therefore, verification of eqs. (81a, b) requires

$$\frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{I2}} = 0 \quad \text{and} \quad \frac{\partial \mathrm{tr}\left(\mathbf{A}_{22}^{-1}\right)}{\partial\theta_{II2}} = 0 \tag{82}$$

If one expands these two expressions, it turns out that they are satisfied by the orthogonality of the two vectors $\overline{\boldsymbol{\psi}}_I$ and $\overline{\boldsymbol{\psi}}_{II}$. In fact, any set of joint angles for θ_{I3} and θ_{II3}, adding up to $\pi/2$, yields the characteristic length $L = 0.01987$m when substituted into eq. (81c), which tallies with the result reported in (Gauthier et al., 2009) using a brute-force approach. Hence, the minimum condition number of the forward-kinematics Jacobian is found as $\kappa_{\overline{\mathbf{A}}} = 1.2196$, occuring at the posture with joint and posture variables

$$[\theta_{I1}, \theta_{II1}, \phi] = [0, \pi, 0]$$

while $\theta_{I3} + \theta_{II3} = \pi/2$. It is noteworthy that a) the minimum condition number reported by Gauthier et al. (2009) is 4.8783 b) the other two independent joint angles, θ_{I2} and θ_{II2}, can be obtained from the solution of the *inverse-displacement problem* reported by Gauthier et al. (2009). If a fully symmetric posture is desired, the joint angles can be chosen as

$$[\theta_{I1}, \theta_{II1}, \theta_{I2}, \theta_{II2}, \theta_{I3}, \theta_{II3}, \phi] = [0, \pi, 0.53\pi, 0.53\pi, \pi/4, \pi/4, 0]$$

which thus determines the home posture sought. This posture is depicted in Fig. 3.

6 Conclusions

In this report the home posture of the McGill Schönflies Motion Generator (SMG) is sought via the minimization of the condition number of the forward-kinematics Jacobian. Although computer algebra can manipulate cumbersome mathematical expressions, it is confined to work with scalar quantities. Indeed, computer algebra is incapable of handling vector and matrix expressions in their invariant form, thereby requiring that the user assume a reference frame and introduce components, which in many cases can lead to extremely cumbersome expressions that prevent calculation of the final numerical results. On the contrary, a coordinate-free formulation of the problem at hand allows for expressions that give insight, and hence, by invoking the intuition of the analyst into the problem, the derivation of the final numerical results.

In the realm of computational kinematics, knowledge from linear algebra proves to be invaluable in manipulating the expressions resulting from the kinematic relations of the system at hand. This is demonstrated here while deriving the first-order normality conditions (FONC) for minimization of the condition number of the forward-kinematics Jacobian of the SMG. As made apparent in the report, the expressions become too cumbersome to prevent the analyst from finding a solution, if the components of the vectors and matrices are substituted at the outset. Nevertheless, by resorting to the relations from the kinematic analysis of the robot, as well as some useful theorems from linear algebra, the FONC of the minimization problem are derived. Considering the algebraic structure of these equations, the symmetry of the robot architecture provides some clues for solving the system of equations thus resulting.

Acknowledgements

The authors would like to thank Canada's Natural Sciences and Engineering Research Council (NSERC) for providing funds to support this research via an *Idea to Innovation Grant*, which allowed the team to produce the experimental platform motivating the work reported here. Further work has been supported under NSERC's *Discovery Grants* program and partly through a James McGill Professorship to the second author.

A Appendix

The coefficients used in this appendix are listed below.

$$\epsilon_{11} = \boldsymbol{\sigma}^T \overline{\boldsymbol{\Psi}}_{II}^T \overline{\boldsymbol{\Psi}}_{II} \boldsymbol{\sigma} \qquad \boldsymbol{\epsilon_{21}} = \overline{\boldsymbol{\Psi}}_{II}^T \overline{\boldsymbol{\Psi}}_{II} \boldsymbol{\sigma} \qquad \varrho = \epsilon_{11} - \boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-1} \boldsymbol{\epsilon_{21}}$$

$$p_4 = \frac{(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{f}_I \times \mathbf{a}_{I2})}{\delta} \qquad p_5 = \frac{(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T (\mathbf{f}_I \times \mathbf{a}_{I2})}{\delta} \qquad p_6 = \frac{-(\boldsymbol{\sigma} \times \boldsymbol{\psi}_{II})^T (\mathbf{f}_I \times \mathbf{a}_{I2})}{\delta}$$

$$p_7 = \frac{(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{f}_I \times \mathbf{a}_{I2})}{\delta} \qquad p_8 = \frac{-(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{k} \times \mathbf{r}_{II14} + \boldsymbol{\sigma})}{\delta} \qquad p_9 = \frac{-(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T (\mathbf{k} \times \mathbf{r}_{II14} + \boldsymbol{\sigma})}{\delta}$$

$$p_{10} = \frac{(\boldsymbol{\sigma} \times \boldsymbol{\psi}_{II})^T (\mathbf{k} \times \mathbf{r}_{II14} + \boldsymbol{\sigma})}{\delta} \qquad p_{11} = \frac{-(\boldsymbol{\psi}_I \times \boldsymbol{\sigma})^T (\mathbf{f}_{II} \times \mathbf{a}_{II2})}{\delta} \qquad p_{12} = \frac{-(\boldsymbol{\psi}_{II} \times \boldsymbol{\psi}_I)^T (\mathbf{f}_{II} \times \mathbf{a}_{II2})}{\delta}$$

$$p_{13} = \frac{(\boldsymbol{\sigma} \times \boldsymbol{\psi}_{II})^T (\mathbf{f}_{II} \times \mathbf{a}_{II2})}{\delta} \qquad p_{14} = \overline{\boldsymbol{\psi}}_I^T \overline{\boldsymbol{\psi}}_{II} \qquad p_{15} = 1 + p_{14}^2$$

$$\begin{aligned}
\boldsymbol{\gamma} &= \overline{\boldsymbol{\psi}}_{II} \times \boldsymbol{\sigma} \\
\boldsymbol{\varpi}_1 &= \overline{\boldsymbol{\psi}}_{II} \times \mathbf{a}_{II5} \\
\boldsymbol{\varpi}_2 &= \overline{\boldsymbol{\psi}}_{II} \times \boldsymbol{\varpi}_1 \\
\boldsymbol{\varpi}_3 &= p_1 \tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\gamma} + p_1 \overline{\boldsymbol{\psi}}_{II} \times (\tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma}) + p_2 \boldsymbol{\varpi}_2 \\
\boldsymbol{\varpi}_4 &= p_4 \tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\gamma} + p_4 \overline{\boldsymbol{\psi}}_{II} \times (\tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma}) + p_5 \boldsymbol{\varpi}_2 \\
\boldsymbol{\varpi}_5 &= \boldsymbol{\gamma} \times \left(\mathbf{k} \times \overline{\boldsymbol{\psi}}_{II} - p_8 \tilde{\boldsymbol{\rho}}_{II3}\right) - \overline{\boldsymbol{\psi}}_{II} \times \left[\left(\mathbf{k} \times \overline{\boldsymbol{\psi}}_{II} - p_8 \tilde{\boldsymbol{\rho}}_{II3}\right) \times \boldsymbol{\sigma}\right] + (1 - p_9)\, \boldsymbol{\varpi}_2 \\
\boldsymbol{\varpi}_6 &= p_{11} \tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\gamma} + p_{11} \overline{\boldsymbol{\psi}}_{II} \times (\tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma}) + p_{12} \boldsymbol{\varpi}_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}_1 &= \mathbf{f}_I \mathbf{e}_I^T + \mathbf{e}_I \mathbf{f}_I^T + \left(\mathbf{k} \times \tilde{\boldsymbol{\rho}}_{I3} + p_3 \overline{\boldsymbol{\psi}}_I\right) \tilde{\boldsymbol{\rho}}_{I3}^T + \tilde{\boldsymbol{\rho}}_{I3} \left(\mathbf{k} \times \tilde{\boldsymbol{\rho}}_{I3} + p_3 \overline{\boldsymbol{\psi}}_I\right)^T + p_1 \overline{\boldsymbol{\psi}}_{II} \tilde{\boldsymbol{\rho}}_{II3}^T + p_1 \tilde{\boldsymbol{\rho}}_{II3} \overline{\boldsymbol{\psi}}_{II}^T \\
\mathbf{Q}_2 &= p_6 \overline{\boldsymbol{\psi}}_I \tilde{\boldsymbol{\rho}}_{I3}^T + p_6 \tilde{\boldsymbol{\rho}}_{I3} \overline{\boldsymbol{\psi}}_1^T + p_7 \overline{\boldsymbol{\psi}}_{II} \tilde{\boldsymbol{\rho}}_{II3}^T + p_7 \tilde{\boldsymbol{\rho}}_{II3} \overline{\boldsymbol{\psi}}_{II3}^T \\
\mathbf{Q}_3 &= \mathbf{f}_{II} \mathbf{e}_{II}^T + \mathbf{e}_{II} \mathbf{f}_{II}^T + \left(\mathbf{k} \times \tilde{\boldsymbol{\rho}}_{II3} + p_8 \overline{\boldsymbol{\psi}}_{II}\right) \tilde{\boldsymbol{\rho}}_{II3}^T + \tilde{\boldsymbol{\rho}}_{II3} \left(\mathbf{k} \times \tilde{\boldsymbol{\rho}}_{II3} + p_8 \overline{\boldsymbol{\psi}}_{II}\right)^T + p_{10} \overline{\boldsymbol{\psi}}_I \tilde{\boldsymbol{\rho}}_{I3}^T + p_{10} \tilde{\boldsymbol{\rho}}_{I3} \overline{\boldsymbol{\psi}}_I^T \\
\mathbf{Q}_4 &= p_{13} \overline{\boldsymbol{\psi}}_I \tilde{\boldsymbol{\rho}}_{I3}^T + p_{13} \tilde{\boldsymbol{\rho}}_{I3} \overline{\boldsymbol{\psi}}_I^T + p_{11} \overline{\boldsymbol{\psi}}_{II} \tilde{\boldsymbol{\rho}}_{II3}^T + p_{11} \tilde{\boldsymbol{\rho}}_{II3} \overline{\boldsymbol{\psi}}_{II3}^T
\end{aligned}$$

The coefficients of the FONC polynomials, as introduced in eq. (80), are listed below.

$$\eta_{12} = \frac{-4}{\varrho^2} \left[2\boldsymbol{\gamma}^T \left(-p_1 \tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma} - p_2 \boldsymbol{\varpi}_1\right) - 2\boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-1} \boldsymbol{\varpi}_3 + \boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-1} \mathbf{Q}_1 \mathbf{A}_{22}^{-1} \boldsymbol{\epsilon_{21}}\right] \tag{83a}$$

$$\begin{aligned}
\eta_{11} = {} & \frac{2}{\varrho} \boldsymbol{\gamma}^T \left(-p_1 \tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma} - p_2 \boldsymbol{\varpi}_1\right) - \frac{\epsilon_{11} + 4\boldsymbol{\epsilon_{21}}^T \left(\mathbf{A}_{22}^{-1}\right)^2 \boldsymbol{\epsilon_{21}}}{\varrho^2} \Big[-2p_1 \boldsymbol{\gamma}^T \left(\tilde{\boldsymbol{\rho}}_{II3} \times \boldsymbol{\sigma}\right) - 2p_2 \boldsymbol{\gamma}^T \boldsymbol{\varpi}_1 \\
& - 2\boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-1} \boldsymbol{\varpi}_3 + \boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-1} \mathbf{Q}_1 \mathbf{A}_{22}^{-1} \boldsymbol{\epsilon_{21}}\Big] + \frac{8}{\varrho} \boldsymbol{\epsilon_{21}}^T \mathbf{A}_{22}^{-2} \boldsymbol{\varpi}_3 - \frac{4}{\varrho} \boldsymbol{\epsilon_{21}}^T \left(\mathbf{A}_{22}^{-1} \mathbf{Q}_1 \mathbf{A}_{22}^{-2} + \mathbf{A}_{22}^{-2} \mathbf{Q}_1 \mathbf{A}_{22}^{-1}\right) \boldsymbol{\epsilon_{21}} \\
& - \frac{32 p_{15}}{\Delta_{22}^2} \left(\overline{\boldsymbol{\psi}}_I \times \overline{\boldsymbol{\psi}}_{II}\right)^T \left[\left(\mathbf{k} \times \overline{\boldsymbol{\psi}}_I - p_3 \tilde{\boldsymbol{\rho}}_{I3}\right) \times \overline{\boldsymbol{\psi}}_{II} - p_1 \overline{\boldsymbol{\psi}}_I \times \tilde{\boldsymbol{\rho}}_{II3}\right] + \frac{16 p_{14}}{\Delta_{22}} \Big[\left(\mathbf{k} \times \overline{\boldsymbol{\psi}}_I - p_3 \tilde{\boldsymbol{\rho}}_{I3}\right)^T \overline{\boldsymbol{\psi}}_{II} \\
& - p_1 \overline{\boldsymbol{\psi}}_I^T \tilde{\boldsymbol{\rho}}_{II3}\Big]
\end{aligned} \tag{83b}$$

$$
\begin{aligned}
\eta_{10} =\ & 2\left(\frac{\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho}+\frac{3}{2}+\frac{2}{\Delta_{22}}p_{15}\right)\boldsymbol{\gamma}^T\left(-p_1\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_2\boldsymbol{\varpi}_1\right)-\frac{\epsilon_{11}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Big[-2\boldsymbol{\gamma}^T\left(p_1\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}\right.\\
& \left.-p_2\boldsymbol{\varpi}_1\right)-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_3+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_1\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Big]+\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_3-\frac{\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_1\mathbf{A}_{22}^{-2}\right.\\
& \left.+\mathbf{A}_{22}^{-2}\mathbf{Q}_1\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}-\frac{8\epsilon_{11}p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_I-p_3\tilde{\boldsymbol{\rho}}_{I3}\right)\times\overline{\boldsymbol{\psi}}_{II}-p_1\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right]\\
& +\frac{4\epsilon_{11}p_{14}}{\Delta_{22}}\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_I-p_3\tilde{\boldsymbol{\rho}}_{I3}\right)^T\overline{\boldsymbol{\psi}}_{II}-p_1\overline{\boldsymbol{\psi}}_I^T\tilde{\boldsymbol{\rho}}_{II3}\right]
\end{aligned}\tag{83c}
$$

$$
\eta_{22}=\frac{-4}{\varrho^2}\left[2\boldsymbol{\gamma}^T\left(-p_7\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_5\boldsymbol{\varpi}_1\right)-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_4+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_2\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\right]\tag{83d}
$$

$$
\begin{aligned}
\eta_{21} =\ & \frac{2}{\varrho}\boldsymbol{\gamma}^T\left(-p_7\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_5\overline{\boldsymbol{\psi}}_{II}\times\mathbf{a}_{II5}\right)-\frac{\epsilon_{11}+4\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Big[-2p_7\boldsymbol{\gamma}^T\left(\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}\right)-2p_5\boldsymbol{\gamma}^T\boldsymbol{\varpi}_1\\
& -2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_4+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_2\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Big]+\frac{8}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_4-\frac{4}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_2\mathbf{A}_{22}^{-2}+\mathbf{A}_{22}^{-2}\mathbf{Q}_2\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}\\
& -\frac{32p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left(-p_6\boldsymbol{\rho}_{I3}\times\overline{\boldsymbol{\psi}}_{II}-p_4\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right)+\frac{16p_{14}}{\Delta_{22}}\left(-p_6\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}-p_4\overline{\boldsymbol{\psi}}_I^T\tilde{\boldsymbol{\rho}}_{II3}\right)
\end{aligned}\tag{83e}
$$

$$
\begin{aligned}
\eta_{20} =\ & 2\left(\frac{\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho}+\frac{3}{2}+\frac{2}{\Delta_{22}}p_{15}\right)\boldsymbol{\gamma}^T\left(-p_7\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_5\boldsymbol{\varpi}_1\right)-\frac{\epsilon_{11}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Big[2\boldsymbol{\gamma}^T\left(-p_7\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}\right.\\
& \left.-p_5\boldsymbol{\varpi}_1\right)-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_4+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_2\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Big]+\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_4-\frac{\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_2\mathbf{A}_{22}^{-2}\right.\\
& \left.+\mathbf{A}_{22}^{-2}\mathbf{Q}_2\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}-\frac{8\epsilon_{11}p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left(-p_6\boldsymbol{\rho}_{I3}\times\overline{\boldsymbol{\psi}}_{II}-p_4\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right)+\frac{4\epsilon_{11}p_{14}}{\Delta_{22}}\left(-p_6\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}\right.\\
& \left.-p_4\overline{\boldsymbol{\psi}}_I^T\tilde{\boldsymbol{\rho}}_{II3}\right)
\end{aligned}\tag{83f}
$$

$$
\eta_{32}=\frac{-4}{\varrho^2}\left\{2\boldsymbol{\gamma}^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\times\boldsymbol{\sigma}-(1-p_9)\,\boldsymbol{\varpi}_1\right]-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_5+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_3\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\right\}\tag{83g}
$$

$$
\begin{aligned}
\eta_{31} =\ & \frac{2}{\varrho}\boldsymbol{\gamma}^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\times\boldsymbol{\sigma}-(1-p_9)\boldsymbol{\varpi}_1\right]-\frac{\epsilon_{11}+4\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Big\{\boldsymbol{\gamma}^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\times\boldsymbol{\sigma}\right.\\
& \left.-(1-p_9)\,\boldsymbol{\varpi}_1\right]-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_5+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_3\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Big\}+\frac{8}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_5-\frac{4}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_3\mathbf{A}_{22}^{-2}\right.\\
& \left.+\mathbf{A}_{22}^{-2}\mathbf{Q}_3\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}-\frac{32p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left[\overline{\boldsymbol{\psi}}_I\times\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)-p_{10}\tilde{\boldsymbol{\rho}}_{I3}\times\overline{\boldsymbol{\psi}}_{II}\right]
\end{aligned}\tag{83h}
$$

$$
+\frac{16p_{14}}{\Delta_{22}}\left[-p_{10}\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}+\overline{\boldsymbol{\psi}}_I^T\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\right]\tag{83i}
$$

$$
\begin{aligned}
\eta_{30} =\ & 2\left(\frac{\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho}+\frac{3}{2}+\frac{2}{\Delta_{22}}p_{15}\right)\boldsymbol{\gamma}^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\times\boldsymbol{\sigma}-(p_9-1)\boldsymbol{\varpi}_1\right]\\
& -\frac{\epsilon_{11}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Big\{2\boldsymbol{\gamma}^T\left[\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\times\boldsymbol{\sigma}-(1-p_9)\boldsymbol{\varpi}_1\right]-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_5\\
& +\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_3\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Big\}+\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_5-\frac{\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_3\mathbf{A}_{22}^{-2}+\mathbf{A}_{22}^{-2}\mathbf{Q}_3\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}\\
& -\frac{8\epsilon_{11}p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left[\overline{\boldsymbol{\psi}}_I\times\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)-p_{10}\tilde{\boldsymbol{\rho}}_{I3}\times\overline{\boldsymbol{\psi}}_{II}\right]+\frac{4\epsilon_{11}p_{14}}{\Delta_{22}}\Big[-p_{10}\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}\\
& +\overline{\boldsymbol{\psi}}_I^T\left(\mathbf{k}\times\overline{\boldsymbol{\psi}}_{II}-p_8\tilde{\boldsymbol{\rho}}_{II3}\right)\Big]
\end{aligned}\tag{83j}
$$

$$\eta_{42} = \frac{-4}{\varrho^2}\left\{2\boldsymbol{\gamma}^T\left(-p_{11}\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_{12}\boldsymbol{\varpi}_1\right)-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_6+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_4\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\right\} \tag{83k}$$

$$\begin{aligned}\eta_{41} = {} & \frac{2}{\varrho}\boldsymbol{\gamma}^T\left(-2p_{11}\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-2p_{12}\boldsymbol{\varpi}_1\right)-\frac{\epsilon_{11}+4\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\Bigg[2\boldsymbol{\gamma}^T\left(-p_{11}\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_{12}\boldsymbol{\varpi}_1\right)\\ & -2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_6+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_4\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\Bigg]+\frac{8}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_6-\frac{4}{\varrho}\boldsymbol{\epsilon_{21}}^T\big(\mathbf{A}_{22}^{-1}\mathbf{Q}_4\mathbf{A}_{22}^{-2}\\ & +\mathbf{A}_{22}^{-2}\mathbf{Q}_4\mathbf{A}_{22}^{-1}\big)\boldsymbol{\epsilon_{21}}-\frac{32p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left(-p_{13}\tilde{\boldsymbol{\rho}}_{I3}\times\overline{\boldsymbol{\psi}}_{II}-p_{11}\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right)\\ & +\frac{16p_{14}}{\Delta_{22}}\left(-p_{13}\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}-p_{11}\overline{\boldsymbol{\psi}}_I^T\tilde{\boldsymbol{\rho}}_{II3}\right)\end{aligned} \tag{83l}$$

$$\begin{aligned}\eta_{40} = {} & 2\left(\frac{\mathbf{a}_{21}'^{\,T}(\mathbf{A}_{22}^{-1})^2\mathbf{a}_{21}'}{\varrho}+\frac{3}{2}+\frac{2}{\Delta_{22}}p_{15}\right)\boldsymbol{\gamma}^T\left(-p_{11}\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}-p_{12}\boldsymbol{\varpi}_1\right)\\ & -\frac{\epsilon_{11}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}}{\varrho^2}\left[2\boldsymbol{\gamma}^T\left(-p_{12}\boldsymbol{\varpi}_1-p_{11}\tilde{\boldsymbol{\rho}}_{II3}\times\boldsymbol{\sigma}\right)-2\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\boldsymbol{\varpi}_6+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\mathbf{Q}_4\mathbf{A}_{22}^{-1}\boldsymbol{\epsilon_{21}}\right]\\ & +\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\varpi}_6-\frac{8\epsilon_{11}p_{15}}{\Delta_{22}^2}\left(\overline{\boldsymbol{\psi}}_I\times\overline{\boldsymbol{\psi}}_{II}\right)^T\left(-p_{13}\tilde{\boldsymbol{\rho}}_{I3}\times\overline{\boldsymbol{\psi}}_{II}-p_{11}\overline{\boldsymbol{\psi}}_I\times\tilde{\boldsymbol{\rho}}_{II3}\right)\\ & -\frac{\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\left(\mathbf{A}_{22}^{-1}\mathbf{Q}_4\mathbf{A}_{22}^{-2}+\mathbf{A}_{22}^{-2}\mathbf{Q}_4\mathbf{A}_{22}^{-1}\right)\boldsymbol{\epsilon_{21}}+\frac{4\epsilon_{11}p_{14}}{\Delta_{22}}\left(-p_{13}\tilde{\boldsymbol{\rho}}_{I3}^T\overline{\boldsymbol{\psi}}_{II}-p_{11}\overline{\boldsymbol{\psi}}_I^T\tilde{\boldsymbol{\rho}}_{II3}\right)\end{aligned} \tag{83m}$$

$$\eta_{52} = \frac{8}{\varrho^2}\left[\epsilon_{11}+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\left(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\gamma}\right)\right] \tag{83n}$$

$$\begin{aligned}\eta_{51} = {} & -\frac{2\epsilon_{11}}{\varrho}+\frac{2\epsilon_{11}}{\varrho^2}\left[\epsilon_{11}+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\left(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\gamma}\right)\right]+\frac{8}{\varrho^2}\left(\epsilon_{11}+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}\right)\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}\\ & +\frac{8}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\left(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\gamma}\right)\end{aligned} \tag{83o}$$

$$\begin{aligned}\eta_{50} = {} & -\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}-2\epsilon_{11}\left(\frac{3}{2}+\frac{2}{\Delta_{22}}p_{15}\right)+\frac{2\epsilon_{11}}{\varrho^2}\left[\epsilon_{11}+\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-1}\left(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\gamma}\right)\right]\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\boldsymbol{\epsilon_{21}}\\ & +\frac{2\epsilon_{11}}{\varrho}\boldsymbol{\epsilon_{21}}^T\mathbf{A}_{22}^{-2}\left(\overline{\boldsymbol{\psi}}_{II}\times\boldsymbol{\gamma}\right)\end{aligned} \tag{83p}$$

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